JOURNAL OF DIFFERENTIAL EQUATIONS 18, 18–28 (1975)

Hereditary Differential Systems with Constant Delays. II. A Class of Affine Systems and the Adjoint Problem*

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Revised July 15, 1973

In this paper we (i) specialize some of the results of Delfour and Mitter (*J. Differential Equations* **12**, 1972, 213–235) to a class of representable affine hereditary differential systems, (ii) introduce the hereditary adjoint system, and (iii) give an integral representation of solutions.

1. INTRODUCTION

The object of this paper is to specialize the results of Part I (cf. Delfour and Mitter [6]) to affine hereditary differential systems. In Section 2 we define the class of representable affine hereditary differential systems which we shall exclusively study in this paper. In Section 3 we specialize the results of Theorem 3.3 in Delfour and Mitter [6]. In Section 4 we introduce the hereditary adjoint system and in Section 5 we exhibit an integral representation of solutions. Some of the results in this paper have been announced in Delfour and Mitter [8]. For earlier results on the theory of affine functional differential equations in the framework of continuous functions, the reader is referred to J. K. Hale [11, 12], A. Halanay [10], and H. T. Banks [2] and the bibliography cited therein. For work on partial differential equations with delay, see Artola [1]. All proofs will be omitted since they are straightforward.

* This research was supported in part at Case Western Reserve University by NSF Grant GK-3714, at the Electronic Systems Laboratory of the Massachusetts Institute of Technology by AFOSR Grant 70-1941, AF-AFOSR Grant 72-2273 and NSF Grant GK-25781, and at the University of Montréal by NRC Grant (Canada) A-8730 and a FCAC Grant (Québec).

Additional Notation

$\mathscr{L}(X, Y)$ denotes the Banach space of all continuous linear maps from a real Banach space X into another Banach space Y when endowed with the natural norm

$$\| A \| = \sup_{|x|_X=1} | Ax |_{Y'}, \ A \in \mathscr{L}(X, Y).$$

When $X = Y$, $\mathscr{L}(X, X)$ is written $\mathscr{L}(X)$ and the identity in $\mathscr{L}(X)$ is denoted by I. The transpose of the linear map A in $\mathscr{L}(X, Y)$ is denoted by A^* (in $\mathscr{L}(Y^*, X^*)$).

Given an integer $n \geqslant 1$ and F a closed or open subset of R^n (R, the real numbers), $C(F, X)$ will denote the Banach space of all bounded continuous maps $F \to X$ endowed with the usual sup norm $\| \ \|_C$. $C_c(F; X)$ is the vector space of all continuous maps $F \to X$ with compact support in F; $\mathscr{L}^p(F; X)$ is the vector space of all m-measurable (m, the Lebesgue measure on R^n) maps $F \to X$ which are p-integrable, $1 \leqslant p < \infty$, or essentially bounded, $p = \infty$; the natural Banach space associated with $\mathscr{L}^p(F; X)$ is denoted by $L^p(F; X)$ and the corresponding L^p-norm by $\| \ \|_p$. We shall very often use for F the sets

$$\mathscr{P}(t_0, t_1) = \{(t, s) \in R^2 \mid t_0 \leqslant s \leqslant t < t_1\} \tag{1.1}$$

for $t_0 \in R$ and $t_1 \in]t_0, \infty]$ or

$$\bar{\mathscr{P}}(t_0, T) = \{(t, s) \in R^2 \mid t_0 \leqslant s \leqslant t \leqslant T\} \tag{1.2}$$

for $t_0 < T < \infty$. When F is equal to $I(a, b) = [a, b] \cap R$ for $a < b$ in $[-\infty, \infty]$, we shall write $C(a, b; X)$, $C_c(a, b; X)$, $\mathscr{L}^p(a, b; X)$ or $L^p(a, b; X)$. Let X be a real Banach space, let Ω be an open subset of R^n, let $1 \leqslant p < \infty$ and let $m \geqslant 0$ be an integer. We denote by $W^{m,p}(\Omega; X)$ the Sobolev space of all (equivalence classes) of functions f in $L^p(\Omega; X)$ such that

$$D^j f \in L^p(\Omega; X), \qquad | j | \leqslant m, \tag{1.3}$$

where j is some tuple of integers $\geqslant 0$

$$j = (j_1, \ldots, j_n), \qquad | j | = j_1 + \cdots + j_n,$$

$$D^j = \frac{\partial^{|j|}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}$$

and $D^j f$ is a derivative in the distribution sense. With the norm

$$\| f \|_{W^{m,p}} = \left(\sum_{|j| \leqslant m} \| D^j f \|_p^p \right)^{1/p}, \tag{1.4}$$

$W^{m,p}(\Omega; X)$ is a Banach space; it is reflexive when $1 < p < \infty$ and X is reflexive. Let $\bar{\mathscr{P}}(t_0, T)^0$ denote the interior of $\bar{\mathscr{P}}(t_0, T)$. Exceptionally we shall write $W^{m,p}(\bar{\mathscr{P}}(t_0, T); X)$ instead of $W^{m,p}(\bar{\mathscr{P}}(t_0, T)^0; X)$. We shall also use the notation $W^{m,p}_{\text{loc}}(\mathscr{P}(t_0, t_1); X)$ for the Fréchet space of all (equivalence classes of) functions f in $L^p_{\text{loc}}(\mathscr{P}(t_0, t_1); X)$ such that

$$D^j f \in L^p_{\text{loc}}(\mathscr{P}(t_0, t_1); X) \qquad \forall \mid j \mid \leqslant m.$$

Similarly we shall use the notation $W^{m,p}_{\text{loc}}(t_0, t_1; X)$ for the Fréchet space of all f in $L^p_{\text{loc}}(t_0, t_1; X)$ such that

$$D^j f \in L^p_{\text{loc}}(t_0, t_1; X).$$

Let $1 \leqslant p < \infty$, let $0 < b \leqslant \infty$ and let E be a finite-dimensional Banach space. Consider the following seminorm defined on $\mathscr{L}^p(-b, 0; E)$:

$$\alpha(f) = (| f(0)|^p + \| f \|_p^p)^{1/p}. \tag{1.5}$$

$M^p(-b, 0; E)$ will denote the quotient space of $\mathscr{L}^p(-b, 0: E)$ by its linear subspace $S = \{f \in \mathscr{L}^p \mid \alpha(f) = 0\}$. It is a Banach space with norm $\alpha(f)$. It is isomorphic to $E \times L^p(-b, 0; E)$. We shall also use the notation

$$B^p(-b, 0; E) = E^N \times M^p(-b, 0; E)$$

for some integer $N \geqslant 1$. For additional details regarding these spaces and their use in hereditary differential system, see Delfour and Mitter [6].

2. Representable Affine Hereditary Differential Systems

Given p, $1 \leqslant p < \infty$, a hereditary differential system is said to be affine when the map $f : [t_0, t_1[\times B^p(-b, 0; E) \to E$ satisfies the hypotheses (CAR-1), (LIP) and (BC) of Theorem 3.3 in Delfour and Mitter [6] and the map $z \mapsto f(t, z)$ is affine for all $t \in [t_0, t_1[$. When f characterizes an affine differential system the hypotheses (CAR-1), (LIP) and (BC) of Theorem 3.3 in Delfour and Mitter [6] reduce to

Hypotheses 2.1. (*Affine Hereditary Differential Systems*).
There exist two maps g and l

$$g : [t_0, t_1[\to E, \; l : [t_0, t_1[\times B^p(-b, 0; E) \to E \tag{2.1}$$

for which

$$f(t, z) = l(t, z) + g(t), \; t \in [t_0, t_1[, \; z \in B^p(-b, 0; E); \tag{2.2}$$

the map g is in $L^1_{loc}(t_0, t_1 : E)$ *and the map l has the following properties*:

(i) *the map* $z \mapsto l(t, z) : B^p(-b, 0; E) \to E$ *is linear for all* $t \in [t_0, t_1[$,

(ii) *the map* $t \mapsto l(t, z) : [t_0, t_1[\to E$ *is m-measurable for all* $z \in B^p(-b, 0; E)$,

(iii) *and there exists a map* $n \in L^q_{loc}(t_0, t_1; R)$, $q^{-1} + p^{-1} = 1$, *such that for almost all* $t \in [t_0, t_1[$

$$| l(t, z)| \leqslant n(t) \| z \|_{B^p}, \; z \in B^p(-b, 0; E). \quad ▮ \tag{2.3}$$

In this paper we shall only deal with a subfamily of the set of affine hereditary differential systems.

DEFINITION 2.2 Let $1 \leqslant p < \infty$, let X be a finite dimensional real Hilbert space and let $E = X$ in Hypotheses 2.1. The members of the Representable class $\mathscr{R}$ are affine hereditary differential systems for which the map l satisfies Hypotheses 2.1 and is of the form

$$\begin{aligned} &l(t, (z_N, ..., z_1, \kappa^{-1}(z_{00}, z_{01}))) \\ &\quad = \sum_{j=1}^{N} A_j(t)\, z_j + A_{00}(t)\, z_{00} + \int_{-b}^{0} A_{01}(t, \theta)\, z_{01}(\theta)\, d\theta, \end{aligned} \tag{2.4}$$

where

$$A_{00}, A_1, ..., A_N : [t_0, t_1[\to \mathscr{L}(X),$$

and

$$A_{01} : [t_0, t_1[\times I(-b, 0) \to \mathscr{L}(X). \quad ▮ \tag{2.5}$$

Definition 2.2 is an implicit one and it is more convenient to start with sufficient conditions for the A's rather than Hypotheses 2.1 for the map l. The following gives a set of sufficient conditions on the A's for which the map l satisfies Hypotheses 2.1. Assume that $A_{00}, A_1, ..., A_N$ are strongly m-measurable and bounded on all intervals of the form $[t_0, t]$ for all t in $[t_0, t_1[$ and $A_{01} : [t_0, t_1[\times I(-b, 0) \to \mathscr{L}(X)$ is strongly m-measurable and bounded on all sets of the form $[t_0, t] \times K(t)$ for all t in $[t_0, t_1[$, where $\{K(t) | \, t \in [t_0, t_1[\}$ is a family of subsets of $I(-b, 0)$ with the following properties:

(i) $K(t_1) \subset K(t_2)$ for all $t_1 \leqslant t_2$;

(ii) $\{\theta \in I(-b, 0) | \, A_{01}(t, \theta) \neq 0\} \subset K(t)$;

(iii) $K(t)$ has finite measure for all t.

3. Fundamental Theorem

In this section we specialize the results of Theorem 3.3 in Delfour and Mitter [6] under the hypotheses at the end of section 2. In the remainder of this paper we shall consider representable systems for which the sufficient conditions given in the previous paragraph are verified and we shall identify an element h of $M^p(-b, 0; X)$ with the pair $(h^0, h^1) = \kappa^{-1}(h)$ in $X \times L^p(-b, 0; X)$. We shall also use the more standard Sobolev space $W^{1,p}$ as space of solutions rather than the space AC^p.

Theorem 3.1. *Let the sufficient conditions on the A's be verified. For some* s, $t_0 \leqslant s < t_1$, *consider the system*

$$\frac{dx(t)}{dt} = A_{00}(t)\, x(t) + \sum_{i=1}^{N} A_i(t) \begin{cases} x(t+\theta_i), & t+\theta_i \geqslant s \\ h^1(t+\theta_i - s), & \textit{otherwise} \end{cases}$$

$$+ \int_{-b}^{0} A_{01}(t,\theta) \begin{cases} x(t+\theta), & t+\theta \geqslant s \\ h^1(t+\theta - s), & \textit{otherwise} \end{cases} d\theta + f(t),$$

$$\textit{a.e. in } [s, t_1], \tag{3.1}$$

$$x(s) = h^0,\ h = (h^0, h^1) \quad \textit{in} \quad M^p(-b, 0; X),$$

where f *is in* $L^p_{\text{loc}}(t_0, t_1; X)$.

(i) *Given the initial datum* h *in* $M^p(-b, 0; X)$ *at time* s, *there exists a unique solution* $\phi(\cdot; s, h, f)$ in $W^{1,p}_{\text{loc}}(s, t_1; X)$ *to Eq.* (3.1).

(ii) *The map*

$$(h, f) \mapsto \phi(\cdot; s, h, f) \colon M^p(-b, 0; X) \times L^p_{\text{loc}}(s, t_1; X) \to W^{1,p}_{\text{loc}}(s, t_1; X) \tag{3.2}$$

is linear and continuous and for all $T > s$ *there exists a constant* $c(T) > 0$ *such that*

$$\| \phi(\cdot; s, h, f) \|_{W^{1,p}_{\text{loc}}(s,T;X)} \leqslant c(T)\, [\| h \|_{M^p} + \| f \|_{L^p(s,T;X)}]. \tag{3.3}$$

(iii) *Given* $\epsilon > 0$, *there exists* $\delta(T) > 0$ *such that*

$$| T - s | < \delta(T) \Rightarrow \max_{[s,T]} | \phi(t; s, h, f) - h^0 | + \| D_t \phi(\cdot; s, h, f) \|_{L^p(s,T;X)} < \epsilon, \tag{3.4}$$

where D_t *indicates the distributional derivative with respect to* t.

(iv) *The map*

$$(t, s) \to \phi(t; s, h, f) \colon \mathscr{P}(t_0, t_1) \to X \tag{3.5}$$

is continuous.

4. Hereditary Adjoint System

In this section we introduce the hereditary adjoint system and the hereditary product and characterize solutions of the adjoint system.

Definition 4.1. Given $T > t_0$ and k^0 in X, the Hereditary adjoint system defined in $[t_0, T]$ with final datum k^0 at time T is defined as follows:

$$\frac{dp(t)}{dt} + A_{00}(t)^* p(t) + \sum_{i=1}^{N} \begin{cases} A_i(t-\theta_i)^* p(t-\theta_i), & t-\theta_i \leqslant T \\ 0, & \text{otherwise} \end{cases}$$

$$+ \int_{-b}^{0} \begin{cases} A_{01}(t-\theta, \theta)^* p(t-\theta), & t-\theta \leqslant T \\ 0, & \text{otherwise} \end{cases} d\theta + g(t)$$

$$= 0, \text{ a.e. in } [t_0, T], \qquad p(T) = k^0 \tag{4.1}$$

where $g \in L^p(t_0, T; X)$. ∎

Remark 1. System (4.1) is similar to system (3.1) up to a change in the direction of time. As a result we have the equivalent of Theorem 3.1. Notice also that the maps $t \mapsto A_{00}(t)^*$, $t \mapsto A_i(t)^*$ $(i = 1,..., N)$ and

$$(t, \theta) \mapsto A_{01}(t, \theta)^*$$

are strongly measurable (cf. Hille and Phillips, [13, Theorem 3.5.3, p. 72, Theorem 2.9.2, p. 36, and a remark, p. 73]) and verify the same hypotheses as A_{00}, $A_i (i = 1,..., N)$ and A_{01} at the end of section 2.

Remark 2. (i) For all k^0 in X and T in $]t_0, t_1[$ there exists a unique solution $\psi(\cdot; T, k^0, g)$ in $W^{1,p}(t_0, T; X)$ to Eq. (4.1).

(ii) The map

$$(k^0, g) \mapsto \psi(\cdot; T, k^0, g) : X \times L^p(t_0, T; X) \to W^{1,p}(t_0, T; X)$$

is linear and continuous and there exists $d(T) > 0$ such that

$$\| \psi(\cdot; T, k^0, g)\|_{W^{1,p}(t_0,T,X)} \leqslant d(T) [| k^0 |_X + \| g \|_{L^p(t_0,T,X)}] \tag{4.2}$$

(iii) The map

$$(T, t) \mapsto \psi(t; T, k^0, g) : \mathscr{P}(t_0, t_1) \to X$$

is continuous.

The next definition and the next proposition establish in what sense system (4.1) is "adjoint" to system (3.1).

DEFINITION 4.2. Given $T > t_0$, h in $M^p(-b, 0; X)$ and p in $W^{1,p}(t_0, T; X)$, the Hereditary product at time t of h and p is a function defined in the following manner:

$$\mathscr{H}(t; T, h, p)$$

$$= (p(t), h^0) + \int_{-b}^{0} \left(\int_{\max\{-b,\alpha+t-T\}}^{\alpha} A_{01}(t+\alpha-\theta, \theta)^* p(t+\alpha-\theta), h^1(\alpha)\right) d\alpha$$

$$+ \int_{-b}^{0} \left(\sum_{i=1}^{N} \begin{cases} A_i(t+\alpha-\theta_i)^* p(t+\alpha-\theta_i), & \alpha+t-T < \theta_i \leqslant \alpha \\ 0, & \text{otherwise} \end{cases}, \right.$$

$$\left. h^1(\alpha)\right) d\alpha. \quad \blacksquare \tag{4.3}$$

Remark. When $p = 2$, we can use a simpler definition. Given $T > t_0$, h and k in $M^2(-b, 0; X)$, the Hereditary product at time t of h and k can be defined as follows:

$$\langle k, h\rangle_t$$

$$= (k^0, h^0) + \int_{-b}^{0} \left(\int_{\max\{-b,\alpha+t-T\}}^{\alpha} A_{01}(t+\alpha-\theta, \theta)^* k^1(\theta-\alpha), h^1(\alpha)\right) d\alpha$$

$$+ \int_{-b}^{0} \left(\sum_{i=1}^{N} \begin{cases} A_i(t+\alpha-\theta_i)^* k^1(\theta_i-\alpha), & \alpha+t-T < \theta_i \leqslant \alpha \\ 0, & \text{otherwise} \end{cases}, h^1(\alpha)\right) d\alpha. \tag{4.4}$$

PROPOSITION 4.3. *Fix $t_0 \leqslant s < T$. Let x and p belong to $W^{1,p}(s, T; X)$ and h to $M^p(-b, 0; X)$. Assume that $x(s) = h^0$. Then*

$$\mathscr{H}(t; T, \tilde{x}(t), p) - \mathscr{H}(s; T, h, p)$$

$$= \int_s^t \left(p(r), \frac{dx(r)}{dr} - A_{00}(r)\, x(r) - \sum_{i=1}^{N} A_i(r) \begin{cases} x(r+\theta_i), & r+\theta_i \geqslant s \\ h^1(r+\theta_i-s,) & \textit{otherwise} \end{cases} \right.$$

$$\left. - \int_{-b}^{0} A_{01}(r, \theta) \begin{cases} x(r+\theta), & r+\theta \geqslant s \\ h^1(r+\theta-s), & \textit{otherwise} \end{cases} d\theta\right) dr$$

$$+ \int_s^t \left(\frac{dp(r)}{dr} + A_{00}(r)^* p(r) + \sum_{i=1}^{N} \begin{cases} A_i(r-\theta_i)^* p(r-\theta_i), & r-\theta_i \leqslant T \\ 0, & \textit{otherwise} \end{cases} \right.$$

$$\left. + \int_{-b}^{0} \begin{cases} A_{01}(r-\theta, \theta)^* p(r-\theta), & r-\theta \leqslant T \\ 0, & \textit{otherwise} \end{cases} d\theta, x(r)\right) dr, \tag{4.5}$$

where $\tilde{x}(t) = (\tilde{x}(t)^0, \tilde{x}(t)^1)$ *is defined from x and its initial datum h at time s by*

$$\tilde{x}(t)^0 = x(t), \ \tilde{x}(t)^1(\theta) = \begin{cases} x(t+\theta), & t+\theta \geqslant s \\ h^1(t+\theta-s), & \text{otherwise} \end{cases}.$$

COROLLARY. *Let x and p be the solutions in* $W^{1,p}(s, T; X)$ *of systems* (3.1) *and* (4.1) *with* $f = 0$ *and* $g = 0$, *respectively. Then*

$$\mathscr{H}(t; T, \tilde{x}(t), p) = \text{constant}, \ s \leqslant t \leqslant T,$$

where the constant solely depends on h, k^0 *and T.* ▮

THEOREM 4.4. *Fix* h^0 *and* k^0 *in X. For each* (t, s) *in* $\mathscr{P}(t_0, t_1)$ *define*

$$x(t, s) = \phi(t; s, (h^0, 0), 0), \ p(s, t) = \psi(s; t, k^0, 0). \tag{4.6}$$

Then

$$(p(s, t), h^0) = (k^0, x(t, s)). \tag{4.7}$$

5. Integral Representation of Solutions

In this section we introduce the operator $\Phi^0(t, s)$ and show that given an initial datum h (resp. k^0) and a function f (resp. g) the solution $\phi(t; s, h, f)$ of (3.1) (resp. $\psi(t; T, k^0, g)$ of system (4.1)) can be expressed in terms of Φ^0, h, f (resp. k^0, g) and the operators A_{01} and A_i ($i = 1,\ldots, N$).

Given (t, s) in $\mathscr{P}(t_0, t_1)$ the continuous linear map $h^0 \mapsto \phi(t; s, (h^0, 0), 0)$ defines an element $\Phi^0(t, s)$ of $\mathscr{L}(X)$ in an obvious manner:

$$\Phi^0(t, s)\, h^0 = \phi(t; s, (h^0, 0), 0). \tag{5.1}$$

PROPOSITION 5.1. *For all* h^0 *in X*

(i) $(t, s) \mapsto \Phi^0(t, s)\, h^0$ *is continuous,*

(ii) $t \mapsto \Phi^0(t, s)\, h^0$ *is the solution in* $W^{1,p}_{\mathrm{loc}}(s, t_1; X)$ *of*

$$\begin{aligned} \frac{\partial}{\partial t}\Phi^0(t, s)h^0 &= A_{00}(t)\, \Phi^0(t, s)h^0 + \sum_{i=1}^{N} A_i(t) \begin{cases} \Phi^0(t+\theta_i, s)h^0, & t+\theta_i \geqslant s \\ 0, & \text{otherwise} \end{cases} \\ &\quad + \int_{-b}^{0} A_{01}(t, \theta) \begin{cases} \Phi^0(t+\theta, s), & t+\theta \geqslant s \\ 0, & \text{otherwise} \end{cases} d\theta, \ \text{a.e. in } [s, t_1[, \\ \Phi^0(s, s) &= I, \end{aligned} \tag{5.2}$$

(iii) $(t, s) \mapsto D_t\Phi^0(t, s)\, h^0$ *is m-measurable and bounded on every compact subsets of* $\mathscr{P}(t_0, t_1)$,

(iv) $s \mapsto \Phi^0(t, s)^* h^0$ *is the solution in* $W^{1,p}(t_0, t; X)$ *of*

$$\frac{\partial}{\partial s} \Phi^0(t, s)^* h^0 + A_{00}(s)^* \Phi^0(t, s)^* h^0$$

$$+ \sum_{i=1}^{N} \begin{cases} A_i(s - \theta_i)^* \Phi^0(t, s - \theta_i)^* h^0, & s - \theta_i \leqslant t \\ 0, & \text{otherwise} \end{cases}$$

$$+ \int_{-b}^{0} \begin{cases} A_{01}(s - \theta, \theta)^* \Phi^0(t, s - \theta)^* h^0, & s - \theta \leqslant t \\ 0, & \text{otherwise} \end{cases} d\theta = 0$$

$$\text{a.e. in } [t_0, t], \tag{5.3}$$

(v) $(t, s) \mapsto D_s\Phi^0(t, s)^* h^0$ *and* $(t, s) \mapsto D_s\Phi^0(t, s) h^0$ *are m-measurable and bounded on every compact subsets of* $\mathscr{P}(t_0, t_1)$.

THEOREM 5.2. (i) *For all h in* $M^p(-b, 0; X)$ *and f in* $L^p_{\text{loc}}(s, t_1; X)$

$$\phi(t; s, h, f) = \Phi^0(t, s)h^0 + \int_{-b}^{0} \Phi^1(t, s, \alpha)h^1(\alpha)\, d\alpha + \int_{s}^{t} \Phi^0(t, r) f(r)\, dr, \tag{5.4}$$

where

$$\Phi^1(t, s, \alpha) = \sum_{i=1}^{N} \begin{cases} \Phi^0(t, s + \alpha - \theta_i)A_i(s + \alpha - \theta_i), & \alpha + s - t < \theta_i \leqslant \alpha \\ 0, & \text{otherwise} \end{cases}$$

$$+ \int_{\max\{-b, \alpha+s-t\}}^{\alpha} \Phi^0(t, s + \alpha - \theta)\, A_{01}(s + \alpha - \theta, \theta)\, d\theta. \tag{5.5}$$

(ii) *For all* k^0 *in X and g in* $L^p(t_0, T; X)$

$$\psi(t; T, k^0, g) = \Phi^0(T, t)^* k^0 + \int_{t}^{T} \Phi^0(r, t)^* g(r)\, dr. \tag{5.6}$$

COROLLARY. (i) *For all h in* $M^p(-b, 0; X)$ *the map*

$$(t, s) \mapsto D_t\phi(t; s, h, f) \tag{5.7}$$

is in $L^p_{\text{loc}}(\mathscr{P}(t_0, t_1); X)$.

(ii) *For all h in* $\mathscr{D} = \{(h(0), h) \mid h \in W^{1,p}(-b, 0; X)\}$

$$(t, s) \mapsto D_s\phi(t; s, h, f) \tag{5.8}$$

is in $L^p_{\text{loc}}(\mathscr{P}(t_0, t_1); X)$ *and*

$$\frac{\partial}{\partial s} \phi(t; s, h, f) = -\, \Phi^0(t, s) \Big[A_{00}(s)\, h(0) + \sum_{i=1}^{N} A_i(s)\, h(\theta_i)$$

$$+ \int_{-b}^{0} A_{01}(s, \theta)\, h(\theta)\, d\theta + f(s)\Big] - \int_{-b}^{0} \Phi^1(t, s, \alpha) \frac{dh}{d\alpha}(\alpha)\, d\alpha. \tag{5.9}$$

Remark 1. It will be convenient to introduce the operator $\Phi(t, s)$ in $\mathscr{L}(M^p(-b, 0; X), X)$ defined as

$$\Phi(t, s)h = \Phi^0(t, s)h^0 + \int_{-b}^{0} \Phi^1(t, s, \alpha)\, h^1(\alpha)\, d\alpha \tag{5.10}$$

6. Final Remarks

In part III of this paper we shall present a theory for hereditary differential systems in "state" form systematically using Sobolev spaces. This theory is very similar to the theory of linear evolution equations as developed by J. L. Lions (cf. Lions [15]). We shall also present a state adjoint theory. This adjoint theory is useful in optimal control problems.

Note added in proof. By choosing $b = +\infty$ in (3.1) it can easily be shown that affine Volterra integro-differential equations can be put in the form of equation (3.1).

References

1. M. Artola, Sur les perturbations des équations d'évolution. Applications à des problèmes de retard, *Annales E.N.S.* **2** (1969), 137–253.
2. H. T. Banks, The representation of solutions of linear functional differential equations, *J. Differential Equation* **5** (1969), 399–410.
3. A. Bielecki, Une remarque sur la méthode de Banach–Cacciopoli–Tikhonov dans la théorie des équations différentielles ordinaires, *Bull. Acad. Polon. Sci.* **4** (1956), 261–264.
4. C. Corduneanu, Sur certaines équations fonctionnelles de Volterra, *Funkcial. Ekvac.* **9** (1966), 119–127.
5. M. C. Delfour, Function spaces with a projective structure, *J. Math. Anal. Applic.* **42** (1973), 554–568.
6. M. C. Delfour and S. K. Mitter, Hereditary differential systems with constant delays. I. General case, *J. Differential Equations* **12** (1972), 213–235.
7. M. C. Delfour and S. K. Mitter, Controllability, observability and optimal feedback control of affine hereditary differential systems, *SIAM J. Control* **10** (1972), 298–328.
8. M. C. Delfour and S. K. Mitter, Systèmes d'équations différentielles héréditaires à retards fixes. Une classe de systèmes affines et le problème du système adjoint, *C. R. Acad. Sci. Paris* **272** (1971), 1109–1112.
9. N. Dunford and J. T. Schwartz, Linear operators. I, Interscience, New York, 1967.
10. A. Halanay, Differential equations; stability, oscillations, time lags, Academic Press, New York, 1966.
11. J. K. Hale, Linear functional-differential equations with constant coefficients, *J. Differential Equations* **2** (1963), 291–319.

12. J. K. HALE, Functional differential equations, Springer-Verlag, New York, 1971.
13. E. HILLE AND R. S. PHILLIPS, Functional analysis and semi-groups, American Mathematical Society, Providence, R.I., 1957.
14. J. L. LIONS, Problèmes aux limites dans les équations aux dérivées partielles, Séminaire de Mathématiques Supérieures, été 1962, Les presses de l'Université de Montréal, Montréal, Canada, 1967.
15. J. L. LIONS, Équations différentielles opérationelles et problèmes aux limites, Springer Verlag, Heidelberg-Berlin, 1961.