OPTIMAL CONTROL OF LINEAR INTEGRAL EQUATIONS

WITH A QUADRATIC COST FUNCTION:

THE INFINITE TIME INTERVAL PROBLEM*

by

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September 1974

Revision February 1975

CRM-454

* This research was supported at the Centre de Recherches Mathématiques by NRC (Canada) Grant A-8730, a FCAC Grant of the Ministry of Education (Québec) and by the France-Québec exchange program.

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1. Preliminaries

In this chapter we present a general theory of the stability and the optimal control of a linear autonomous system defined over an infinite time horizon, where the cost function to be minimized is quadratic. This theory simultaneously covers important classes of partial differential equations and differential delay equations.

Necessary and sufficient conditions are given to insure stability, the Lyapunov equation is derived following R. DATKO [2]'s approach and examples are given. For the control problem, we introduce the WONHAM's concept of stabilizability and use the results of the previous chapters to study the asymptotic behavior of the feedback operator $P_T(t)$. From this, we establish the existence of a constant feedback law which minimizes the cost function and show that it is characterized by an algebraic Riccati equation. We finally characterize the asymptotic behavior of the optimal solutions under various conditions and use the concept of observability.

Notations and Terminology. Let X and Y be two real Hilbert spaces with norms $|\ |_X$ and $|\ |_Y$ and inner product $(\ ,\ )_X$ and $(\ ,\ )_Y$. The space of all continuous linear maps $T : X \to Y$ endowed with the *natural norm*

$$|T| = \sup\{|Tx|_Y \mid |x|_X \leq 1\}$$

will be denoted $L(X,Y)$; when endowed with the strong operator topology, it will be denoted $L_s(X,Y)$. When $X = Y$ we shall use the notation $L(X)$ and $L_s(X)$. The transposed operator of T in $L(X)$ is an element of $L(X)$ which will be denoted T^*. T is self adjoint if $T^* = T$; a self adjoint operator T is positive, $T \geq 0$, (resp. T is positive - definite, $T > 0$), if for all x in X $(x,Tx) \geq 0$, (resp. for all $x \neq 0$, $(x,Tx) > 0$).

2. System Description.

In this section, we specialize the linear controlled system (S) of Chapter 3 (Cf. section 2, Def. 1) to the autonomous case. Let X (resp. U) be a real Hilbert space endowed with the inner product (,) (resp. $(\, , \,)_U$) and norm $|\cdot|$ (resp. $|\cdot|_U$).

Definition 2.1. (i) Let G belong to $L(U,X)$ and let the $\Lambda : [0,\infty[\to L(X)$ be a strongly continuous semigroup:

a) $\Lambda(t+s) = \Lambda(t)\Lambda(s)$, $t \geq 0$, $s \geq 0$,

b) $\Lambda(0) = I$ (identity in $L(X)$),

c) $\forall x \in X$, the map $t \mapsto \Lambda(t)x : [0,\infty) \to X$ is continuous.

(ii) The *state* $\xi(t;s,x_0,v)$ of system (S) at time $t > s$ with initial datum x_0 at time $s \geq 0$ and control function v in $L^2_{loc}(s,\infty;U)$ is defined as follows:

$$\xi(t;s,x_0,v) = \Lambda(t-s)x_0 + \int_s^t \Lambda(t-r)Gv(r)dr. \tag{2.1}$$

(iii) Given T, $0 < T < \infty$, the *state* $\pi(t;T,x_T)$ of the adjoint system (S*) at time t, $0 \leq t \leq T$, with final datum x_T at time T is defined as follows:

$$\pi(t;T,x_T) = \Lambda(T-t)^*x_T \ .\square \tag{2.2}$$

Proposition 2.2. The quantity

$$\omega_0 = \operatorname*{Inf}_{t>0} \frac{\log|\Lambda(t)|}{t} \tag{2.3}$$

is finite or equals $-\infty$, and

$$\omega_0 = \lim_{t\to\infty} \frac{\log|\Lambda(t)|}{t} \ . \tag{2.4}$$

For each $\mu > \omega_0$ there is a constant M_μ such that

$$|\Lambda(t)| < M_\mu e^{\mu t}, \quad t \geq 0. \tag{2.5}$$

Proof. Cf. DUNFORD-SCHWARTZ [1], Cor. 5, p. 619. □

Definition 2.3. For $h > 0$ the linear operator F_h is defined by the formula

$$F_h x = \frac{\Lambda(h)x-x}{h}, \quad x \text{ in } X. \tag{2.6}$$

Let $\mathcal{D}(F)$ be the set of all x in X for which the limit,

$$\lim_{h\to 0} F_h x,$$

exists and define the operator F with domain $\mathcal{D}(F)$ by the formula

$$Fx = \lim_{h\to 0} F_h x, \quad \forall x \text{ in } \mathcal{D}(F). \tag{2.7}$$

The operator F with domain $\mathcal{D}(F)$ is called the *infinitesimal generator* of the semigroup Λ.

Proposition 2.4. (i) the set $\mathcal{D}(F)$ is a linear mainifold which is dense in X and F is a closed linear operator on $\mathcal{D}(F)$.

(ii) If x is in $\mathcal{D}(F)$, then $\Lambda(t)x$ is in $\mathcal{D}(F)$, $0 < t < \infty$

$$\frac{d}{dt}\Lambda(t)x = F\Lambda(t)x = \Lambda(t)Fx. \tag{2.8}$$

(iii) If x is in $\mathcal{D}(F)$, then

$$[\Lambda(t)-\Lambda(s)]x = \int_s^t \Lambda(u)Fx\,du,$$

for $0 < s < t < \infty$.

(iv) If $t > 0$ and g is a Lebesque integrable function continuous at t, then

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} g(u)\Lambda(u)x\,du = g(t)\Lambda(t)x. \tag{2.9}$$

(v) The resolvent of F, $\rho(F)$, lies in $\{\lambda\varepsilon C \mid \mathrm{Re}\lambda > \omega_0\}$.

(vi) If $\mathcal{D}(F)$ is endowed with the inner product and norm

$$((x,y)) = (x,y) + (Fx,Fy), \; \| x \| = ((x,x))^{\frac{1}{2}}, \tag{2.10}$$

it becomes a Hilbert space and the canonical injection i of $\mathcal{D}(F)$ in X is dense and continuous.

Proof. Cf. DUNFORD-SCHWARTZ [1] (Lemma 7, p. 619). □

Notation. In the sequel we shall denote by V the domain $\mathcal{D}(F)$ of F endowed with the Hilbert space topology induced by the inner product (2.10). Using the notation and definitions of Proposition 2.4, we denote by X' (resp. V') the topological dual of X (resp. V). We identify the elements of X and X' and denote by $i^* : X \to V'$ the adjoint map of i:

$$\langle v, i^*x \rangle = (iv,x), \; \forall v \in V, \; \forall x \in X$$

where $\langle \cdot,\cdot \rangle$ denotes the duality of V and V'.

3. Formulation of the problem.

Let x be the state of system (S) with initial condition x_o at time 0 and control function v in $L^2_{loc}(0,\infty;U)$. Let Y be a real Hilbert space and let H belong to $\mathcal{L}(X,Y)$. We associate with system (S) the cost function

$$J(v,x_o) = \int_0^\infty [|Hx(t)|_Y^2 + (v(t),Nv(t))_U]dt, \tag{3.1}$$

where N belongs to $\mathcal{L}(U)$ and

$$N^* = N,\ \exists\ \ c > 0 \text{ such that } \forall\ \ u \in U\ (u,Nu)_U \geq c|u|_U^2. \tag{3.2}$$

Our objective is to show that under certain stabilizability hypotheses, there exists a unique u in $L^2_{loc}(0,\infty;U)$ which minimizes the cost function $J(v,x_o)$ over all control functions v in $L^2_{loc}(0,\infty;U)$. We shall also show that the minimizing control u can be synthesized via a constant feedback law

$$u(t) = -N^{-1}G^*P\ x(t), \tag{3.3}$$

where P is a solution of a certain algebraic Riccati equation which characterizes the minimum of the cost function.

4. Stability and stabilizability.

In this section we study the behavior at ∞ of the solutions of the uncontrolled system (S) (that is, $v = 0$). We introduce four types of stability, we derive the Lyapunov equation, we introduce the concept of observability and show how theses concepts are interconnected.

Definition 4.1. The uncontrolled system (S) is said to be

(i) weakly stable if for all x in X, $\Lambda(t)x$ goes to 0 in X weak,

(ii) strongly stable if for all x in X, $\Lambda(t)x$ goes to 0 in X strong,

(iii) L^2-stable if for all x in X, the map $t \mapsto \Lambda(t)x$ belongs to $L^2(0,\infty;X)$,

(iv) exponentially stable if there exists $\alpha > 0$ and $M \geq 1$ such that

$$\forall \quad t \geq 0, \quad \forall \quad x \in X, \quad |\Lambda(t)x| \leq Me^{-\alpha t}. \square$$

When X is finite dimensional the four types of stability are equivalent. When X is infinite dimensional L^2-stability and strong stability are not equivalent as can be seen from the following example of R. DATKO [2].

Example. Let $X = \ell_2$ the Hilbert space of all infinite sequences $x = (x_1,\ldots,x_n,\ldots)$ such that

$$\sum_{n=1}^{\infty} x_n^2 < \infty.$$

Consider the semigroup of operators in $\mathcal{L}(\ell^2)$

$$\Lambda(t)x = (e^{-t}x_1, e^{-t/2}x_2,\ldots,e^{-t/n}\, x_n,\ldots).$$

It can be shown that

(i) $\forall\, x \in \ell^2, \quad \Lambda(t)x \to 0 \quad \text{as} \quad t \to \infty$

(ii) $\forall\, t, \quad |\Lambda(t)| = 1, \quad \omega_o = 0,$

(iii) $Fx = -(x_1, \frac{x_2}{2}, \ldots, \frac{x_n}{n}, \ldots), \quad \mathcal{D}(F) = \ell^2$

(iv) spectrum $(F) = \{-\frac{1}{n} \mid n = 1,2,\ldots\} \cup \{0\}$.

A more general problem could also be formulated. Given a Hilbert space Y with norm $|\cdot|_Y$ and a transformation H in $\mathcal{L}(X,Y)$, what can we say about the stability of the map $t \mapsto H\Lambda(t)x$ or $t \mapsto \Lambda(t)x$. The first result is a characterization of L^2-stability. It uses the following lemma

Lemma 4.2. Let $\{U_n\}$ be an increasing sequence of self adjoint elements of $\mathcal{L}(X)$. If $\sup\{|U_n| : n \to \infty\} < \infty$, there exists a self adjoint element U of $\mathcal{L}(X)$ such that $U_n \to U$ in the strong operator topology.

Proof. Cf. KANTOROVICH-AKILOV p. 189, RIESZ-SZ-NAGY p. 261.□

Proposition 4.3. The following statements are equivalent:

(i) for all x in X

$$(4.1)\qquad \int_0^\infty |H\Lambda(t)x|_Y^2 dt < \infty;$$

(ii) for all x and y in X

$$(4.2)\qquad \lim_{t\to\infty} \int_0^t (H\Lambda(t)x,\, H\Lambda(t)y)_Y dt$$

exists and the operator B defined by

$$(4.3)\qquad (Bx,y) = \int_0^\infty (H\Lambda(t)x,\, H\Lambda(t)y)_Y dt$$

is a positive self adjoint element of $\mathcal{L}(X)$;

(iii) there exists a positive self adjoint element D of $\mathcal{L}(X)$ such that

$$(4.4)\qquad F^*Di + i^*DF + i^*H^*Hi = 0 \quad \text{in} \quad \mathcal{L}(V,V').$$

Proof. (i) ⇒ (ii) We define B(t) in $\mathcal{L}(X)$ as follows:

$$(4.5)\qquad (B(t)x,y) = \int_0^t (H\Lambda(s)x,\, H\Lambda(s)y)_Y ds,\ t > 0.$$

By definition B(t) belongs to $\mathcal{L}(X)$, $B(t)^* = B(t) \geq 0$ and $B(t_2)-B(t_1) \geq 0$ for all $t_2-t_1 \geq 0$. By hypothesis for all x in X.

$$\lim_{t\to\infty} (B(t)x,x)$$

exists. But by symmetry

$$2(B(t)x,y) = (B(t)x,x) + (B(t)y,y) - (B(t)(x-y),x-y)$$

and for all x and y in X

$$\lim_{t\to\infty} (B(t)x,y)$$

exists. By the principle of uniform boundedness

$$\sup_{t>0} |B(t)| < \infty.$$

Let $B_n = B(n)$. Then by Lemma 4.2 there exists a positive self-adjoint element B of $\mathcal{L}(X)$ such that for all x in X

$$\lim_{n\to\infty} B_n x = Bx.$$

But for all x

$$(x,Bx) = \lim_{n\to\infty} (x,B_n x) = \lim_{t\to\infty} (x,B(t)x).$$

By symmetry and positivity (4.3) is true.

(ii) ⇒ (iii). Pick x and y in $\mathcal{D}(F)$. Then by (4.5) for all x in $\mathcal{D}(F)$

$$(4.6)\qquad (B(t)Fx,x) + (B(t)x,Fx) = \int_0^t [(H\Lambda(s)Fx,H\Lambda(s)x)_Y + (H\Lambda(s)x,H\Lambda(s)Fx)_Y]ds$$

$$= \int_0^t \frac{d}{ds} |H\Lambda(s)x|_Y^2 \, ds$$

$$= |H\Lambda(t)x|_Y^2 - |Hx|_Y^2.$$

As a result for all x in $\mathcal{D}(F)$

$$\lim_{t\to\infty} |H\Lambda(t)x|_Y^2 = |Hx|_Y^2 + \lim_{t\to\infty}[(B(t)Fx,x) + (B(t)x,Fx)]$$

$$= |Hx|_Y^2 + (BFx,x) + (Bx,Fx).$$

However we know that for all x in X

$$\int_0^\infty |H\Lambda(t)x|_Y^2 \, dt < \infty$$

and necessarily

$$\underline{\lim}_{t\to\infty} |H\Lambda(t)x|_Y^2 = 0.$$

Since the limit exists, it is necessarily equal to its $\underline{\lim}$ and we obtain

$$\forall \quad x \in \mathcal{D}(F) \quad , \quad \lim_{t\to\infty} |H\Lambda(t)x|_Y = 0.$$

If we now rewrite (4.6) with x and y in $\mathcal{D}(F)$ we obtain, by going to the limit, equation (4.4) with $D = B$.

(iii) $\Rightarrow$ (i) For all x and y in $\mathcal{D}(F)$, $\Lambda(t)x$ and $\Lambda(t)y$ belong to $\mathcal{D}(F)$ and

$$(F\Lambda(t)x,D\Lambda(t)y) + (D\Lambda(t)x,F\Lambda(t)y) + (H\Lambda(t)x, H\Lambda(t)y)_Y = 0$$

$$\text{(4.7)} \qquad \frac{d}{dt}(\Lambda(t)x,D\Lambda(t)y) + (H\Lambda(t)x, H\Lambda(t)y)_Y = 0.$$

For all x in $\mathcal{D}(F)$ and all $t \geq 0$.

$$\text{(4.8)} \qquad \int_0^t |H\Lambda(s)x|_Y^2 \, ds = (x,Dx) - (\Lambda(t)x,D\Lambda(t)x)$$

$$\leq (x,Dx)$$

By density of $\mathcal{D}(F)$ in x inequation (4.8) is true for all x in X. This concludes the proof of the proposition.□

Corollary 4.4. Assume system (4.4) has a positive self-adjoint solution D. Then, $D \geq B$, for all x in X the map $t \mapsto (\Lambda(t)x, D\Lambda(t)x)$ is a monotonically decreasing function of t and for all x and y in X

$$\lim_{t\to\infty} (\Lambda(t)x, D\Lambda(t)y) = (x,Dy) - (x,By). \tag{4.9}$$

Proof. All this is clear from identity (4.8).□

Corollary 4.5. Under the hypotheses of Proposition 4.3 any of the statements implies that

$$\forall \ x \in \mathcal{D}(F), \lim_{t\to\infty} H\Lambda(t)x = 0, \text{ and, } \forall \ x \in X, \lim_{t\to\infty} B\Lambda(t)x = 0. \tag{4.10}$$

Proof. Cf. Proof of Proposition 4.3 and RIESZ-SZ-NAGY p. 261.□

We now specialize the previous results to the case $X = Y$ and $M = I_X$.

Theorem 4.6. The following statements are equivalent:

(i) The uncontrolled system (S) is L^2-stable;

(ii) For all x and y in X

$$\lim_{t\to\infty} \int_0^t (\Lambda(t)x, \Lambda(t)y)dt \tag{4.11}$$

exists and the operator B defined by

$$(Bx,y) = \int_0^\infty (\Lambda(t)x, \Lambda(t)y)dt \tag{4.12}$$

is a positive self-adjoint element of $\mathcal{L}(X)$;

(iii) There exists a positive self-adjoint element D of $\mathcal{L}(X)$ such that

$$F^*Di + i^*DF + i^*i = 0; \tag{4.13}$$

(iv) $$\omega_0 = \operatorname*{Inf}_{t>0} \frac{\log |\Lambda(t)|}{t} < 0;$$

(v) There exist $\mu < 0$ and $M \geq 1$ such that for all x in X

$$|\Lambda(t)x| \leq M \exp(\mu t) \ |x|. \tag{4.14}$$

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) is true by Proposition 4.3.

(v) $\Rightarrow$ (i) is clear. (i) $\Rightarrow$ (iv) By R. DATKO [2, Lemma 3] for all x in X

$$\lim_{t\to\infty} \Lambda(t)x = 0$$

and by the Principle of Uniform Boundedness there exists a constant $C \geq 1$ such that

$$\forall \quad t \geq 0 \quad |\Lambda(t)| \leq C.$$

We also know that there exists a constant $C_1 > 0$ such that

$$\forall \quad x \in X, \int_0^\infty |\Lambda(t)x|^2 dt \leq C_1 |x|^2.$$

Pick ρ, $0 < \rho < C^{-1}$, and define

$$t_x(\rho) = \sup\{t: \forall \quad s \in [0,t], \quad |\Lambda(s)x| \geq \rho|x|\}.$$

The quantity $t_x(\rho)$ is finite since $\Lambda(t)x$ goes to zero as t goes to infinity. As a result

$$t_x(\rho)\cdot\rho^2|x|^2 \leq \int_0^{t_x(\rho)} |\Lambda(t)x|^2 dt \leq \int_0^\infty |\Lambda(t)x|^2 dt \leq C_1^2|x|^2$$

and necessarily

$$t_x(\rho) \leq \left(\frac{C_1}{\rho}\right)^2 = \tau.$$

Thus for all $t > \tau$ and all x

$$|\Lambda(t)x| \leq |\Lambda(t-t_x(\rho))||\Lambda(t_x(\rho))x| \leq C\rho|x| = \rho'|x| \Rightarrow |\Lambda(t)| \leq \rho', \ 0 < \rho' < 1.$$

By definition of ω_0

$$\forall \quad t \geq 0, \quad |\Lambda(t)| \geq e^{\omega_0 t},$$

and necessarily

$$\forall \quad t \geq 0,\ 1 > \rho' \geq |\Lambda(t)| \geq e^{\omega_0 t} \Rightarrow \omega_0 < 0.$$

(iv) $\Rightarrow$ (v). By contradiction. Assume that $\omega_0 \geq 0$. Then

$$|\Lambda(t)| \leq M\ e^{\mu t} \Rightarrow \omega_0 \leq \frac{\log |\Lambda(t)|}{t} \leq \frac{\log M}{t} + \mu$$

$$\Rightarrow \omega_0 = \operatorname*{Inf}_{t>0} \frac{\log |\Lambda(t)|}{t} \leq \operatorname*{Inf}_{t>0} \left| \frac{\log M}{t} + \mu \right| = \mu < 0. \square$$

Remark. (i) Theorem 4.6 establishes the equivalence between L^2-stability and exponential stability. It is originally due to R. DATKO [2].

(ii) A. PAZY also proved that given p, $1 \leq p < \infty$, exponential stability is equivalent to L^p-stability:

$$\forall \quad x \in X, \quad \int_0^\infty |\Lambda(t)x|^p\ dt < \infty.$$

Proposition 4.3 characterizes the "L^2-stability" of the map $t \mapsto H\Lambda(t)x$ and Theorem 4.6 the L^2-stability of the map $t \mapsto \Lambda(t)x$. But is it possible under appropriate hypotheses on H and Λ to conclude to some form of stability for Λ from the "L^2-stability" of the map $t \mapsto H\Lambda(t)x$.

Proposition 4.7. Assume that for all x in X

$$\int_0^\infty |H\Lambda(t)x|_Y^2\ dt < \infty. \tag{4.15}$$

If there exists constants $\alpha > 0$ and β such that for all x in X

$$\int_0^\infty |H\Lambda(t)x|_Y^2\ dt \geq \alpha \int_0^\infty |\Lambda(t)x|^2\ dt - \beta|x|^2 , \tag{4.16}$$

then Λ is L^2-stable.

Proof. Under the above hypothesis

$$\int_0^\infty |\Lambda(t)x|^2 dt \le \frac{1}{\alpha}\int_0^\infty |H\Lambda(t)x|_Y^2 dt + \frac{\beta}{\alpha}|x|^2.$$

$$\le \alpha^{-1}(c+\beta)|x|^2.$$

If $c+\beta\le 0$, then for all x and all t $\Lambda(t)x$ is identically zero and Λ is trivially L^2-stable. If $c+\beta>0$, Λ is L^2-stable by definition. □

Corollary 4.8. Under hypothesis (4.15) Λ will be L^2-stable if there exists a constant $c>0$ such that

(4.17) $$\forall\ x \in X \quad |Hx|_Y \ge c|x|. \quad \square$$

Remark. The strange looking condition (4.16) finds its application in the Stability Theory of hereditary differential systems.

Condition (4.16) is very strong and it is desirable to find weaker conditions. To do this we notice that we can conclude from Corollary 4.5 that

(4.18) $$\forall\ x \in X, \quad \lim_{t\to\infty} B\Lambda(t)x = 0.$$

If there exists some constant $b>0$ such that

(4.19) $$\forall\ x \in X \quad (Bx,x) \ge b|x|^2$$

(by symmetry of B, it is also invertible), then it is clear that Λ is strongly stable. We would also reach the same conclusion if

(4.20) $$\forall\ \{x_n\} \text{ in } X \text{ such that } \{Bx_n\} \to 0 \text{ in } X, \text{ then } \{x_n\} \to 0 \text{ in } X.$$

But the above condition is equivalent to the invertibility since B is self adjoint (cf. J. DUGUNGJI, p. 186 Definition 5.4, 5.5, p. 218 Theorem 6.2, 6.3 and p. 302 Theorem 5.2). We shall see that the invertibility of the operator B is too strong a condition for infinite dimensional systems. To fix this, it will be necessary to use the concepts of stabilizability, detectability and observability, first introduced by W.M. WONHAM [1].

Definition 4.9.

(i) The pair (F,H) is observable if the map

$$x \to H\Lambda(\cdot)x : X \to L^2_{loc}(0,\infty;Y) \tag{4.21}$$

is injective; (ii) the pair (F,H) is strongly observable if there exists a constant $c > 0$ such that

$$\forall\ x \in X, \int_0^\infty |H\Lambda(t)x|_Y^2 dt \geq c|x|^2; \tag{4.22}$$

(iii) the pair (F,H) is uniformly strongly observable if there exists a time $T > 0$ and a constant $c_T > 0$ such that

$$\forall\ x \in X, \int_0^T |H\Lambda(t)x|_Y^2 dt \geq c_T|x|^2. \blacksquare \tag{4.23}$$

The above three concepts are equivalent in the finite dimensional case; in the infinite dimensional case the following ordering is obvious

uniform strong observability ⇒ strong observability ⇒ observability.

Proposition 4.10. The operator B is invertible if and only if the uncontrolled system (S) is L^2-stable with respect to H (that is,

$$\forall\ x, \int_0^\infty |H\Lambda(t)x|_Y^2 dt) \tag{4.24}$$

and the pair (F,H) is strongly observable. In this situation $\{\Lambda(t)\}$ is strongly stable.

Proof. By definition. □

Unfortunately the results of Proposition 4.10 put some very restrictive conditions on the semigroup $\{\Lambda(t)\}$ as can be seen from the following results of A. PAZY [1].

Proposition 4.11. (i) If the uncontrolled system (S) is L^2-stable with respect to H and if the pair (F,H) is strongly observable, then there exist a time $T > 0$ and a constant $c > 0$ such that

$$(4.25) \qquad \forall\, x \in X, \quad |\Lambda(T)x| \geq c|x|.$$

(ii) If condition (4.25) is verified for some $T>0$, there exists a constant $m>0$ such that

$$(4.26) \qquad \forall\, x, \quad \int_0^T |\Lambda(t)x|^2 dt \geq m|x|^2.$$

Moreover for each $T>0$ there exists $c'>0$

$$(4.27) \qquad \forall\, t \in [0,T], \quad \forall\, x, \quad |\Lambda(t)x| \geq c'|x|.$$

Proof. (i) For any $\tau > 0$

$$(Bx,x) = \int_0^\infty |H\Lambda(t)x|_Y^2 dt = \int_0^\tau |H\Lambda(t)x|_Y^2 dt + \int_\tau^\infty |H\Lambda(t)x|_Y^2 dt$$

$$= \int_0^\tau |H\Lambda(t)x|_Y^2 dt + (B\Lambda(\tau)x, \Lambda(\tau)x).$$

By the Uniform Boundedness Principle,

$$\exists\, d > 0,\ \forall\, t \in [0,1], \quad |H\Lambda(t)x|_Y \leq d|x|.$$

For $0 < \tau \leq 1$

$$c|x|^2 \leq \tau\, d^2 |x|^2 + |B|\ |\Lambda(\tau)x|^2$$

and there exists $\tau \leq \tau^0 = \min\{1, c/2d^2\}$ such that $\forall\, x \quad |\Lambda(\tau)x| \geq \frac{c}{2}|x|$.

(ii) For all t in [0,T] and all x

$$c|x| \le |\Lambda(T)x| \le |\Lambda(T-t)||\Lambda(t)x| \le \sup_{[0,T]} |\Lambda(s)||\Lambda(t)x|$$

and there exists k>0 such that

$$\forall\ t \in [0,T],\quad \forall\ x,\quad |\Lambda(t)x| \ge k|x|.$$

Pick t in [T,2T]

$$\begin{aligned}\forall\ t \in [T,2T],\quad \forall\ x \in X,\quad |\Lambda(t)x| &= |\Lambda(T)\Lambda(t-T)x| \\ &\ge c|\Lambda(t-T)x| \ge c^2|x|.\end{aligned}$$

As a result for all T>0 there exists c'>0 such that

$$\forall\ t \in [0,T],\quad \forall\ x \in X,\quad |\Lambda(t)x| \ge c|x|$$

and inequality (4.28) follows immediately. □

The conditions of Proposition 4.10 are very strong as can be seen from Proposition 4.11. They say that for each t $\Lambda(t)$ has a continuous left inverse. Very few semigroups enjoy such a property (which is independent of H). This means that, in general we cannot expect that B be invertible. We introduce the following definition for completeness.

Definition 4.12. The uncontrolled system (S) is said to be degenerate if there exists T>0 and q, $0 \ne q \in X$, such that

(4.28) $$\forall\ x \in X,\quad (q,\Lambda(T)x) = 0. \quad \square$$

We shall say that the uncontrolled system (S) is non-degenerate or complete if (S) is not degenerate. This situation will occur if and only if for all $t\ge 0$ the image of $\Lambda(t)$ is dense in X. Notice that if the uncontrolled system (S) is complete and if condition (4.25) is verified the semi-group $\{\Lambda(t)\}$ can be readily extended to a group. This considerably limits the class of semigroup which are simultaneously L^2-stable, observable and non-degenerate.

Remark 1. It will later be seen that the semi-group associated with a linear hereditary differential system HDS can never be a group. According to Proposition 4.11, the map $t \mapsto H\Lambda(t)$ can never be simultaneously L^2-stable, strongly observable and non-degenerate. However we know linear HDS which are L^2-stable and non-degenerate. Thus the concept of strong observability seems to be "too strong" for linear HDS, but we do not really know how "pertinent" the concept of observability is.

Remark 2. When X is finite dimensional, the semi-group $\{\Lambda(t)\}$ is necessarily a group (cf. R. BELLMAN, p. 167), L^2-stability coincides with strong and weak stability and observability and strong observability are characterized by the rank condition

$$\text{Rank}[M^*, M^*F^*, \ldots, M^*(F^*)^{d-1}] = d$$

where d is the dimension of X (cf. R.E. KALMAN [1], [2]).

Since the invertibility of the operator B is generally too strong a condition for infinite dimensional systems we have to go to the concepts of stabilizability and detectability as introduced by W.M. WONHAM [1].

Definition 4.13. Let Y be a Hilbert space and let H belong to $\mathcal{L}(X,Y)$.

(i) The pair (F,G) is said to be stabilizable with respect to H if

$$(4.29) \qquad \forall\, x_0, \quad \exists\, v \in L^2(0,\infty;U), \quad \int_0^\infty |Hx(t;x_0,v)|_Y^2 dt < \infty,$$

where $x(\cdot;x_0,v)$ is the state of system (S) with initial condition x_0 at time 0 and control function v in $L^2_{loc}(0,\infty;U)$.

(ii) The pair (F,G) is said to be stabilizable if

$$\forall\ x_0,\quad \exists\ v \in L^2(0,\infty;U),\quad \int_0^\infty |x(t;x_0,v)|^2 dt < \infty. \tag{4.30}$$

(iii) The pair (F,G) is said to be stabilizable by feedback with respect to H if there exists an operator K in $\mathcal{L}(X,U)$ such that

$$\forall\ x_0,\quad \int_0^\infty |Hx(t;x_0)|_Y^2 dt < \infty, \tag{4.31}$$

where $x(t;x_0)$ is the solution of the closed loop system

$$\begin{cases} \dot{x}(t) = (F+GKi)x(t) \\ x(0) = x_0. \end{cases} \tag{4.32}$$

(iv) The pair (F,G) is said to be stabilizable by feedback if there exists an operator K in $\mathcal{L}(X,U)$ such that

$$\forall\ x_0,\quad \int_0^\infty |x(t;x_0)|^2 dt < \infty, \tag{4.33}$$

where $x(t;x_0)$ is the solution of (4.32).

(iv) The pair (F,H) is said to be detectable if the pair (F^*,H^*) is stabilizable by feedback. □

Remark. In section 5 (Corollary 5.5) we will show that stabilizability and stabilizability by feedback of the pair (F,G) are equivalent concepts.

Proposition 4.14. (J. ZABCZYK [1]). If the uncontrolled system (S) is L^2-stable with respect to H (that is,

$$\forall\ x,\quad \int_0^\infty |H\Lambda(t)x|_Y^2 dt < \infty\ ), \tag{4.34}$$

and if the pair (F,H) is detectable, then F is L^2-stable and equation (4.4) has a unique positive solution.

<u>Proof</u>. (i) Let $x(t) = \Lambda(t)x$, where $\{\Lambda(t)\}$ denotes the semigroup generated by F. By hypothesis there exists S in $\mathcal{L}(Y,X)$ such that the semigroup generated by $F^* + i^*H^*S^*$ be L^2-stable. Hence the semigroup $\{\Phi(t)\}$ generated by $F + SHi$ is also L^2-stable. We can now rewrite the equation

$$\dot{x}(t) = Fx(t) \tag{4.35}$$

in the form

$$\dot{x}(t) = (F+SHi)x(t) - SHix(t) \tag{4.36}$$

or equivalently

$$x(t) = \Phi(t)x - \int_0^t \Phi(t-s)SHx(s)ds. \tag{4.37}$$

Finally

$$\left[\int_0^\infty |x(t)|^2 dt\right]^{\frac{1}{2}} \le \left[\int_0^\infty |\Phi(t)x|^2 dt\right]^{\frac{1}{2}} + \left[\int_0^\infty \left|\int_0^t \Phi(t-s)SHx(s)ds\right|^2 dt\right]^{\frac{1}{2}}. \tag{4.38}$$

But there exist $\alpha > 0$ and $M \ge 1$ such that (cf. Theorem 4.6)

$$\forall\ x,\ |\Phi(t)x| \le Me^{-\alpha t}|x| \tag{4.39}$$

and the second term on the right hand side of inequality (4.38) can be majored by

$$\int_0^\infty \left[\int_0^t Me^{-\alpha(t-s)}|S|\,|Hx(s)|_Y ds\right]^2 dt. \tag{4.40}$$

If we introduce the following functions defined on $\mathbb{R}$

$$f(s) = \begin{Bmatrix} M|S|e^{-\alpha s}, & s \geq 0 \\ 0 \quad , & \text{otherwise} \end{Bmatrix}, \quad g(s) = \begin{Bmatrix} |Hx(s)|_Y, & s \geq 0 \\ 0 \quad , & \text{otherwise} \end{Bmatrix}, \tag{4.41}$$

we know that f belongs to $L^1(0,\infty;\mathbb{R})$ and g belongs to $L^2(0,\infty;\mathbb{R})$. Thus we can apply a result of DUNFORD-SCHWARTZ [2, Lemma 1(c)]:

$$|f*g|_2 \leq |f|_1 \ |g|_2,$$

where f*g is the convolution of f and g which is defined by the equation

$$(f*g)(t) = \int_{\mathbb{R}} f(t-s)g(s)ds.$$

Here we clearly obtain

$$(f*g)(t) = \int_0^t M|S|e^{-\alpha(t-s)}|Hx(s)|_Y ds. \qquad \square$$

It is a natural question to ask in what way Proposition 4.16 is related to Proposition 4.10 or in other words what is the connection between detectability of the pair (F,H) and observability of the pair (F,H). We give the following partial result.

<u>Proposition 4.15</u>. If the pair (F,H) is uniformly strongly observable by H, then the pair (F*,H*) is stabilizable (hence the pair (F,H) is detectable by Corollary 5.5 in section 5).

<u>Proof</u>. Uniform strong observability by H means that the map

$$M_T x = \int_0^T \Lambda(t)^* H^* H \Lambda(t) x \, dt \tag{4.42}$$

is invertible for some $T>0$. As a result for any x we can choose a control

$$y(s) = - H\Lambda(T-s)M_T^{-1}\Lambda(T)^* x$$

in $L^2(0,T;Y)$ and

$$x(T) = \Lambda(T)^* x + \int_0^T \Lambda(T-s)^* H^* y(s) ds = 0.$$

After time T we set the control y equal to 0. Hence by definition the pair (F^*,H^*) is stabilizable. □

Of course we would like to relate strong observability of the pair (F^*,H^*) and stabilizability of the pair (F^*,H^*). We have the following picture

$$\begin{array}{rcl} & & (F,H) \ \text{observable} \\ & & \Uparrow \\ & & (F,H) \ \text{strongly observable} \\ & & \Uparrow \\ (F,H) \ \text{detectable} & \Leftarrow & (F,H) \ \text{uniformly strongly observable} \end{array}$$

We now get back to Theorem 4.6 and the L^2-stability of the semigroup $\{\Lambda(t)\}$. We investigate the connection between L^2-stability and the position of the spectrum of the operator F in the complex plane.

<u>Proposition 4.16</u>. (M. DELFOUR [2]) Let

$$\sum(F) = \begin{cases} \sup[\operatorname{Re} \lambda : \lambda \in \sigma(F)], & \sigma(F) \neq \emptyset \\ -\infty & , \ \sigma(F) = \emptyset \end{cases} . \tag{4.43}$$

(i) In general $\Sigma(F) \le \omega_0$.

(ii) A sufficient condition for $\Sigma(F) = \omega_0$ is that

$$(4.44) \qquad C\sigma(\Lambda(1)) \subset \{\exp \lambda : \lambda \in \sigma(F)\} \cup \{0\},$$

where $C\sigma(\Lambda(1))$ is the continuous spectrum of $\Lambda(1)$.

Proof. Given T in $\mathcal{L}(X)$, the spectral radius theorem says that

$$\lim_{n\to\infty} |T^n|^{1/n} = \rho(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

If we apply this theorem to $\Lambda(1)$

$$\log \rho(\Lambda(1)) = \sup\{|\lambda| : \lambda \in \sigma(\Lambda(1))\}.$$

But by the spectral mapping theorem (cf. HILLE-PHILLIPS [1] p. 247 Thm. 16.7.1),

$$(4.45) \qquad \{e^{\lambda} : \lambda \in \sigma(F)\} \subset \sigma(\Lambda(1))$$

and necessarily

$$(4.46) \qquad \sup\{|e^{\lambda}| : \lambda \in \sigma(F)\} \le \sup\{|\xi| : \xi \in \sigma(\Lambda(1))\}.$$

But

$$|e^{\lambda}| = e^{\mathrm{Re}\,\lambda}$$

and

$$\sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(F)\} = \log \sup\{e^{\mathrm{Re}\,\lambda} : \lambda \in \sigma(F)\} \le \log \rho(\Lambda(1)) = \omega_0.$$

(ii) In general (cf. HILLE-PHILLIPS [1])

$$P\sigma(\Lambda(1)) \cup R\sigma(\Lambda(1)) \subset \{e^{\lambda} : \lambda \in P\sigma(F)\} \cup \{0\}$$

and under hypothesis (4.44)

$$\{0\} \cup \{e^{\lambda}:\lambda \in \sigma(F)\} \supset \sigma(\Lambda(1)). \tag{4.47}$$

In general equation (4.46) can be rewritten as

$$\sup\{0,\{|e^{\lambda}|:\lambda \in \sigma(F)\}\} \leq \sup\{|\xi|:\xi \in \sigma(\Lambda(1))\}. \tag{4.48}$$

In view of (4.47), we also have inequality (4.48) in the opposite direction. Hence under hypothesis (4.44) $\Sigma(F) = \omega_0$. □

Corollary 4.17. (i) The spectrum $\sigma(F)$ of F lies entirely in $\{\lambda \in C|\text{Re } \lambda \leq \omega_0\}$.
(ii) If, in addition F is a bounded operator or $C\sigma(\Lambda(1)) \subset \{\exp \lambda:\lambda \in \sigma(F)\} \cup \{0\}$, the "spectrum condition" [there exists $\omega<0$ such that $\Sigma(F) \leq \omega$] is necessary and sufficient for the L^2-stability of system (S). □

In general it is not possible to infer the L^2-stability of the semigroup from the position of the spectrum of its infinitesimal generator. A counterexample can be found in HILLE-PHILLIPS [1, p. 665]. An even more striking result has been obtained by J. ZABCZYK [2].

Proposition 4.18. (J. ZABCZYK [2]). Given any two real numbers $\omega < \omega_0$ there exists a C_0 semigroup $\{\Lambda(t)\}$ on a Hilbert space X such that $\Sigma(F) = \omega$ and $|\Lambda(t)| = e^{\omega_0 t}$. □

It would be useful to completely characterize semigroups for which $\Sigma(F) = \omega_0$. Other conditions on the resolvent can be found in M. SLEMROD [4].

5. Infinite time quadratic cost control problem.

If we consider the problem of section 3 in the time interval [0,T] we can exhibit an operator $P_T(t)$ as in Chapter 3. In this section we shall show that under a certain stabilizability hypothesis the operator $P_T(t)$ "converges" to a constant operator P. This operator will be used to construct the optimal solution to the infinite time control problem. It will also be shown that P is the solution of a certain algebraic Riccati equation and that the properties of P are connected with the behaviour at infinity of the semigroup associated with the closed-loop system.

5.1. Asymptotic behaviour of $P_T(t)$.

We first recall some results we obtained for the optimal control problem in the finite time interval [s,t] for some s in [0,T[. Here the state of system (S) is

$$\xi_s(t) = \Lambda(t-s)h + \int_s^t \Lambda(t-r)Gv(r)dr, \quad s \le t \le T, \tag{5.1}$$

for h in X and v in $L^2(s,T;U)$ and the cost function is

$$J_T^s(v,x_0) = \int_s^t [|H\xi_s(t)|_Y^2 + (v(t),Nv(t))_U]dt. \tag{5.2}$$

It was shown in the previous chapter that there exists a unique weakly continuous operator $P_T(t)$ defined in [0,T] such that

$$\begin{aligned}(P_T(t)h,\overline{h}) = \int_t^T &[(H^*H\Lambda_T(r-t)h,\Lambda_T(r-t)\overline{h}) \\ &+ (P_T(r-t)\Lambda_T(r-t)h,RP_T(r-t)\Lambda_T(r-t)\overline{h})]dr,\end{aligned} \tag{5.3}$$

for all t in $[0,T]$, all h and $\overline{h}$ in X (here $R = GN^{-1}G^*$) and

$$\Lambda_T(t)k = \Lambda(t)k - \int_0^t \Lambda(t-r)RP_T(r)\Lambda_T(r)kdr,$$

for all t in $[0,T]$ and all k in X. It can be shown that

$$\operatorname*{Inf}_{v\in L^2(s,T;U)} J_T^s(v,h) = (h,P_T(s)h)$$

and that there is a unique element u in $L^2(s,T;U)$ which minimizes $J_T(v,h)$ over all v in $L^2(s,T;U)$. Moreover, the control function u can be synthesized via *the time varying feedback law* $-N^{-1}G^*P_T$:

$$u(t) = -N^{-1}G^*P_T(t)\xi_s(t).$$

It is not difficult but fundamental to notice that for all s in $[0,t[$ and all h in X.

$$\operatorname*{Inf}_{v\varepsilon L^2(s,T;U)} J_T^s(v,h) = \operatorname*{Inf}_{w\varepsilon L^2(0,T-s;U)} J_{T-s}^0(w,h).$$

As a result, $P_{T_1}(t_1) = P_{T_2}(t_2)$ for all $T_1 \geq t_1 \geq 0$ and $T_2 \geq t_2 \geq 0$ such that $T_1 - t_1 = T_2 - t_2$.

Proposition 5.1. Assume that the pair (F,G) is stabilizable with respect to H, then (i)

1) there exists a constant $c > 0$ such that

$$(5.4) \qquad |P_T(t)| \leq c \text{ for all } T \geq t \geq 0$$

2) there exists a unique positive self-adjoint element P of $L(X)$

such that for all $t \geq 0$ and all x in X

$$\lim_{t \leq T \to \infty} P_T(t)x = Px. \tag{5.5}$$

(ii) For all x and y in X

$$(Px,y) = \int_0^\infty ([H^*H + PRP]\Lambda_P(t)x, \Lambda_P(t)y)\,dt, \tag{5.6}$$

where Λ_p is the strongly continuous semigroup generated by the infinitesimal generator $F\text{-}RPi$.

Proof. (i) By hypothesis, for all h there exists v in $L^2(0,\infty;U)$ such that $J(v,h) < \infty$. In particular for all T and h

$$(P_T(0)h,h) = J_T^0(u,h) \leq J_T^0(v,h) \leq J(v,h) < \infty,$$

where u is the optimal control in $[0,T]$. But since the system is time-invariant

$$(P_T(s)h,h) = J_T^s(u,h) = J_{T-s}^0(\overline{u},h) = (P_{T-s}(0)h,h),$$

where u (resp. $\overline{u}$) is the optimal control in $[s,T]$ (resp. $[0,T-s]$), and for all h and all $0 \leq s < T$

$$(P_T(s)h,h) < \infty.$$

By symmetry and positivity we even have for all h and k in X and all $0 \leq s \leq T$

$$|(P_T(s)h,k)| < \infty.$$

By the Uniform Boundedness Principle for each h there exists $c(h) > 0$

such that for all $0 \le s < T$

$$|P_T(s)h| \le c(h).$$

Again by the same Principle there exists $c > 0$ such that for all $0 \le s < T$

$$(5.7) \qquad |P_T(s)| \le c.$$

Since system (S) is autonomous and the cost function (5.2) is defined in terms of constant matrices, we obtain for all pairs (s_1,T_1) and (s_2,T_2) such that

$$0 \le T_1 - s_1 \le T_2 - s_2, \quad s_1 \ge 0, \ s_2 \ge 0$$

$$(5.8) \qquad (P_{T_1}(s_1)h,h) = \operatorname{Inf}\left\{J_{T_1}^{s_1}(v_1,h) \,|\, v_1 \in L^2(s_1,T_1;U)\right\}$$

$$\le \operatorname{Inf}\left\{J_{T_2}^{s_2}(v_2,h) \,|\, v_2 \in L^2(s_2,T_2;U)\right\}$$

$$\le (P_{T_2}(s_2)h,h).$$

As a result for all $s \ge 0$ the family $\{P_T(s) \,|\, T \ge s\}$ of positive self-adjoint elements of $\mathcal{L}(X)$ has the following properties:

1) $P_{T_2}(s) \ge P_{T_1}(s)$, $T_2 \ge T_1 \ge s$;

2) there exists $c > 0$ such that $|P_T(s)| \le c$ for all $T \ge s$. In view of Lemma 4.2, there exists a positive self-adjoint linear operator $P(s)$ in $L(X)$ such that $P_T(s) \to P(s)$ in the strong topology as $T \to \infty$. Moreover, for all pairs (s_1,T_1) and (s_2,T_2) such that

$$(5.9) \qquad 0 \le T_1 - s_1 = T_2 - s_2 , \ s_1 \ge 0, \ s_2 \ge 0,$$

we necessarily obtain in view of (5.8)

$$P_{T_1}(s_1) = P_{T_2}(s_2).$$

For arbitrary h in X, $s_1 \geq 0$, $s_2 \geq 0$ and T_1, T_2 as in (5.9)

$$P(s_1)h = \lim_{T_1 \to \infty} P_{T_1}(s_1)h = \lim_{T_1 \to \infty} P_{T_1 - s_1 + s_2}(s_2)h \tag{5.10}$$

$$= \lim_{T_2 \to \infty} P_{T_2}(s_2)h = P(s_2)h.$$

Let $P = P(0)$. We have shown that for all $t \geq 0$ and h in X.

$$P(t)h = \lim_{t \leq T \to \infty} P_T(t)h = \lim_{T \to \infty} P_T(0)h = Ph. \tag{5.11}$$

(ii) Fix $t_1 > 0$ and $y \varepsilon L^1(0,t_1;X)$. Let $T > t_1$ and

$$f_T(t) = P_T(t)y(t), \quad f(t) = Py(t) \text{ in } [0,t_1]. \tag{5.12}$$

In view of (5.7) f_T and f belong to $L^1(0,t_1;X)$. Moreover, for all $T > t_1$ f_T and f are bounded almost everywhere by the L^1-function

$$cy(t)$$

and by virtue of (5.11) for almost all t in $[0,t_1]$

$$f_T(t) = P_T(t)y(t) \to f(t) = Py(t)$$

as $T \to \infty$. By Lebesque dominated convergence theorem $f_T \to f$ in $L^1(0,t_1;X)$. Let x_T be defined by the equation

$$x_T(t) = \Lambda(t)h - \int_0^t \Lambda(t-s)RP_T(s)x_T(s)ds \tag{5.13}$$

$$(5.14)\qquad x(t) = \Lambda_P(t)h = \Lambda(t)h - \int_0^t \Lambda(t-s)RP\Lambda_P(s)hds$$

and $y_T(t) = x_T(t) - x(t)$ in $[0,t_1]$. Then

$$y_T(t) = -\int_0^t \Lambda(t-s)R[P_T(s)x_T(s) - P\Lambda_P(s)h]ds$$

$$= -\int_0^t \Lambda(t-s)RP_T(s)(x_T(s) - \Lambda_P(s)h)ds$$

$$-\int_0^t \Lambda(t-s)R[P_T(s)-P]\Lambda_P(s)hds$$

$$|y_T(t)| \le c_1(t_1)\int_0^t |y_T(s)|ds - c_2(t_1)\int_0^t |[P_T(s)-P]\Lambda_P(s)h|ds.$$

This implies that there exists $c(t_1) > 0$ such that

$$(5.15)\qquad ||y_T||_{C(0,t_1;X)} \le c(t_1)\int_0^{t_1} |[P_T(s)-P]\Lambda_P(s)h|ds.$$

In view of our previous results, the right hand side of (48) goes to zero as $T \to \infty$. Hence

$$(5.16)\qquad x_T(t) \to x(t) \text{ uniformly in } [0,t_1].$$

Moreover, this also implies that $x_T(t)$ is uniformly bounded in $[0,t_1]$ by a constant independent of T.

Consider identify (5.3) with $t = 0$.

$$(5.17)\qquad (P_T(0)h,\bar{h}) = \int_0^T [(H^*Hx_T(s),\bar{x}_T(s)) + (P_T(s)x_T(s),RP_T(s)\bar{x}_T(s))]ds,$$

where $\overline{x}_T$ is the solution of (5.13) with $\overline{h}$ in place of h. We know that the left hand side of (5.17) converges to P as T goes to infinity. We shall prove that the right hand side of (5.17) converges to

$$\int_0^\infty [(H^*H\Lambda_P(s)h,\Lambda_P(s)\overline{h}) + (P\Lambda_P(s)h,RP\Lambda_P(s)\overline{h})]ds$$

as T goes to infinity. For this purpose, we define

$$g_T(t) = \begin{cases} (H^*Hx_T(t),x_T(t)) + (P_T(t)x_T(t),RP_T(t)x_T(t)), & 0 \le t \le T \\ 0 & , T < t < \infty \end{cases}$$

$$g(t) = (H^*H\Lambda_P(t)h,\Lambda_P(t)h) + (P\Lambda_P(t)h,RP\Lambda_P(t)h),\ 0 \le t < \infty.$$

In view of (5.11) and (5.16):

$$g_T(t) \to g(t) \text{ pointwise in } [0,\infty) \text{ as } T \to \infty.$$

By Fatou's lemma

$$\text{(5.18)} \qquad \int_0^\infty g(t)dt = \int_0^\infty \lim_{T\to\infty} g_T(t)dt \le \liminf \int_0^\infty g_T(t)dt$$

$$\le \lim(\Pi_T(0)h,h) = (\Pi h,h).$$

But if we consider the constant feedback law

$$v_P(t) = -N^{-1}G^*Px(t),$$

where x is as defined in equation (5.14), we have for all $T > 0$

$$\text{(5.19)} \qquad (\Pi_T(0)h,h) \le \int_0^T [(H^*Hx(t),x(t)) + (Nv_P(t),v_P(t))_U]dt$$

and necessarily

$$(5.20)\qquad (\Pi h,h) \le \lim_{T\to\infty} \int_0^T g(t)dt = \int_0^\infty g(t)dt.$$

As a result equality holds in inequality (5.18). By symmetry and positivity this yields (5.6). □

5.2. Solution to the Control Problem.

We now look at the problem of section 3. We want to minimize the cost function

$$(5.21)\qquad J(v_0,x_0) = \int_0^\infty [|Hx(t)|_Y^2 + (v(t),Nv(t))]dt$$

over all control functions v in $L^2_{loc}(0,\infty;U)$ under the hypothesis that System (S) is stabilizable with respect to H.

Theorem 5.2. Assume that the pair (F,G) is stabilizable with respect to H. Then for each x_0 in X there exists a control function u in $L^2_{loc}(0,\infty;U)$ such that

$$(5.22)\qquad \text{Inf } \{J(v,x_0)\,|\,v\in L^2_{loc}(0,\infty;U)\} = J(u,x_0)$$

and

$$(5.23)\qquad u(t) = -N^{-1}G^*P\Lambda_P(t)x_0,$$

where P and Λ_P are as defined in Proposition 5.1(ii).

Proof. It is clear that the control function u defined by (5.23) is an element of $L^2_{loc}(0,\infty;U)$. Pick any v in $L^2_{loc}(0,\infty;U)$. Then for all v in $L^2_{loc}(0,\infty;U)$ and for all T > 0

$$(x_0,P_T(0)x_0) = \min\ \{J_T(v,x_0)\,|\,v\in L^2(0,T;U)\}$$
$$\le \int_0^T [(H^*Hy(s),y(s)) + (Nv(s),v(s))]ds,$$

where y corresponds to the control function v.

By going to the limit

$$(x_0,Px_0) \le \int_0^\infty [(H^*Hy(s),y(s)) + (Nv(s),v(s))]ds.$$

But if u is defined by (5.23)

$$\begin{aligned}(Nu(s),u(s)) &= (N[-N^{-1}B^*P\Lambda_P(s)x_0],[-N^{-1}B^*P\Lambda_P(s)x_0]) \\ &= (BN^{-1}B^*P\Lambda_P(s)x_0,P\Lambda_P(s)x_0) \\ &= (RP\Lambda_P(s)x_0,P\Lambda_P(s)x_0)\end{aligned}$$

and in view of equation (5.6)

$$(x_0,Px_0) = \int_0^\infty [(H^*H\Lambda_P(t)x_0,\Lambda_P(t)x_0) + (Nu(t),u(t))]dt.$$

Hence for all x_0 in X and all v in $L^2_{loc}(0,\infty;U)$

$$J(u,x_0) \le J(v,x_0). \;\Box$$

5.3. <u>Riccati Equation for P.</u>

In this section we derive a Riccati equation for the feedback operator P and present conditions under which it has a unique positive self-adjoint solution in $\mathcal{L}(X)$.

<u>Theorem 5.3.</u> Assume that the pair (F,G) is stabilizable with respect to H.

(i) P is a solution of the equation

(5.24) $$i^*PF + F^*Pi + i^*[H^*H-PRP]i = 0 \text{ in } \mathcal{L}(V,V')$$

and any other positive self-adjoint solution $\overline{P}$ in $\mathcal{L}(X)$ is such that $\overline{P} \ge P$.

Moreover, the evolution operator Λ_P for the closed loop system

(5.25) $$\frac{dx}{dt}(t) = (F-RP)x(t), \; x(0) = x$$

has the property

(5.26) $$\forall x \in X, \; P\Lambda_P(t)x \to 0 \text{ as } t \to \infty.$$

(ii) If there exists a positive self-adjoint solution $\overline{P}$ of equation (5.24) with the property

(5.27) $$\forall x \in V, \; \Lambda_{\overline{P}}(t)x \to 0 \text{ as } t \to \infty,$$

then equation (5.24) has a unique positive self-adjoint solution which is necessarily equal to P.

(iii) Let $\bar{P}$ be a positive self-adjoint solution of equation (5.24). If there exists a constant $c > 0$ such that

(5.28) $$\forall x \in X, \; (\bar{P}x,x) \geq c|x|^2 ,$$

then condition (5.27) is verified and necessarily $\bar{P} = P$.

Proof. (i) By proposition 5.1. (ii) We know that for all x in X

$$\int_0^\infty ([H^*H+PRP]\Lambda_P(t)x,\Lambda_P(t)x)dt < \infty.$$

In view of equation (5.6) and Proposition 4.3 (iii), P is a positive self-adjoint solution in $\mathcal{L}(X)$ of the equation

(5.29) $$([F-RPi]x,Piy) + (Pix,[F-RPi]y) + ([H^*H+PRP]ix,iy) = 0$$

for all x and y in $\mathcal{D}(F-RPi) = \mathcal{D}(F)$. Finally the above equation can be rewritten in the form of equation (5.24). Let $\overline{P}$ be another positive self-adjoint solution of (5.24) in $\mathcal{L}(X)$ and denote by $\overline{\Lambda}$ the strongly continuous semigroup generated by $F-R\overline{P}i$. Then

$$(\overline{\Lambda}(t)x,\overline{P}\overline{\Lambda}(t)y) - (x,\overline{P}y) = -\int_0^t([H^*H+\overline{P}R\overline{P}]\overline{\Lambda}(s)x,\Lambda(s)y)ds$$

and for all $t \geq 0$

$$(5.30) \qquad (x,\overline{P}x) = \int_0^t([H^*H+\overline{P}R\overline{P}]\overline{\Lambda}(s)x,\overline{\Lambda}(s)x)ds + (\overline{\Lambda}(t)x,\overline{P}\overline{\Lambda}(t)x)$$

$$\geq \int_0^t([H^*H+\overline{P}R\overline{P}]\overline{\Lambda}(s)x,\overline{\Lambda}(s)x)ds.$$

If we go to the limit

$$(x,\overline{P}x) \geq \int_0^\infty([H^*H+\overline{P}R\overline{P}]\overline{\Lambda}(s)x,\overline{\Lambda}(s)x)ds.$$

The right hand side of the above equation in equal to the cost function $J(\overline{v},x)$, where $\overline{v}$ is the feedback control function

$$\overline{v}(t) = -N^{-1}G^*\overline{P}\overline{\Lambda}(t)x.$$

But by virtue of Theorem 5.2 and Proposition 5.1 (ii) for all x in X

$$(x,\overline{P}x) \geq J(\overline{v},x) \geq \text{Inf }\{J(v,x)\,|\,v\varepsilon L_{1oc}(0,\infty;U)\} = (x,Px).$$

We rewrite (5.30) with P and Λ_p in place of $\overline{P}$ and $\overline{\Lambda}$. In view of identity (5.6), for all x in X, $(\Lambda_p(t)x,P\Lambda_p(t)x)$ decreases monotonically to zero as t goes to infinity.

(ii) Let Π be an arbitrary positive self-adjoint solution of (5.24). Let Λ_Π and $\overline{\Lambda}$ be the strongly continuous semigroups generated by $F-R\Pi i$ and $F-R\overline{P}i$, respectively. Then for all x and y in V

$$(Fx,(\overline{P}-\Pi)y) + ((\overline{P}-\Pi)x,Fy) - (R\overline{P}x,\overline{P}y) + (R\Pi x,\Pi y) = 0.$$

The above equation can be rewritten as follows:

$$(5.31)\qquad ((F-R\Pi)x,(\overline{P}-\Pi)y)) + ((\overline{P}-\Pi)x,(F-R\overline{P})y) = 0.$$

Let h and k belong to $\mathcal{D}(F)$, $x = \Lambda_\Pi(t)h$ and $y = \overline{\Lambda}(t)k$ in (5.31) then

$$\frac{d}{dt}(\Lambda_\Pi(t)h,(\overline{P}-\Pi)\overline{\Lambda}(t)k) = ((F-R\Pi)\Lambda_\Pi(t)h,(\overline{P}-\Pi)\overline{\Lambda}(t)k)$$

$$+ ((\overline{P}-\Pi)\Lambda_\Pi(t)h,(F-R\overline{P})\overline{\Lambda}(t)k) = 0$$

and

$$(5.32)\qquad (\Lambda_\Pi(t)h,(\overline{P}-\Pi)\overline{\Lambda}(t)k) = (h,(\overline{P}-\Pi)k).$$

By hypothesis for all k, $\overline{\Lambda}(t)k$ goes to zero as t goes to infinity and necessarily $\Pi = \overline{P}$. As a result the positive self-adjoint solution is unique and equal to P.

(iii) From part (i) $(\Lambda_P(t)x,P\Lambda_P(t)x)$ goes to zero as t goes to ∞. But for all $t \geq 0$.

$$(P\Lambda_P(t)x,\Lambda_P(t)x) \geq c|\Lambda_P(t)x|^2$$

and (5.27) is verified with P in place of $\overline{P}$. □

Remark. Notice that we only require the hypothesis of stabilizability of the pair (F,G) with respect to H to conclude to the existence of a solution to the operator Riccati equation. This generalizes the result of W.M. WONHAM [1], where the detectability of the pair (F,H) and the stabilizability of the pair (F,G) were required.

In the previous theorem we have shown that if the pair (F,G) is stabilizable with respect to H, then the Riccati equation (5.24) has a positive self-adjoint solution P in $\mathcal{L}(X)$ and the strongly continuous semi-group $\{\Lambda_P(t)\}$ generated by F-RPi has the property

$$(5.34)\qquad \forall\, x \in V,\ P\Lambda_P(t)x \to 0 \text{ as } t \to \infty.$$

We now prove the converse of the above result.

Theorem 5.4. If there exists a positive self-adjoint solution P to the Riccati equation (5.24) and if the strongly continuous semigroup $\{\Lambda_P(t)\}$ generated by F-RPi has the property (5.34) then the pair (F,G) is stabilizable by feedback with respect to H via the feedback law

$$v(t) = -N^{-1}G^*Px(t). \tag{5.35}$$

Conversely if the pair (F,G) is stabilizable with respect to H, there exists a positive self-adjoint solution P to the Riccati equation (5.24) and the semigroup generated by F-RPi has the property (5.34).

Proof. (i) It is easy to show that for all $t > 0$ and x in V that

$$\frac{d}{dt}(\Lambda_P(t)x, P\Lambda_P(t)x) + ([H^*H+PRP]\Lambda_P(t)x, \Lambda_P(t)x) = 0$$

and using property (5.34)

$$(x,Px) = \int_0^\infty ([H^*H+PRP]\Lambda_P(t)x, \Lambda_P(t)x)dt.$$

By density the above identity is true for all x in X. It corresponds to the control function

$$u(t) = -N^{-1}G^*P\Lambda_P(t)x$$

and this means that we have stabilizability with respect to H.

(ii) The converse is true by Theorem 5.3. □

Corollary 5.5. The pair (F,G) is stabilizable with respect to H if and only if it is stabilizable by feedback with respect to H. □

6. Stability of the closed loop system.

In the previous section we have seen that when System (S) is stabilizable with respect to H, there exists a positive self adjoint operator P in $\mathcal{L}(X)$ such that

$$\forall \quad x \in V, \quad P\Lambda_p(t)x \to 0 \quad \text{as} \quad t \to \infty \tag{6.1}$$

(cf. Theorem 5.3). In this section we study the stability of the semigroup Λ_p and attempt to find conditions on Λ, G and H under which we can conclude to some type of stability for the semigroup Λ_p (cf. Definition 4.1). We have seen a simpler version of this problem in section 4 (cf. Propositions 4.10 and 4.16) and shown that it is related to the concept of observability (cf. Definition 4.9).

In section 6.1 we give a strong condition for the L^2-stability of the semigroup Λ_p; in section 6.2 we show that under the hypothesis of detectability of the pair (F,H) we can conclude to the L^2-stability of the semigroup Λ_p.

6.1. Sufficient condition for L^2-stability of Λ_p.

The following result is a straightforward consequence of Proposition 4.7.

Proposition 6.1. Assume that the pair (F,G) is stabilizable with respect to H. If there exist $\alpha > 0$ and β such that

$$\forall \quad x \in X, \quad \int_0^\infty |H\Lambda_p(t)x|_Y^2 \, dt \geq \alpha \int_0^\infty |\Lambda_p(t)x|^2 dt - \beta |x|^2 , \tag{6.2}$$

then equation (5.24) has a unique positive self adjoint solution in $\mathcal{L}(X)$ which is equal to P and the semigroup Λ_p is L^2-stable.

Proof. Cf. Proposition 4.7 and Theorem 5.3.□

Corollary 6.2. If there exists a constant $c > 0$ such that

$$\forall\ x \in X,\ |Hx|_Y \geq c|x|, \tag{6.3}$$

then Λ is L^2-stable. □

6.2. Sufficient condition for L^2-stability of Λ_p in terms of the detectability of the pair (F,H).

Theorem 6.3. (J. ZABCZYK [1]). Assume that the pair (F,G) is stabilizable with respect to H. If the pair (F,H) is detectable, then equation (5.24) has a unique positive self-adjoint solution in $\mathcal{L}(X)$ which is precisely P and the semigroup Λ_p is L^2-stable.

Proof. Let $x(t) = \Lambda_p(t)x$. By hypothesis there exists S in $\mathcal{L}(Y,X)$ such that the semigroup generated by $F^* + i^*H^*S^*$ be L^2-stable. Hence the adjoint semigroup, that is the semigroup $\{\Phi(t)\}$ generated by $F + SHi$ is L^2-stable. We can now rewrite the differential equation

$$\dot{x}(t) = (F-RPi)x(t) \tag{6.4}$$

in the form

$$\dot{x}(t) = (F+SHi)x(t) - (SH+RP)ix(t) \tag{6.5}$$

or equivalently

$$x(t) = \Phi(t)x - \int_0^t \Phi(t-s)(SH+RP)x(s)ds. \tag{6.6}$$

Finally

$$(6.7) \qquad \left[\int_0^\infty |x(t)|^2 dt\right]^{\frac{1}{2}} \le \left[\int_0^\infty |\Phi(t)x|^2 dt\right]^{\frac{1}{2}} + \left[\int_0^\infty \left|\int_0^t \Phi(t-s)(SH+RP)x(s)ds\right|^2 dt\right]^{\frac{1}{2}}.$$

But $\{\Phi(t)\}$ is L^2-stable and there exist $\alpha > 0$ and $M \ge 1$ (cf. Theorem 4.6) such that

$$(6.8) \qquad \forall\ x, \quad |\Phi(t)x| \le Me^{-\alpha t}|x|$$

and the second term on the right hand side of inequality (6.7) can be majored by

$$(6.9) \qquad \int_0^\infty \left\{\int_0^t Me^{-\alpha(t-s)}[|S|\,|Hx(s)|_Y + |GN^{-1}|\,|G^*Px(s)|_U]ds\right\}^2 dt.$$

As in the proof of Proposition 4.16 we introduce the following two functions defined on $\mathbb{R}$

$$f(s) = \left\{\begin{matrix} M(|S|+|GN^{-1}|)e^{-\alpha s}, & s \ge 0 \\ 0 & , \text{ otherwise}\end{matrix}\right\},$$

$$g(s) = \left\{\begin{matrix} |Hx(s)|_Y + G^*Px(s)\ _U, & s \ge 0 \\ 0 & , \text{ otherwise}\end{matrix}\right\}$$

and apply the result of DUNFORD-SCHWARTZ [2, Lemma 1(c)] since f and g belong to $L^1(0,\infty;\mathbb{R})$ and $L^2(0,\infty;\mathbb{R})$, respectively. □

Remark. The above theorem further generalize the result of W.M. WONHAM [1] who required observability of the pair (F,H) for uniqueness of solution to the operator Riccati equation.

In order to further characterize the operator P we give conditions under which P is invertible.

Theorem 6.4. Assume that the pair (F,G) is stabilizable with respect to H. (i) If the pair (F,H) is uniformly strongly observable, then P is invertible and Λ_P is L^2-stable. (ii) If P is invertible, then the pair (F,H) is strongly observable.

Proof. We use the fact that a positive self-adjoint operator P in $\mathcal{L}(X)$ is invertible if and only if

$$\exists\ c > 0, \quad \forall\ x \in X, \quad (Px,x) \geq c|x|^2.$$

(i) The proof is by contradiction. Assume that condition (6.4) is not verified. There exists a sequence $\{x_i\}$ in X, $|x_i| = 1$, $i = 1,2,\ldots$, such that $(Px_i,x_i) \to 0$ as i goes to ∞. But for all x

$$(Px,x) = \int_0^\infty [|H\Lambda_p(t)x|_Y^2 + (PRP\Lambda_p(t)x,\ \Lambda_p(t)x)]dt,$$

where $R = GN^{-1}G^*$. This means that the maps

$$t \mapsto H\Lambda_p(t)x_i \quad \text{and} \quad t \mapsto u_i(t) = -N^{-1}G^*P\Lambda_p(t)x_i$$

which belong to $L^2(0,\infty;X)$ converge to 0 in $L^2(0,\infty;X)$ as i goes to infinity. Pick any $T > 0$, then there exists $c > 0$ such that

$$|u_i(t)| \leq c \quad \text{in} \quad [0,T]$$

and

$$\begin{aligned}\Lambda_p(t)x_i - \Lambda(t)x_i &= -\int_0^t \Lambda(t-s)GN^{-1}G^*P\Lambda_p(s)x_i ds \\ &= \int_0^t \Lambda(t-s)G\ u_i(s)ds.\end{aligned}$$

It is readily seen that for all t in [0,T]

$$|\Lambda_p(t)x_i - \Lambda(t)x_i| \le [\int_0^T |\Lambda(t)G|^2 dt]^{\frac{1}{2}} [\int_0^T |u_i(s)|^2 ds]^{\frac{1}{2}}$$

$$\le c_T \|u_i\|_2$$

for some constant $c_T > 0$. This means that for all $T > 0$

$$\lim_{i\to\infty} [\Lambda_p(t)x_i - \Lambda(t)x_i] = 0 \quad \text{uniformly in } [0,T].$$

As a result

$$\lim_i \|H\Lambda(\cdot)x_i\|_{L^2(0,T;Y)} \le \lim_i \|H\Lambda_p(\cdot)x_i\|_{L^2(0,T;Y)}$$

$$+ \lim_i \|H[\Lambda(\cdot)x_i - \Lambda_p(\cdot)x_i]\|_{L^2(0,T;Y)}$$

$$= 0$$

and for all $T > 0$

$$\lim_{i\to\infty} \|\pi_T M x_i\|_{L^2(0,T;Y)} = 0 ,$$

where π_T is the projection of $L^2_{loc}(0,\infty;Y)$ onto $L^2(0,T;Y)$. This means that the map M cannot have a continuous left inverse and contradicts the uniform strong observability by H (cf. Definition 4.9(iii)). The L^2-stability of Λ_p follows from Proposition 4.15 and Theorem 6.3.

(ii) For all x in X

$$c|x|^2 \le (Px,x) = J(u,x) \le J(0,x) = \int_0^\infty |H\Lambda(t)x|_Y^2 \, dt. \square$$

Remark. Proposition 6.4 is not "optimal". It only shows that

$$(\text{USO}) \Rightarrow P \ge c \Rightarrow (\text{SO})$$

and does not give a necessary and sufficient condition which would characterize the invertibility of the operator P. Something between (SO) and (USO) should completely characterize this property.

7. Relationships between controllability and stabilizability.

Assume for a moment that $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, $Y = \mathbb{R}^r$ and that the operators F and G are matrices of dimensions $n\times n$ and $n\times m$, respectively. This is the so-called finite dimensional case, where the concept of controllability is defined and characterized as follows:

Definition 7.1. The pair (F,G) is said to be controllable if

$$(7.1) \qquad \forall\, x_0 \in X, \quad \exists\, T > 0, \quad \exists\, v \in L^2(0,T;U) \text{ such that } x(T;x_0,v) = 0,$$

where $x(t;x_0,v)$ is the solution of the differential equation

$$(7.2) \qquad \begin{cases} \dot{x}(t) = Fx(t) + Gv(t), & t \geq 0 \\ x(0) = x_0. & \square \end{cases}$$

Theorem 7.2. The following conditions are equivalent:

(i) (F,G) controllable;

(ii) Given any spectrum σ of a real $n\times n$ matrix, there exists an $m\times n$ matrix K such that the spectrum, $\sigma(F+GK)$, of F+GK is exactly σ;

(iii) $\mathrm{Rank}[G,FG,\ldots,F^{n-1}G] = n$. $\square$

This theorem now says that when the pair (F,G) is controllable it is necessarily stabilizable by feedback. The converse is obviously not true.

When X, U and Y are infinite dimensional spaces. Definition 7.1 can be retained, but conditions (ii) and (iii) are difficult to generalize. However the following straightforward result remains true.

Theorem 7.3. The pair (F,G) is stabilizable if the pair (F,G) is controllable. $\square$

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