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# CONTROLLABILITY AND OBSERVABILITY FOR INFINITE-DIMENSIONAL SYSTEMS*

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**Abstract.** This paper systematically studies the notions of controllability and observability for an affine abstract system defined in a Hilbert space with initial data, controls and observations also belonging to Hilbert spaces. Necessary and sufficient conditions are obtained in that framework and the duality property is studied. This theory can find applications in the study of "boundary controllability" and "boundary observability" for parabolic partial differential equations. Specific results have been obtained for affine hereditary differential systems defined in the $M^2$-space framework (cf. Delfour and Mitter [1], [2], [5]).

**1. Introduction.** This paper systematically studies the notions of controllability and observability for an affine abstract system defined in a Hilbert space with initial data, controls and observations also belonging to Hilbert spaces. Necessary and sufficient conditions are obtained in that framework and the duality property is studied. In fact, the notion of controllability was chosen in such a way that the duality between the two notions reduces to the notions of topological duality and adjoint map. The choice of this framework was motivated by the work of H. O. Fattorini [8], [9], S. K. Mitter [13], V. Jurdjevic [10] and J. L. Lions [12], and the main ideas all arise from the study of partial differential equations.

This theory finds applications in the study of the notions of "boundary controllability" and "boundary observability" for parabolic partial differential equations. But it is in the theory of affine hereditary differential systems (HDS) that the most interesting applications are found. It is well known that the state space of a HDS is infinite-dimensional. When such a system is studied in the framework of the space $M^2(-b, 0; H)$, its state space is a Hilbert space (cf. Delfour and Mitter [1], [2], [3], [4]). Therefore it was possible to adapt techniques developed by J. L. Lions [12] for optimal control problems with a quadratic cost (cf. Delfour and Mitter [5], [6]). It was also possible to use the results of this paper to study the various notions of controllability and observability for affine HDS in the framework of the space $M^2(-b, 0; H)$ (cf. Delfour and Mitter [5]).

*Notations and terminology.* Given two real linear spaces $X$ and $Y$ and a linear map $T:X \to Y$, the image of $T$ in $Y$ will be denoted by $\mathrm{Im}(T)$ and the kernel of $T$ in $X$ by $\mathrm{Ker}(T)$. Let $H$ and $K$ be two Hilbert spaces and $T:H \to K$ be a continuous linear map. The adjoint of $T$ will be denoted $T^*(\in \mathscr{L}(K, H))$. When $H = K$ we shall write $T \geqq 0$ for a positive operator ($(Tx|x) \geqq 0$ for all $x$) and $T > 0$ for a positive definite operator ($(Tx|x) > 0, x \neq 0$). The identity map in $\mathscr{L}(H)$ is

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written $I$. The restriction of the map $x:[0, \infty[ \to X$ to the interval $[0, t]$ is denoted $\pi_t x$ for all $t \in ]0, \infty[$.

**2. Definitions.** Let $D$, $U$, $X$, $Y$ be real Hilbert spaces with norm $|\cdot|$ and inner product $(\cdot|\cdot)$ indexed by the appropriate space. The space $D$ is the *space of data* (or *initial states*), $U$ is the *space of controls*, $L^2_{\text{loc}}(0, \infty; U)$ is the *space of control maps*, $X$ is the *space of evolution* and $Y$ is the *space of observations*. Consider the *affine system* $\mathscr{A}$ at time $t \in [0, \infty[$ which is characterized by the *evolution map*

$$(1) \qquad (d, u) \mapsto \phi(t; d, u) = F(t)d + S(t)\pi_t(u) + g(t)$$

defined on $D \times L^2_{\text{loc}}(0, \infty; U)$ with values in $X$, where

(i) for all $d \in D$, $t \mapsto F(t)d:[0, \infty[ \to X$ is continuous;

(ii) $t \mapsto g(t):[0, \infty[ \to X$ is continuous;

(iii) $S(0) = 0$, $S(t) \in \mathscr{L}(L^2(0, t; U), X)$ for all $t \in ]0, \infty[$;

(iv) for all $u \in L^2_{\text{loc}}(0, \infty; U)$, $t \mapsto S(t)\pi_t(u):[0, \infty[ \to X$ is continuous.

*Remark.* $D$ can be thought of as the *state space* of system $\mathscr{A}$. When $X = D$ the above formulation is similar to Kalman's definition of a dynamical system, where (1) is the *state transition map.* In general (that is, $X \neq D$), (1) is not a state transition map; it only describes the evolution of the system. This occurs in the theory of hereditary differential equations where the space in which the system evolves is not the state space. Finally, the space $X$ should not be confused with the space of observations $Y$ which can be different from both $X$ and $D$.

For system $\mathscr{A}$ we define an *observer* $\bar{Z}(t)$ *at time* $t \in ]0, \infty[$ as an element of $\mathscr{L}(L^2(0, t; X), L^2(0, t; Y))$; the *observation* $\zeta(t, d, u)$ at time $t$ is given by $\zeta(t, d, u) = \bar{Z}(t)\pi_t(\phi(\cdot; d, u))$.

DEFINITION 1. Let $T$, $0 < T < \infty$, be fixed.

(i) The data $d \in D$ is *controllable at time* $T$ to a point $x \in X$ if there exists a sequence of control maps $\{u_n\}$ in $L^2(0, T; U)$ such that $\phi(T; d, u_n) \to x$; $d$ is said to be *strictly controllable at time* $T$ to $x$ if there exists a control map $u$ in $L^2(0, T; U)$ such that $\phi(T; d, u) = x$. System $\mathscr{A}$ is said to be *controllable* (*strictly controllable*) *at time* $T$ if all points of $D$ are controllable (strictly controllable) at time $T$ to all points of $X$.

(ii) Given $u \in L^2(0, T; U)$, a state $d \in D$ is said to be *observable at time* $T$ if $d$ can be uniquely determined from a knowledge of $u$ and the *observation map* $\zeta(T, d, u)$; system $\mathscr{A}$ is said to be *observable at time* $T$ if all states in $D$ are observable at time $T$.

DEFINITION 2. The data $d \in D$ is *controllable* (*strictly controllable*) *to the origin* if there exists a finite time $T > 0$ for which $d$ is controllable (strictly controllable) at time $T$ to the origin. System $\mathscr{A}$ is said to be *controllable* (*strictly controllable*) *to the origin* if all points of $D$ are controllable (strictly controllable) to the origin.

PROPOSITION 3. *When $X$ is finite-dimensional the notions of controllability and strict controllability in Definitions* 1 *and* 2 *coincide.*

*Proof.* All subspaces of a finite-dimensional Hilbert space are closed.

In general, the notions of controllability at time $T$ and controllability to the origin are not equivalent. Very often the sufficient conditions for controllability to the origin are obtained in the following way.

PROPOSITION 4. *$\mathscr{A}$ is controllable to the origin if there exists a finite time $T > 0$ for which system $\mathscr{A}$ is controllable at time $T$.*

In this paper we shall limit our investigation to the problem of controllability at time $T$. Definition 2 was introduced for completeness.

**3. Main results.** We first derive necessary and sufficient conditions for the various notions introduced in Definition 1. Prior to our main theorem we need three lemmas and a definition.

LEMMA 5. *Let $H$ and $K$ be Hilbert spaces and let $\{T(t)|t \in [0, \infty[\}$ be a family of elements of $\mathscr{L}(H, K)$. Assume that for all $h \in H$ the map $t \mapsto T(t)h$ is continuous. Then $T \in L^{\infty}_{\mathrm{loc}}(0, \infty; \mathscr{L}(H, K))$.*

*Proof.* The proof is a straightforward adaptation of the proof of a similar lemma in [7, Lemma 3, p. 616].

DEFINITION 6. Let $T > 0$ be finite. The maps $\bar{F}(T): D \to L^2(0, T; X)$ and $\bar{S}(T): L^2(0, T; U) \to L^2(0, T; X)$ are defined by $(\bar{F}(T)d)(t) = F(t)d$ and $(\bar{S}(T)u)(t) = S(t)\pi_t(u)$.

LEMMA 7. *The maps $\bar{F}(T)$ and $\bar{S}(T)$ are linear and continuous. In particular, the map $(d, u) \mapsto \pi_T(\phi(\cdot\,; d, u))$ (resp. $(d, u) \mapsto \zeta(T, d, u))$ defined in $D \times L^2(0, T; U)$ with values in $L^2(0, T; X)$ (resp. $L^2(0, T; Y)$) is affine and continuous.*

*Proof.* By Lemma 5, $\bar{F}(T) \in L^{\infty}(0, T; \mathscr{L}(D, X))$. Hence $\|\bar{F}(T)d\|_{L^2} \leqq \|\bar{F}(T)\|_{L^\infty}|d|_D$. The proof is identical for $\bar{S}(T)$. The remainder of the lemma is now obvious.

LEMMA 8. *Let $H$ and $K$ be Hilbert spaces and let $\Lambda \in \mathscr{L}(H, K)$ be given. The following statements are equivalent:*

(i) *$\Lambda$ is injective (resp. has a dense image in $K$);*

(ii) *$\Lambda^*$ has a dense image in $H$ (resp. is injective);*

(iii) *$\Lambda^*\Lambda > 0$ (resp. $\Lambda\Lambda^* > 0$).*

*Proof.* (i) $\Leftrightarrow$ (ii). $\operatorname{Ker}\Lambda = (\operatorname{Im}\Lambda^*)^{\perp} = (\overline{\operatorname{Im}\Lambda^*})^{\perp}$, where $^{\perp}$ denotes the orthogonal complement in $H$. Hence $\overline{\operatorname{Im}\Lambda^*} = H$ if and only if $\operatorname{Ker}\Lambda = 0$.

(i) $\Leftrightarrow$ (iii). By definition, $\operatorname{Ker}\Lambda^*\Lambda \supset \operatorname{Ker}\Lambda$. Conversely, $h \in \operatorname{Ker}\Lambda^*\Lambda$ implies that $|\Lambda h|_K^2 = (h|\Lambda^*\Lambda h) = 0$. Hence $\operatorname{Ker}\Lambda^*\Lambda \subset \operatorname{Ker}\Lambda$. When $\Lambda$ is injective, $\Lambda^*\Lambda$ is injective. By symmetry, $\Lambda^*\Lambda > 0$.

THEOREM 9.

(i) *$\mathscr{A}$ is controllable at time $T$ if and only if the image of $S(T)$ is everywhere dense in $X$.*

(ii) *$\mathscr{A}$ is strictly controllable at time $T$ if and only if the map $S(T)$ is surjective.*

(iii) *$\mathscr{A}$ is observable at time $T$ if and only if the composite map $\bar{Z}(T)\bar{F}(T)$ is injective.*

*Proof.* (i) Let the image of $S(T)$ be everywhere dense in $X$. Then for arbitrary $d \in D$ and $x \in X$ there exists a sequence $\{u_n\}$ in $L^2(0, T; U)$ such that $S(T)u_n \to x - F(T)d - g(T)$, that is, $\phi(T; d, u_n) \to x$. Conversely, for arbitrary $y \in X$, let $h = 0$ and $x = g(T) + y$. Since $\mathscr{A}$ is controllable at time $T$ there exists $\{u_n\}$ in $L^2(0, T; U)$ such that $\phi(T; 0, u_n) \to x = g(T) + y$, that is, $S(T)u_n \to y$. Hence the image of $S(T)$ is everywhere dense in $X$.

(ii) When $S(T)$ is surjective, $\mathscr{A}$ is clearly strictly controllable at time $T$. Conversely, for arbitrary $y \in X$, let $h = 0$ and $x = g(T) + y$; there exists $u \in L^2(0, T; U)$ such that $g(T) + y = x = S(T)u + g(T)$. Hence $S(T)$ is surjective.

(iii) Let $\bar{Z}(T)\bar{F}(T)$ be injective. For arbitrary $d \in D$ consider the observation $\zeta(T, d, 0)$ and assume that there exists $d' \neq d$ such that $\zeta(T, d, 0) = \zeta(T, d', 0)$. Then $\bar{Z}(T)(\bar{F}(T)(d - d')) = 0$ in contradiction with our initial hypothesis. Conversely, if the system is observable at time $T$, then for all $d \neq 0$,

$$\bar{Z}(T)(\bar{F}(T)d + \pi_T(g)) \neq \bar{Z}(T)(\pi_T(g)).$$

But this means that $\bar{Z}(T)(\bar{F}(T)d) \neq 0$ and that the composite map $\bar{Z}(T)\bar{F}(T)$ is injective.

COROLLARY 10. *The following conditions are equivalent*:

(i) $\mathscr{A}$ *is controllable* (*observable*) *at time* $T$;

(ii) $\overline{\mathrm{Im}\,(S(T))} = X$ ($\bar{Z}(T)\bar{F}(T)$ *is injective*);

(iii) $(S(T))^*$ *is injective* ($\overline{\mathrm{Im}\,(F(T)^*Z(T)^*)} = D$);

(iv) $S(T)(S(T))^* > 0$ $(\bar{F}(T)^*\bar{Z}(T)^*\bar{Z}(T)\bar{F}(T) > 0)$.

*Remarks.* (i) The evolution space $X$ is the direct sum of the closed linear subspace $X_c = \overline{\mathrm{Im}(S(T))}$ and its orthogonal complement $X_u$. Similarly, the state space $D$ is the direct sum of the closed linear subspace $D_u = \mathrm{Ker}(\bar{Z}(T)\bar{F}(T))$ and its orthogonal complement $D_0$ which is isomorphic to the quotient $D/D_u$. There exist linear maps

$$\mathscr{F}(T): D_0 \oplus D_u \to X_c \oplus X_u \quad \text{and} \quad \mathscr{S}(T): L^2(0, T, U) \to X_c \oplus X_u$$

such that system $\mathscr{A}$ at time $T$ is equivalent to the following canonical system:

$$(d, u) \mapsto \mathscr{F}(T)d + \mathscr{S}(T)u : (D_0 \oplus D_u) \times L^2(0, T; U) \to X_1 \oplus X_2.$$

(ii) $X_c$ (resp. $X_u$) is usually referred to as the *controllable* (resp. *uncontrollable*) part of $\mathscr{A}$. Similarly, $D_0$ (resp. $D_u$) is referred to as the *observable* (*unobservable*) part of $\mathscr{A}$.

(iii) When $\mathscr{A}$ is strictly controllable at $T$, $S(T)$ is surjective. Thus it is a topological isomorphism by the open-mapping theorem. In particular, when a point $h \in D$ is strictly controllable to a point $x \in X$, all points in a neighborhood of $h$ are strictly controllable to points in a neighborhood of $x$.

The duality between the notions of observability and controllability is a consequence of Lemma 8.

DEFINITION 11. (i) The *controlled dual system* $\mathscr{A}_c^*$ of $\mathscr{A}$ is defined at time $t \in [0, T]$ by the map

$$y \mapsto \chi(t; y) = \int_t^T F(s)(\bar{Z}(T)^*y)(s)\, ds : L^2(0, T; Y) \to D.$$

(ii) The *observed dual system* $\mathscr{A}_0^*$ of $\mathscr{A}$ is defined by the map

$$x \mapsto \psi(x) = S(T)^*x : X \to L^2(0, T; U).$$

*Remark.* For dual systems, $Y$ is the space of controls, $D$ is the space of evolution, $X$ is the space of data and $U$ is the space of observations.

DEFINITION 12. Assume that the following hypotheses are satisfied:

(i) the operator $S(T)$ has the integral representation

$$S(T)u = \int_0^T R(T, t)B(t)u(t)\, dt,$$

where $B \in L^\infty(0, T; \mathscr{L}(U, D))$, $R(T, t) \in \mathscr{L}(D, X)$ and the map $t \mapsto R(T, t)d : [0, T] \to X$ is continuous for all $d$ in $D$;

(ii) there exists $Z \in L^\infty(0, T; \mathscr{L}(X, Y))$ such that

$$(\bar{Z}(T)x)(t) = Z(t)x(t) \quad \text{a.e. in } [0, T] \quad \text{for all } x \in L^2(0, T; X).$$

The simultaneously controlled and observed adjoint system $\mathscr{A}^*$ is defined as follows:

$$\phi^*(t; x, y) = R(T, t)^* x + \int_t^T F(s)^* Z(s)^* y(s)\, ds \quad \text{(evolution map)},$$

$$\zeta^*(t; x, y) = B(t)^* \phi^*(t; x, y) \quad \text{(observation map)}.$$

*Remarks.* (i) Notice that by Lemma 8, the maps $t \mapsto R(T, t) : [0, T] \to \mathscr{L}(D, X)$ and $s \mapsto F(s)^* : [0, T] \to \mathscr{L}(X, D)$ are in $L^\infty(0, T; \mathscr{L}(D, X))$ and $L^\infty(0, T; \mathscr{L}(X, D))$. Thus the above hypotheses make sense. Moreover, when $x = 0$ system $\mathscr{A}^*$ coincides with $\mathscr{A}_c^*$ and when $y = 0$ it coincides with $\mathscr{A}_0^*$.

(ii) For system $\mathscr{A}^*$ the direction of time has been reversed and we must speak of controllability and observability at time 0 (zero) instead of at time $T$.

THEOREM 13. *System $\mathscr{A}$ is controllable (resp. observable) at time $T$ if and only if system $\mathscr{A}^*$ is observable (resp. controllable) at time* 0.

*Proof.* The proof follows from Definitions 11 and 12, Lemma 8 and Corollary 10.

## REFERENCES


[1] M. C. DELFOUR AND S. K. MITTER, *Hereditary differential systems with constant delays. I—General case*, Centre de recherches mathématiques, Université de Montréal, Canada, 1970; submitted to J. Differential Equations.

[2] ———, *Hereditary differential systems with constant delays. II—A class of affine systems and the adjoint problem*, Centre de recherches mathématiques, Université de Montréal, Canada, 1970; submitted to J. Differential Equations.

[3] ———, *Systèmes d'équations différentielles héréditaires à retards fixes. Théorèmes d'existence et d'unicité*, C.R. Acad. Sci. Paris, 272 (1971), pp. 382–385.

[4] ———, *Systèmes d'équations différentielles héréditaires à retards fixes. Une classe de systèmes affines et le problème du système adjoint*, Ibid., 272 (1971), pp. 1109–1112.

[5] ———, *Controllability, observability and optimal feedback control of affine hereditary differential systems*, this Journal, 10 (1972).

[6] ———, *Le contrôle optimal de systèmes gouvernés par des équations différentielles héréditaires affines*, C.R. Acad. Sci. Paris, 272 (1971), pp. 1715–1718.

[7] N. DUNFORD AND J. T. SCHWARTZ, *Linear Operators. I*, Interscience, New York, 1967.

[8] H. O. FATTORINI, *Some remarks on complete controllability*, this Journal, 4 (1966), pp. 686–694.

[9] ———, *On complete controllability of linear systems*, J. Differential Equations, 3 (1967), pp. 391–402.

[10] V. JURDJEVIC, *Abstract control systems: Controllability and observability*, this Journal, 8 (1970), pp. 424–439.

[11] R. E. KALMAN, *Mathematical description of linear dynamical systems*, this Journal, 1 (1963), pp. 152–192.

[12] J. L. LIONS, *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*, Dunod, Paris, 1968; English transl., Springer-Verlag, Berlin, New York, 1971.

[13] S. K. MITTER, *Optimal control of distributed parameter systems*, Joint Automatic Control Conference (Boulder, Colo., 1969), ASME, New York, 1969, pp. 13–48.