OPTIMAL CONTROL OF LINEAR HEREDITARY SYSTEMS WITH QUADRATIC CRITERION

by

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ABSTRACT

Systems governed by retarded functional differential equations are studied in the context of the Delfour-Mitter M^2 space setting. An exact, explicit closed form solution to a differential-delay equation with one delay is exhibited. The optimal control problem with quadratic cost on a finite or infinite time interval is considered and solved completely. The optimal control and the optimal cost are expressed in terms of an M^2 operator $\Pi(t)$ which is the unique solution of a Riccati (differential) operator equation. In the tracking problem, we have in addition a M^2-valued function $\tilde{g}(t)$ for which a differential equation is established. From these two differential equations, it is possible to deduce the first order differential equations satisfied by the matrix valued functions $\Pi_{00}(t)$, $\Pi_{01}(t,\alpha)$, $\Pi_{11}(t,\theta,\alpha)$ and the vector valued functions $g_0(t)$, $g_1(t,\theta)$ appearing in the expressions for the optimal control and the optimal cost. This coupled system of differential equations is not solved explicitly. Instead, in the autonomous case, we demonstrate an approximation technique based upon the eigenfunctions of $\mathcal{A}$ and which reduces to the quadratic criterion problem for systems governed by ordinary differential equations. An application of the various results is made to Kalechi's differential-delay equation governing the rate of investment in a capitalistic economy.

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Notation

Set of real numbers denoted by R, set of complex numbers by $\mathbb{C}$.

Let X,Y be topological vector spaces.

We denote by $\mathcal{L}(X,Y)$ the set of all continuous linear maps X into Y.

In the case $X = Y$, we write $\mathcal{L}(X)$ instead of $\mathcal{L}(X,Y)$.

Let H,K be real Hilbert spaces.

The inner product of two elements $x,y \in H$ is denoted by

$(x,y)_H$

and the norm of an element $x \in H$ is denoted by

$||x||_H = (x,x)_H$

In the case $H = R^n$, we denote the inner product of two elements $x = (x_1 \ldots x_n)$, $y = (y_1 \ldots y_n)$ by

$$(x,y) = \sum_{i=1}^{n} x_i y_i$$

For any $\Lambda \in \mathcal{L}(H,K)$, the adjoint in $\mathcal{L}(H,K)$ is denoted by Λ^*. In the case $H = K$, Λ is said to be self-adjoint if $\Lambda = \Lambda^*$.

Table of Contents

Abstract 2

Acknowledgements 3

Notation 5

Chapter 1 Introduction 9

1.1 Hereditary Systems in the Physical World 10

1.2 History of functional differential equations 23

1.3 Development of Control Theory for R.F.D.E. 26

1.4 Brief Outline 31

Chapter 2 Mathematical Preliminaries 34

2.1 Existence Uniqueness and Continuity of solutions of R.F.D.E. 34

2.2 Representation of solutions of R.F.D.E. 40

2.3 Continuity and Differentiability in M^2; State evolution equation 53

2.4 Semigroup of operators 66

2.5 Hereditary adjoint equation; Hereditary product 73

2.6 Linear bounded operators and unbounded differential operators in M^2 82

2.7 Advantages of M^2 over C 98

Chapter 3 Finite Time Quadratic Criterion Optimal Control Problem 101

3.1 Formulation of the control problem 101

3.2 Existence and characterization of the optimal control 105

3.3 Decoupling optimality pair of equations 112

3.4 The operator $\Pi(t)$ and the optimal cost and optimal control 121

3.5 Operator Riccati differential equation for $\Pi(t)$ 131

3.6 Finite time tracking problem 144

Chapter 4 Infinite Time Regulator Problem for Autonomous R.F.D.E. 159

4.1 Formulation of the control problem 160

4.2 L^2-stability; Stabilizability 161

4.3 Asymptotic behavior of $\Pi_T(t)$ 176

4.4 Operator Riccati equation for Π 189

4.5 Example 196

Chapter 5 Approximate Optimal Control for Autonomous R.F.D.E. 199

5.1 Decomposing M^2 into eigenspace and complementary subspace 201

5.2 Finite time regulator problem 214

5.3 Finite time tracking problem 223

5.4 Infinite time problem 228

5.5 Completeness question 232

5.6 Example 256

Chapter 6 Applications, Suggestions for Further Research, Conclusions 268

6.1 The business cycle, or a control theorist looks at economics 268

6.2 Suggestions for further research, Conclusions 277

Appendix 1 279

Appendix 2 295

Appendix 3 303

Appendix 4 312

Appendix 5 316

References 334

Biographical Note 345

Chapter 1

Introduction

Physical processes involving discrete subsystems are usually described by ordinary differential equations. The underlying assumption - implicit or explicit - is that the interactions between the subsystems will be instantaneous. In practice and in theory however, this will not be the case. The dynamics of mechanical, electrical, hydraulic and pneumatic devices involve non-zero time delays. The special theory of relativity sets an upper limit to the speed with which subsystems can communicate - this upper limit being the speed of light (or radio waves). Thus for subsystems stationed on the earth that interact through radio waves, the delay will be so small that for all practical purposes it can be ignored. This is not the case for space travel. Radio waves take $1\frac{1}{4}$ seconds to travel from a control center on the earth to a space vehicle orbitting the moon and another $1\frac{1}{4}$ seconds to come back. If and when a space vehicle is sent to Jupiter, the delay could be up to 40 minutes. It would not be prudent to ignore such a large delay.

It should be pointed out that it might be possible that a delay in a dynamical system is harmless in the sense that the asymptotic properties of the "delayed" dynamical system are similar to that of the "undelayed" dynamical system. Driver [24] has an interesting discussion on this point.

1.1 Hereditary Systems in the Physical World

A hereditary system is a system whose dynamics depends in some predetermined manner upon the past history of the system. Hereditary systems can be adequately described by functional differential equations. A functional differential equation of retarded type (R.F.D.E.) is one in which the derivative $\dot{x}(t)$ of the state at time t is specified as a functional of the past values of the state x over some time interval $[t-a, t]$. A functional differential equation of neutral type (N.F.D.E.) is one in which $\dot{x}(t)$ is specified as a functional of the past values of x and $\dot{x}$ over some time interval $[t-a, t]$.

Hereditary systems occur naturally in the physical world. The following are some examples:

a) Biological populations

A simple model of a biological population can be found in Cooke [11].

Let $x(t)$ be the number of individuals in a population at time t, τ the gestation period and σ the life span.

Then the functional differential equation governing the growth of the population may be taken to be

$$\dot{x}(t) = \alpha\{x(t-\tau) - x(t-\tau-\sigma)\} \tag{1-1}$$

where α is some constant.

A slightly more sophisticated mathematical description of a fluctuating population of organisms (for instance bacteria) is given in Cunningham [13]

$$\dot{x}(t) = \alpha x(t) - \beta x(t-\tau)x(t) \tag{1-2}$$

where α and β are positive constants.
Equation (1-2) is also applicable to potentially explosive chemical reactions.

b) Learning Theory

In studying problems associated with pattern discrimination, learning, memory and recall in learning theory, Grossberg [33] has used a system of nonlinear functional differential equations describing cross correlated flows in a signed directed graph to model neural mechanisms. His equations are

$$\dot{x}_i(t) = -\alpha_i x_i(t) + \sum_{m=1}^{n} [x_m(t-\tau_{m_i}) - \Gamma_{m_i}]^+ \beta_{m_i} z_{m_i}(t) + C_i(t) \quad (1\text{-}3)$$

$$\dot{z}_{jk} = -\gamma_{jk} z_{jk}(t) + \delta_{jk}[x_j(t-\tau_{jk}) - \Gamma_{jk}]^+ x_k(t) \quad (1\text{-}4)$$

where $C_i(t)$ is the i^{th} input stimulus

$x_i(t)$ is the i^{th} stimulus trace or short term memory trace

$z_{jk}(t)$ is the $(j,k)^{th}$ memory trace or the long term memory trace recording the pairing of the j^{th} and k^{th} events

Γ_{jk} is the $(j,k)^{th}$ signal threshold

τ_{jk} is the time lag or reaction time between signal sent at j and received at k.

α_i, γ_{jk}, β_{jk} are structural parameters and

$[A]^+ = \max(0,A)$.

c) Number theory

Wright [80] came across the functional differential equation

$$\dot{x}(t) = -\alpha x(t-1)[1 + x(t)] \quad (1\text{-}5)$$

in studying the distribution of primes. Let $\pi(z)$ denote the number of primes less than z. Putting $w(z) = \frac{\pi(z)}{z} \log z - 1$, $\log z = 2^t$, $\alpha = \log 2$, we can heuristically show that $w(t)$ satisfies equation (1-5). In [79], Wright proved that for $0 \leq \alpha \leq \frac{3}{2}$, $w(t) \to 0$ as $t \to \infty$, and from this, the prime number theorem

$$\pi(z) \sim z/\log z \quad \text{as} \quad z \to \infty$$

follows.

d) Two body problem

Denote the position vector at time t of two particles i,j by $r_i(t)$ and $r_j(t)$. Assuming that there is no radiation reaction term, that electromagnet effects propagate at speed c and that the force between the two particles is entirely of an electromagnetic nature, Driver [23] derived the equation

$$\frac{d}{dt}|r_i(t) - r_j(t)| = \left(r_i(t) - r_j(t-\tau(t)), \dot{r}_i(t) - \dot{r}_j(t-\tau(t))\right) \Big/ |r_i(t) - r_j(t)| \left\{c - \left(r_i(t) - r_j(t-\tau(t)), \dot{r}_j(t-\tau(t))\right)\right\} \tag{1-6}$$

where $\tau(t) = |r_i(t) - r_j(t)|/c$.

e) Nuclear reactors

The dynamics of a nuclear reactor has been investigated by Ergen [28] who proposed the functional differential equation

$$\dot{x}(t) = -\frac{c}{a}\int_{-a}^{0} (a+\theta)\exp\{x(t+\theta) - 1\}d\theta \qquad (1\text{-}7)$$

where $x(t)$ is the logarithm of the reactor power, a is the transit time and c is a constant. This functional differential equation arises out of the fact that neutrons are given off some time after the fission that caused them and hence the reactor dynamics depends on its history over some time interval. Further mathematical analysis of equation (1-7) has been carried out in Nohel [63] and Levin and Nohel [55].

f) Rocket engines

The phenomenon of rough burning in a liquid propellant rocket motor can be attributed to the time delay between the instant when the liquid is injected into the combustion chamber and the instant when it is burned into hot gas. A detailed discussion is given in Tsien [76] chapter 8, where by linearizing about the steady state condition, he obtains the functional differential equation

$$\frac{dp}{dt} + (1-n)p(t) + np(t-a) = 0 \qquad (1\text{-}8)$$

where p is the dimensionless deviation from steady pressure

t is the dimensionless time variable

a is the dimensionless constant time lag of combustion

and n is a constant

g) Ship stabilization

In studying problems arising out of stabilizing a ship by means of displacing ballast between two tanks connected by a tube equipped with a propeller pump, Minorsky obtained the functional differential equation

$$m\ddot{x}(t) + r\dot{x}(t) + qx(t-\tau) + kx(t) = 0 \qquad (1\text{-}9)$$

where x is the angular displacement of the ship and m, r, q, k are constants.

h) Transmission line

It is well known, Cooke [12], that a particular initial boundary value problem for a hyperbolic partial differential equation can be replaced by an associated neutral functional differential equation. This observation originally arose out of the study of the transmission line problem, Brayton [7]. The basic idea is that the solution to the wave equation can be expressed as a linear combination of two waves, one travelling to the right, $\phi(x-ct)$ and the other travelling to the left $\psi(x+ct)$. Since they travel with speed c, they will take a finite time to travel from one end of the line to the other. Hence what is happening at one end of the line will depend upon what happened at the other end some finite time back in the past. More specifically, let us consider the flow of electricity in a lossless transmission line with ends at $x = 0$ and $x = 1$. The governing partial differential equations will be

$$\frac{\partial v(x,t)}{\partial x} = -L\frac{\partial i(x,t)}{\partial t} + e(x,t) \qquad (1\text{-}10)$$

$$\frac{\partial i(x,t)}{\partial x} = -C\frac{\partial v(x,t)}{\partial t} \qquad (1\text{-}11)$$

where $i(x,t)$ is the current flowing in the line at point x and time t, $v(x,t)$ the voltage across the line at x and t, L the inductance per unit length and C the capacitance per unit length.

The initial conditions are

$$v(x,0) = v_0(x) \tag{1-12}$$

$$i(x,0) = i_0(x) \tag{1-13}$$

where $v_0(x)$ and $i_0(x)$ are differentiable functions of x.

The boundary conditions are

$$-\, v(0,t) = r_0 i(0,t) + \ell_0 \frac{di(0,t)}{dt} - u_0(t) \tag{1-14}$$

$$v(1,t) = r_1 i(1,t) + \ell_1 \frac{di(1,t)}{dt} + u_1(t) \tag{1-15}$$

For $t > \frac{1}{\tau}$ where $\tau = 1/(LC)^{\frac{1}{2}}$ define

$$\psi_1(t) = \int_{t-\frac{1}{\tau}}^{t} \{e(\tau(\alpha-t)+1,\alpha)/\sqrt{L}\}d\alpha \tag{1-16}$$

$$\psi_2(t) = \int_{t}^{t-\frac{1}{\tau}} \{e(\tau(t-\alpha),\alpha)/\sqrt{L}\}d\alpha \tag{1-17}$$

Also define $y_1(t) = \sqrt{C}\, v(0,t) + \sqrt{L}\, i(0,t)$ (1-18)

$y_2(t) = -\, \sqrt{C}\, v(1,t) + \sqrt{L}\, i(1,t)$

For $t > \frac{1}{\tau}$, $y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ satisfies the functional differential equation

$$\dot{y}_1(t) + \frac{2L^{1/2}}{l_0}\left(\frac{1}{2C^{1/2}} + \frac{r_0}{2L^{1/2}}\right)y_1(t) + \frac{2L^{1/2}}{l_0}\left(\frac{r_0}{2L^{1/2}} - \frac{1}{2C^{1/2}}\right)y_2\left(t-\frac{1}{\tau}\right) + \dot{y}_2\left(t-\frac{1}{\tau}\right)$$
$$-\left[\left(\frac{r_0}{2L^{1/2}} - \frac{1}{2C^{1/2}}\right)\frac{2L^{1/2}}{l_0}\Psi_2(t) + \dot{\Psi}_2(t)\right] - \frac{2L^{1/2}}{l_0}\mu_0(t) = 0 \tag{1-19}$$

$$\dot{y}_2(t) + \frac{2L^{1/2}}{l_1}\left(\frac{1}{2C^{1/2}} + \frac{r_1}{2L^{1/2}}\right)y_2(t) + \frac{2L^{1/2}}{l_1}\left(\frac{r_1}{2L^{1/2}} - \frac{1}{2C^{1/2}}\right)y_1\left(t-\frac{1}{\tau}\right) + \dot{y}_1\left(t-\frac{1}{\tau}\right)$$
$$+\left[\left(\frac{r_1}{2L^{1/2}} - \frac{1}{2C^{1/2}}\right)\frac{2L^{1/2}}{l_1}\Psi_1(t) + \dot{\Psi}_1(t)\right] - \frac{2L^{1/2}}{l_1}\mu_1(t) = 0 \tag{1-20}$$

with initial conditions on $[0,\frac{1}{\tau}]$ given by

$$\dot{y}_1(t) + \frac{2L^{1/2}}{l_0}\left(\frac{r_0}{2L^{1/2}} + \frac{1}{2C^{1/2}}\right)y_1(t) = -\left\{\frac{2L^{1/2}}{l_0}\left(\frac{r_0}{2L^{1/2}} - \frac{1}{2C^{1/2}}\right) + \frac{d}{dt}\right\}$$
$$\left\{-\sqrt{C}\,v_0(\tau t) + \sqrt{L}\,i_0\tau(t) + \int_0^t \left\{e(\tau(t-\alpha),\alpha)/\sqrt{L}\right\}d\alpha \tag{1-21}$$

$$\dot{y}_2(t) + \frac{2L^{1/2}}{l_1}\left(\frac{r_1}{2L^{1/2}} + \frac{1}{2C^{1/2}}\right)y_2(t) = -\left\{\frac{2L^{1/2}}{l_1}\left(\frac{r_1}{2L^{1/2}} - \frac{1}{2C^{1/2}}\right) + \frac{d}{dt}\right\}$$
$$\left\{\sqrt{C}\,v_0(1-\tau t) + \sqrt{L}\,i_0(1-\tau t) + \int_0^t \left\{e(1-\tau t+\tau\alpha,\alpha)/\sqrt{L}\right\}d\alpha\right\} \tag{1-22}$$

$$y_1(0) = \sqrt{C}\times v_0(0) + \sqrt{L}\,i_0(0) \; : \; y_2(0) = -\sqrt{C}\,v_0(0) + \sqrt{L}\,i_0(0) \tag{1-23}$$

The last example (h) illustrates the point that physical problems described by partial differential equations can also be described (by making further approximations or by making an equivalence transformation) by functional differential equations, sometimes with a gain in simplicity. So for instance, a valve in a Diesel engine lifts in response to a pressure wave generated by the piston. It is easier to describe the motion of the valve by introducing a time delay rather than attempting to treat the entire problem of the motion of the valve and the gas flow in the cylinder.

An extensive bibliography listing further examples can be found in Choksy [9]. In chapter 6 we will discuss in detail Kalechi's functional differential equation model for the rate of investment in an economy. But now, we will discuss the strange and mysterious role that time delays play in four everyday occurrences: two of a physiological nature, speech and sight, and two of a mechanical nature, the electric bell and the thermostat.

Speech is the most complicated act (Fry [32]) that a human being is capable of performing. It involves an intricate coordination of the pharynx and the muscles of the chest, larynx and face and with very precise timing. Hence it demands a very complex control mechanism. Part of this control mechanism will be a feedforward open loop control, but there is also a control through a feedback loop. This auditory feedback takes place along the bones of a person's skull and reaches the ear along this pathway. Incidentally because the skull bones have a different frequency characteristic than air, a person never hears his voice the same way others do. This auditory feedback is vitally important for successful speech. Thus adults who become deaf in later life continue to speak normally, but after a while their speech becomes incoherent. Young babies - deaf or normal - all pass through a babbling stage. Normal babies hear their babble and go on to refine it into speech. Deaf babies never do, and have to get special training in order to learn how to speak.

There is a certain time delay associated with this auditory feedback, and it is possible to set up an experiment in which this delay is varied, Lee [52]. By getting a person to speak into a microphone connected to ear phones placed on the person's head and turning up the volume sufficiently to mask the bone conducted sound, the auditory feedback is transferred to the ear phones. The feedback signal can be delayed by recording it and playing it back after a lapse of time. For a certain time delay (usually 1/10 second) the person is unable to speak, starts stammering and stuttering and finally gives up in utter frustration. It should be noted in passing that stammers are usually able to speak fluently when subjected to this experiment.

The second phenomenon is known as the Pulfrich pendulum effect, Arden and Weale [2], in honour of its discoverer. It can be demonstrated with a bare minimum of equipment: a darkened glass and a string attached to a weight to form a pendulum. The pendulum is set swinging in a straight arc normal to the direction of sight and one eye, say the left, is covered with the darkened glass. The bob will appear to describe an ellipse - not a straight arc. An explanation of the strange phenomenon goes as follows. By reducing the light, the left eye has become dark adapted and messages relayed to the brain are delayed relative to the right eye. This delay causes the left eye to perceive the bob slightly in the past and in a different spatial position than the right eye. Now the brain calculates distance from the disparity in the images of the two eyes (This is true for distances up to twenty feet. Beyond that another mechanism comes into play). Under these conditions, the brain will interpret the usual input to be that of a bob describing an ellipse.

The third phenomenon has to do with the electric bell which would not work but for a delay in its mechanism. The magnetic force exerted by the electromagnet does not appear and disappear instantly on the operation of the interrupter contact. If this were not the case, i.e. if the self induction in the electromagnet appeared and disappeared instantaneously when the current is on and off, the hammer would strike the gong in a very feeble manner if it did so at all. The derivation of a functional differential equation describing (approximately) the motion of the hammer can be found in Norkin [64] and is given by

$$m\ddot{x}(t) + r\dot{x}(t) + kx(t) + cx(t-a) = 0 \tag{1-24}$$

where $x(t)$ is the displacement of the hammer at time t, m, r, k, c are constants and $cx(t-a)$ is an approximation to the force acting on the hammer.

In any heating system equipped with a thermostat, there will be an unavoidable delay in response to a change in temperature. It is well known that this time delay can cause the system to oscillate indefinitely rather than settling down.

The functional differential equations of examples (d) and (h) were of neutral type. All the other functional differential equations were of retarded type .

1.2 History of functional differential equations

The preceding examples should have provided enough motivation to study the qualitative features of functional differential equations. Euler was the first mathematician to study functional differential equations, [29] and he did so in connection with the problem of the general form of curves similar to their own evolutes. Later in [30], he looked for solutions of functional differential equations of the form $e^{\lambda t}$. This is basically the same method that we will exploit in chapter 5, though in keeping with the modern style in mathematics the approach we use will be roundabout and convoluted so as to obscure its basic simplicity. A number of other mathematicians, J. Bernoulli, Poisson, Cauchy, Laplace, Condorcet tackled functional differential equations in the latter half of the eighteenth and the first half of the nineteenth centuries. The problem was neglected in the latter half of the nineteenth century and did not attract the attention of mathematicians until 1911, with the publication of a paper by Schmidt [72] who treated a fairly general class of differential-difference equations. Thereafter, a number of mathematicians, Hilb, Bochner, Pitt, Bruwier, Volterra treated various aspects of functional differential equations. In particular, Volterra [78] considered the functional differential equation

$$\ddot{x}(t) + cx(t) = \int_{-a}^{0} F(\theta)x(t+\theta)d\theta \tag{1-25}$$

and obtained conditions guaranteeing the stability of the solution.

But it was not until the nineteen forties that the problem was properly formulated and theorems on the existence, uniqueness and continuity of the solution of a functional differential equation were exhibited (see Myskis [60]). In the nineteen fifties, the standard approach to functional differential equations was to use the Laplace transform to obtain a series solution or a solution by definite integrals. Closely tied to that approach was the study of the distribution of the characteristic roots in the complex plane. A good account of the state of the art then can be found in books by

Bellman and Cooke [5] and Pinney [66].

Then roundabout 1960, arising out of some difficulties he had in studying the stability of functional differential equations using Lipaunov functions, Krasovskii [48] pointed out that the natural concept of a state for a functional differential equation is not the value of x at time t, but the restriction of x to the interval $[t-a,t]$. In other words, the state space should be a function space and not R^n. In this setting it is possible to bring the tools and techniques of functional analysis (spectral, analytic, topological and semigroup methods) to bear on a study of the problem and liberate it from the Laplace transform and complex analysis. A popular choice for the function state space has been the space of continuous functions and within this context a full treatment of functional differential equations of retarded type has been given in Hale's book [36]. The state of the art is less developed for functional differential equations of neutral type. Recently, Delfour and Mitter [18], [19] have proposed the setting of the problem in the function space $M^2(-a,0;R^n)$ which will be described in more detail in chapter 2.

1.3 Development of Control Theory for R.F.D.E.

From the control theorist's point of view, it is not enough to know the qualitative features of a dynamical system. How the system will respond to different controlling inputs and which control results in the best behavior given some pre-ordained criterion is of immense interest. But first of all, the control theorist must have an adequate knowledge of the qualitative features. It is for that reason why we shall not discuss the control theory of systems governed by N.F.D.E. in this thesis, and why we restrict discussion to systems governed by R.F.D.E. Also we shall take the action of the control on the system to be instantaneous; we shall not consider systems in which there is a delay in the control.

The time optimal control problem for systems governed by R.F.D.E. has been dealt with in Oguztoreli [65] and Chyung and Lee [10]. Oguztoreli treated the case where the control restraint set was a hypercube and Chyung and Lee considered the more general case where the control restraint set is compact. In brief, the solution proceeds as follows. Working in R^n, the set of attainability at time t is shown to be convex, compact and varies continuously with t. This enables one to prove the existence and a maximal principle for the time optimal control. If normality conditions are satisfied, this time optimal control will be unique. The maximal principle is in terms of the solution to the hereditary adjoint equation. As in the case of ordinary differential equations without delay, the time optimal control will be bang-bang.

The main concern of this thesis is the solution of the quadratic criterion optimal control problem for systems governed by R.F.D.E. The first definitive paper on the topic was written by Krasovskii [46] in 1961 and is entitled "On the analytic construction of an optimal control in a system with time lag". Krasovskii considered the R.F.D.E.

$$\frac{dx}{dt} = A_{00}x(t) + A_1x(t-a) + Bv(t) \tag{1-26}$$

$$x(\theta) = h(\theta) \qquad \theta \in [-a,\theta]$$

where $v \in R$

and the quadratic cost functional

$$C(v;h) = \int_0^\infty \{(x(t),x(t)) + v^2(t)\}dt \tag{1-27}$$

Working in $C(-a,0;R^n)$ the space of continuous functions, considering a Liapunov functional that would ensure the stability of the system (and hence that the problem was well-posed) and using dynamic programming techniques Krasovskii obtained an optimal feedback control of the form

$$u(t) = -B^*\{\Pi_{00}x(t) + \int_{-a}^{0} \Pi_{00}(\alpha)x(t+\alpha)d\alpha\} \tag{1-28}$$

Krasovskii's work was extended by Ross and Flugge-Lotz who considered the slightly more general case of R^m controls. In terms of the initial function h, they were able to express the minimal cost as

$$\inf_{v} C(v;h) = (h(0),\Pi_{00}h(0)) + 2\int_{-a}^{0}d\alpha(h(0),\Pi_{01}(\alpha)h(\alpha)) + \int_{-a}^{0}d\theta\int_{-a}^{0}d\alpha(h(\theta),\Pi_{11}(\theta,\alpha)h(\alpha)) \tag{1-29}$$

and they were able to characterize Π_{00}, $\Pi_{01}(\alpha)$, $\Pi_{11}(\theta,\alpha)$ by a coupled set of first order differential equations

$$A_{00}^*\Pi_{00} + \Pi_{00}A_{00} - \Pi_{00}R\Pi_{00} + \Pi_{01}^*(0) + \Pi_{01}(0) + Q = 0 \tag{1-30}$$

$$\frac{d\Pi_{01}(\alpha)}{d\alpha} = A_{00}^*\Pi_{01}(\alpha) - \Pi_{00}R\Pi_{01}(\alpha) + \Pi_{11}(0,\alpha) \quad -a \leq \theta \leq 0$$

(1-31)

$$\Pi_{01}(-a) = \Pi_{00}A_1$$

$$\left(\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\theta}\right)\Pi_{11}(\theta,\alpha) = -\Pi_{01}^{*}(\theta)R\Pi_{01}(\alpha) \qquad -a\le\theta\le 0;\ -a\le\alpha\le 0 \qquad (1\text{-}32)$$

$$\Pi_{11}(-1,\alpha) = A_1^{*}\Pi_{01}(\alpha); \quad \Pi_{11}(\alpha,-1) = \Pi_{01}^{*}(\alpha)A_1$$

where $R = BN^{-1}B^{*}$.

The existence and uniqueness of an optimal control in the approach used by Ross and Flugge-Lotz depends upon the existence and uniqueness of a solution to equations (1-30), (1-31) and (1-32). This approach was extended by Eller, Aggarwal and Banks, [27] Kushner and Barnea, [50] Alekal, Brunovsky, Chyũng and Lee, [1] and Mueller, [62] to deal with the finite time quadratic criterion optimal control problem for system governed by non-autonomous R.F.D.E.

The dynamic programming in the space of continuous functions approach to the quadratic criterion is unsatisfactory for the following reasons:

(i) The class of admissible controls is $\{u\ ;\ u(t) = u(\tilde{x}(t))\}$ i.e. the control at time t is linear map of the system state $\tilde{x}(t) \in C(-a,0;R^n)$ into R^m. This is an unnecessary restriction, though as luck would have it the optimal control does indeed turn out to be a linear functional of the state.

(ii) The existence and uniqueness of an optimal control depends upon the existence and uniqueness of the solution to a complicated coupled set of first order partial differential equations with Riccati type features which in the infinite time case reduces to equations (1-30), (1-31) and (1-32).

(iii) In general, dynamic programming does not lend itself to a rigorous mathematical approach (see for example a discussion in Krasovskii [49) though it yields the right answer to optimal control problems.

(iv) There is a complete, satisfactory and standard solution to the quadratic criterion optimal control problem for systems governed by linear ordinary differential equations which gives the optimal control in feedback form with the gain matrix satisfying a matrix Riccati differential equation. None of its features passes over into the solution of the optimal control problem for systems governed by R.F.D.E. using the dynamic programming approach.

1.4 Brief outline

Our approach will be different. Instead of taking initial data in $C(-a,0;R^n)$ the space of continuous functions, following Delfour and Mitter [21], we will take initial data in the space $M^2(-a,0;R^n)$. In Chapter 2, we will quote existence, uniqueness and continuity theorems for the solutions of R.F.D.E. from Delfour and Mitter [18] and in general prove results that will be used in later chapters.

In chapter 3, we use the Lions' direct method [83] to obtain a necessary and sufficient condition for the existence of an optimal control to the finite time quadratic criterion problem. The optimal control is characterized by means of a coupled duo of equations, the R.F.D.E. and its hereditary adjoint equation. We can decouple these two equations to obtain the optimal control in feedback form, and we can express the minimum cost as a quadratic functional of the initial data. This leads to the study of an operator $\Pi(t) : M^2 \to M^2$ for which we derive an operator Riccati differential equation. From this equation we can deduce the coupled set of first order partial differential equations satisfied by $\Pi_{00}(t)$, $\Pi_{01}(t,\alpha)$, $\Pi_{11}(t,\theta,\alpha)$. Whatever advantages working in the function space $M^2(-a,0;R^n)$ might have over the function space $C(-a,0;R^n)$, it does not lead to less taxing and tedious computations. Indeed it seems that no matter what you do, the derivation of the coupled set of first order partial differential equations satisfied by $\Pi_{00}(t)$, $\Pi_{01}(t,\alpha)$, $\Pi_{11}(t,\theta,\alpha)$ involves hideous calculations. These calculations have been tucked into the appendices. Finally in chapter 3, we consider the tracking problem (i.e. we have a forcing term in the R.F.D.E.) whose solution is a modification of the solution for the regulator problem (i.e. have no forcing term in the R.F.D.E.)

In chapter 4, we consider the infinite time autonomous regulator quadratic criterion problem. We introduce the concept of stabilizability in order to ensure that the problem is well posed. Again we obtain the existence of an optimal control in feedback form and the minimum cost as a quadratic functional of the initial data. We derive an operator Riccati equation for an operator $\Pi : M^2 \to M^2$ and from this we deduce the coupled set of differential equations satisfied by Π_{00}, $\Pi_{01}(\alpha)$, $\Pi_{11}(\theta,\alpha)$. It should be noted in passing that our approach to the finite and infinite time quadratic criteria optimal control problems is similar to the usual approach used for systems governed by linear ordinary differential equations.

If we could decouple the coupled set of first order partial differential equations satisfied by $\Pi_{00}(t)$, $\Pi_{01}(t,\alpha)$, $\Pi_{11}(t,\theta,\alpha)$, we would be in a better position of solving those equations to give an explicit and complete solution to the optimal control problem. But alas this does not seem to be possible, and a solution to the equations even in the simplest possible case - the one dimensional infinite time problem - seems well nigh impossible. So approximate we must, and we do so in chapter 5 by considering the solution of the R.F.D.E. in the $M^2(-a,0;R^n)$ function space on an eigenspace of $M^2(-a,0;R^n)$. Fortunately, when this is done, the approximate control problem reduces to a quadratic criterion optimal control problem in R^j where j is the number of eigenfunctions spanning the eigenspace. This way, we reduce the problem to one whose solution is well known. We can show that the optimal control obtained this way is close to the exact optimal control in the sense that as $j \to \infty$, the approximate optimal control approaches the exact optimal control.

Finally in chapter 6, we apply the results of the previous chapters to a model of the rate of investment in a capitalistic economy proposed by Kalechi [40] in 1935.

Chapter 2

Mathematical Preliminaries

In this chapter, we shall establish some results which will be used in succeeding chapters.

2.1 Existence, Uniqueness and Continuity of solutions of R.F.D.E.

We consider the linear R.F.D.E. defined on $[t_0,T]$

$$\frac{dx}{dt} = A_{00}(t)x(t) + \sum_{i=1}^{N} A_i(t)x(t+\theta_i) + \int_{-a}^{0} A_{01}(t,\theta)x(t+\theta)d\theta+f(t)$$

$$x(t_0+\theta) = h(\theta) \quad \theta \in [-a,0],\ h(\cdot) \text{ initial data} \tag{2-1}$$

where $N \geq 1$ is an integer, $0 < a < \infty$

$-a = \theta_N < \theta_{N-1} < \dots < \theta_1 < \theta_0 = 0 \qquad i = 1,\dots N$

$f \in L^2(t_0,T;R^n)$, $A_{00}(\cdot)$, $A_i(\cdot)$ are elements of

$L^2(t_0,T; \mathcal{L}(R^n))$

$A_{01}(\cdot,\cdot) \in L^2(t_0,T;-a,0; \mathcal{L}(R^n))$

Equivalently (2-1) can be written in the integral form (and this is the form in which the existence and uniqueness theorems are proved)

$$x(t) = h(0) + \int_{t_0}^{t} A_{00}(s)x(s)ds + \sum_{i=1}^{N} \int_{t_0}^{t} A_i(s)x(s+\theta_i)ds$$

$$+ \int_{t_0}^{t} ds \int_{-a}^{0} d\theta \; A_{01}(s,\theta)x(s+\theta) + \int_{t_0}^{t} f(s)ds \qquad (2\text{-}2)$$

$$x(t_0+\theta) = h(\theta) \qquad \theta \in [-a,0]$$

Remarks

1. The term $A_i(t)x(t+\theta_i)$ in (2-1) gives rise to a concentrated delay (also known as a transportation lag) taking effect at time t and arising at time $t+\theta_i$. The term $\int_{-a}^{0} A_{01}(t,\theta)x(t+\theta)d\theta$ gives rise to a distributed delay taking effect at time t and arising out of the history of the system on the interval $[t-a,t]$.

2. Later on, for technical reasons $A_{00}(t)$, $A_i(t)$, $A_{01}(t,\cdot)$ to be piecewise continuous and continuous from the right.

First of all, we have to say something about the existence, uniqueness and continuity of the solution of (2-1) with respect to the initial data h taken to lie in some function space. The usual choice for this function space is $C(-a,0;R^n)$ the set of continuous functions mapping $[-a,0]$ into R^n with the supremum norm. Looking for solutions in the space $C(t_0,T;R^n)$ Hale [36] pp. 13-23 proves existence, uniqueness and continuity of the solution with respect to the initial data.

The prominent feature of R.F.D.E. is that the solution will be smoother than the initial data. Also we can have a solution to (2-1) with the initial data h discontinuous. Indeed all we have to specify of the initial data h is the value $h(0)$ and h as a measurable and integrable map $[-a,0] \to R^n$.

This is the motivation for the introduction of the space $M^2(-a,0;R^n)$ (Delfour and Mitter [18], [19], Delfour [17]) which is arrived at in the following manner: Take $\mathcal{L}_0^2(-a,0;R^n)$ the vector space of all Lebesgue measurable and square integrable maps $h : [-a,0] \to R^n$ with $h(0)$ well defined and impose the semi-norm

$$||h|| = \{|h(0)|^2 + \int_{-a}^{0} |h(\theta)|^2 d\theta\}^{\frac{1}{2}} \qquad (2\text{-}3)$$

Define the linear subspace $\mathcal{S}$ of $\mathcal{L}_0^2(-a,0;R^n)$ by

$$\mathcal{S} = \{h; ||h|| = 0\} \qquad (2\text{-}4)$$

$M^2(-a,0;R^n)$ is defined to be the quotient space of $\mathcal{L}_0^2(-a,0;R^n)$ by $\mathcal{S}$.

$M^2(-a,0;R^n)$ with the norm

$$||h||_{M^2} = \{|h(0)|^2 + \int_{-a}^{0} |h(\theta)|^2 d\theta\}^{\frac{1}{2}} \tag{2-5}$$

and inner product

$$(h,k)_{M^2} = (h(0),k(0)) + \int_{-a}^{0} (h(\theta),k(\theta))d\theta \tag{2-6}$$

is a Hilbert space and is isometrically isomorphic to $R^n \times L^2(-a,0;R^n)$.

When there is no possibility of confusion arising, we shall denote $M^2(-a,0;R^n)$ by M^2.

If we take $M^2(-a,0;R^n)$ to be the space of initial data, our solution will be absolutely continuous and with derivative in $L^2(t_0,T;R^n)$. Hence we look for a solution in the function space $AC^2(t_0,T;R^n)$, the vector space of all absolutely continuous maps $[t_0,T] \to R^n$ with derivative in $L^2(t_0,T;R^n)$ and norm

$$||x||_{AC^2} = \{|x(t_0)|^2 + \int_{t_0}^{T} |\frac{dx(t)}{dt}|^2 dt\}^{\frac{1}{2}} \tag{2-7}$$

$AC^2(t_0,T;R^n)$ is a Hilbert space.

With $M^2(-a,0;R^n)$ as the space of initial data and $AC^2(t_0,T;R^n)$ as the space in which a solution is sought, Delfour and Mitter [18] establish the following result which is stated as a theorem.

Theorem 2A Delfour and Mitter [18]
With initial data $h \in M^2(-a,0;R^n)$, the R.F.D.E. has a unique solution $x \in AC^2(t_0,T;R^n)$. Denoting this solution by $\phi(t;t_0,h,f)$ and defining

$$\mathscr{P}(t_0,T) = \{(t,s);\ t,s \in [t_0,T],\ t \geq s\}$$

we have

(i) for fixed t_0 the map

$$\begin{aligned}&(h,f) \mapsto \phi(\cdot;t_0,h,f)\\ &M^2(-a,0;R^n) \times L^2(t_0,T;R^n) \to AC^2(t_0,T;R^n)\end{aligned} \qquad (2\text{-}8)$$

is bilinear and continuous.

(ii) for fixed h,f the map

$$\begin{aligned} &(t,s) \mapsto \phi(t;s,h,f) \\ &\mathscr{P}(t_0,T) \to R^n \end{aligned} \tag{2-9}$$

is continuous.

In chapter 4, we shall consider the autonomous R.F.D.E. defined on $[0,\infty)$

$$\begin{aligned} &\frac{dx}{dt} = A_{00}x(t) + \sum_{i=1}^{N} A_i x(t+\theta_i) + \int_{-a}^{0} A_{01}(\theta)x(t+\theta)d\theta+f(t) \\ &x(\theta) = h(\theta),\ \theta \in [-a,0] \end{aligned} \tag{2-10}$$

where A_{00}, $A_i \in \mathcal{L}(R^n)$ $i = 1,\dots N$

$A_{01}(\cdot) \in L^2(-a,0;\mathcal{L}(R^n))$, $f \in L^2_{\ell oc}(0,\infty;R^n)$

Corollary 1

The autonomous R.F.D.E. (2-10) has a unique solution $\phi(\cdot;h,f)$ in $AC^2_{\ell oc}(0,\infty;R^n)$ and the map

$$\begin{aligned} &(h,f) \mapsto \phi(\cdot;h,v) \\ &M^2(-a,0;R^n) \times L^2_{\ell oc}(0,\infty;R^n) \to AC^2_{\ell oc}(0,\infty;R^n) \end{aligned} \tag{2-11}$$

is bilinear and continuous.

2.2 Representation of solutions of R.F.D.E.

Banks [3] gives a representation of solutions to R.F.D.E. (2-1) when the initial data lies in $C(-a,0;R^n)$. For the initial data lying in $M^2(-a,0;R^n)$ we have the result

Theorem 2B Delfour and Mitter [19]

The solution to the R.F.D.E. (2-1) can be written in the form

$$\begin{aligned}\phi(t;t_0,h,f) &= \Phi^0(t,t_0)h(0) + \sum_{i=1}^{N} \int_{\theta_i}^{\min(0,t-t_0+\theta_i)} d\alpha\ \Phi^0(t,t_0+\alpha-\theta_i)A_i(t_0+\alpha-\theta_i)h(\alpha) \\ &+ \int_{-a}^{0} d\alpha \int_{\max(-a,\alpha-t+t_0)}^{\alpha} d\beta\ \Phi^0(t,t_0+\alpha-\beta)A_{01}(t_0+\alpha-\beta,\beta)h(\alpha) \qquad (2\text{-}12) \\ &+ \int_{t_0}^{t} \Phi^0(t,s)f(s)ds\end{aligned}$$

or more compactly

$$\phi(t;t_0,h,f) = \Phi^0(t,t_0)h(0) + \int_{-a}^{0} \Phi^1(t,t_0,\alpha)h(\alpha) + \int_{t_0}^{t} \Phi^0(t,s)f(s)ds \qquad (2\text{-}13)$$

where $\Phi^0(t,s) \in \mathcal{L}(R^n)$ $t,s \in [t_0,T]$, $t \geq s$ and satisfies the matrix R.F.D.E.

$$\frac{\partial \Phi^0(t,s)}{\partial t} = A_{00}(t)\,\Phi^0(t,s) + \sum_{i=1}^{N} A_i(t)\,\Phi^0(t+\theta_i,s) + \int_{-a}^{0} A_{01}(t,\theta)\,\Phi^0(t+\theta,s)\,d\theta \tag{2-14}$$

$\Phi^0(s,s) = I, \quad \Phi^0(s+\theta,s) = 0 \quad \theta \in [-a,0)$

The mapping $\quad (t,s) \mapsto \Phi^0(t,s)$

$$\mathscr{P}(t_0,T) \to \mathcal{L}(R^n) \tag{2-15}$$

is continuous and the mapping

$$\Phi^0 : [t_0,T] \times [t_0,T] \to \mathcal{L}(R^n) \tag{2-16}$$

(where $\Phi^0(t,s) = 0$ for $t < s$) is an element of $L^2(t_0,T;t_0,T;\mathcal{L}(R^n))$

$\Phi^1(t,t_0,\alpha) \in \mathcal{L}(R^n)$ and

$$\Phi^1(t,t_0,\alpha) = \sum_{i=1}^{N} \begin{cases} \Phi^0(t,t_0+\alpha-\theta_i)A_i(t_0+\alpha-\theta_i) & \alpha+t_0-t<\theta_i<\alpha \\ 0 & \text{otherwise} \end{cases}$$

$$+ \int_{\max(-a,\alpha-t+t_0)}^{\alpha} d\beta\, \Phi^0(t,t_0+\alpha-\beta)A_{01}(t_0+\alpha-\beta,\beta) \tag{2-17}$$

Proof. Consider the R.F.D.E. on $[s,T]$ where $s \in [t_0,T]$

$$\frac{dx}{dt} = A_{00}(t)x(t) + \sum_{i=1}^{N} A_i(t)x(t+\theta_i) + \int_{-a}^{0} A_{01}(t,\theta)x(t+\theta)d\theta \tag{2-18}$$

$$x(s) = h(0) \qquad x(s+\theta) = 0 \qquad \theta \in [-a,0)$$

From theorem 2A, (2-18) has a unique solution $x(t;s)$ and for fixed t, $t \geq s$, the map

$$h(0) \to x(t;s) \tag{2-19}$$

is linear and continuous and we can write

$$x(t;s) = \Phi^0(t,s)h(0)$$

clearly $\Phi^0(t,s)$ satisfies (2-14) since $x(t;s) = \Phi^0(t,s)h(0)$ satisfies (2-18). Also the continuity of the map

$$(t,s) \mapsto \Phi^0(t,s)$$

$$\mathscr{P}(t_0,T) \to \mathscr{L}(R^n)$$

follows from (ii) of Theorem 2A.

Now let us consider the R.F.D.E.

$$\frac{dx}{dt} = A_{00}(t)x(t) + \sum_{i=1}^{N} A_i(t)x(t+\theta_i) + \int_{-a}^{0} A_{01}(t,\theta)x(t+\theta)\,d\theta + f(t)$$

$$x(t_0+\theta) = 0\ , \qquad \theta \in [-a,0] \tag{2-20}$$

or equivalently the integral equation

$$x(t) = \int_{t_0}^{t} A_{00}(r)x(r)dr + \sum_{i=1}^{N} \int_{t_0}^{t} A_i(r)x(r+\theta_i)dr$$

$$+ \int_{t_0}^{t} dr \int_{-a}^{0} d\theta\ A_{01}(r,\theta)x(r+\theta) + \int_{t_0}^{t} f(r)dr \tag{2-21}$$

$$x(t_0+\theta) = 0 \qquad \theta \in [-a,0]$$

$\Phi^0(t,s)$ will satisfy the matrix integral equation

$$\Phi^0(t,s) = I + \int_{s}^{t} A_{00}(r)\Phi^0(r,s)dr + \sum_{i=1}^{N} \int_{s-\theta_i}^{t} A_i(r)\Phi^0(r+\theta_i,s)dr$$

$$+ \int_{-a}^{0} d\theta \int_{s-\theta}^{t} dr\ A_{01}(r,\theta)\Phi^0(r+\theta,s) \tag{2-22}$$

$$\Phi^0(s+\theta,s) = 0 \qquad \theta \in [-a,0)$$

We want to show that

$$y(t) = \begin{cases} \int_{t_0}^{t} \Phi^0(t,s)f(s) & t \geq t_0 \\ & \\ 0 & t < t_0 \end{cases} \tag{2-23}$$

satisfies the integral equation (2-21).

Clearly it satisfies the initial conditions.

Substituting (2-23) into the left hand side of (2-21) we obtain

$$\int_{t_0}^{t} dr \int_{t_0}^{r} ds\ A_{00}(r)\Phi^0(r,s)f(s)$$

$$+ \sum_{i=1}^{N} \int_{t_0}^{t} dr \int_{t_0}^{r+\theta_i} ds\ A_i(r)\Phi^0(r+\theta_i,s)f(s)$$

$$+ \int_{t_0}^{t} dr \int_{-a}^{0} ds \int_{t_0}^{r+\theta} ds\ A_{01}(r,\theta)\Phi^0(r+\theta,s)f(s) + \int_{t_0}^{t} f(s)ds$$

$$= \int_{t_0}^{t} dr \int_{t_0}^{t} ds\ A_{00}(r)\Phi^0(r,s)f(s)$$

$$+ \sum_{i=1}^{N} \int_{t_0}^{t} dr \int_{t_0}^{t} ds\ A_i(r)\Phi^0(r+\theta_i,s)f(s)$$

$$+ \int_{t_0}^{t} dr \int_{-a}^{0} d\theta \int_{t_0}^{t} ds\ A_{01}(r,\theta)\Phi^0(r+\theta,s)f(s) + \int_{t_0}^{t} f(s)ds$$

since $\Phi^0(r,s) = 0$ for $r < s$

$$= \int_{t_0}^{t} ds \int_{t_0}^{t} dr\ A_{00}(r)\Phi^0(r,s)f(s)$$

$$+ \sum_{i=1}^{N} \int_{t_0}^{t} ds \int_{t_0}^{t} dr\ A_i(r)\Phi^0(r+\theta_i,s)f(s)$$

$$+ \int_{t_0}^{t} ds \int_{-a}^{0} d\theta \int_{t_0}^{t} dr\ A_{01}(r,\theta)\Phi^0(r+\theta,s)f(s) + \int_{t_0}^{t} f(s)ds$$

interchanging order of integration by Fubini's theorem

$$= \int_{t_0}^{t} ds \int_{s}^{t} dr\ A_{00}(r)\Phi^0(r,s)f(s)$$

$$+ \sum_{i=1}^{N} \int_{t_0}^{t} ds \int_{s-\theta_i}^{t} dr\ A_i(r)\Phi^0(r+\theta_i,s)f(s)$$

$$+ \int_{t_0}^{t} ds \int_{-a}^{0} d\theta \int_{s-\theta}^{t} A_{01}(r,\theta)\Phi^0(r+\theta,s)f(s) + \int_{t_0}^{t} f(s)ds$$

using fact that $\Phi^0(r,s) = 0 \quad r < s$

$$= \int_{t_0}^{t} ds \Big\{ I + \int_{s}^{t} dr\, A_{00}(r)\, \Phi^{0}(r,s) + \sum_{i=1}^{N} \int_{s-\theta_i}^{t} dr\, A_i(r)\, \Phi^{0}(r+\theta_i, s)$$

$$+ \int_{-a}^{0} d\theta \int_{s-\theta}^{t} dr\, A_{01}(r,\theta)\, \Phi^{0}(r+\theta, s)\Big\} f(s)$$

$$= \int_{t_0}^{t} ds\, \Phi^{0}(t,s)\, f(s) = y(t)$$

Hence $y(t)$ does indeed satisfies integral equation (2-21) and from uniqueness, it must be the solution of (2-21).

Now define

$$\bar{h}(t) = \sum_{i=1}^{N} \begin{cases} A_i(t)h(t-t_0+\theta_i) & t-a \leq t+\theta_i \leq t_0 \\ 0 & \text{otherwise} \end{cases}$$

$$+ \begin{cases} \int_{-a}^{t_0-t} A_{01}(t,\theta)h(t-t_0+\theta)d\theta & -a \leq t_0-t \leq 0 \\ 0 & \text{otherwise} \end{cases} \qquad (2\text{-}24)$$

Clearly $\bar{h} \in L^2(t_0,T;R^n)$.

Now consider the R.F.D.E.

$$\frac{dx}{dt} = A_{00}(t)x(t) + \sum_{i=1}^{N} A_i(t)x(t+\theta_i) + \int_{-a}^{0} A_{01}(t,\theta)x(t+\theta)d\theta + \bar{h}(t)$$

$$x(t_0+\theta) = 0 \qquad \theta \in [-a, 0] \qquad (2\text{-}25)$$

whose solution is denoted by $x_1(t)$

and the R.F.D.E.

$$\frac{dx}{dt} = A_{00}(t)x(t) + \sum_{i=1}^{N} A_i(t)x(t+\theta_i) + \int_{-a}^{0} A_{01}(t,\theta)x(t+\theta)d\theta$$

$$x(t_0) = 0, \quad x(t_0+\theta) = h(\theta) \qquad \theta \in [-a,0) \tag{2-26}$$

whose solution is denoted by $x_2(t)$.

Let $z(t) = x_1(t) - x_2(t)$. Then z satisfies the R.F.D.E.

$$\frac{dx}{dt} = A_{00}(t)x(t) + \sum_{i=1}^{N} A_i(t)x(t+\theta_i) + \int_{-a}^{0} A_{01}(t,\theta)x(t+\theta)d\theta + \bar{h}(t)$$

$$x(t_0) = 0 \qquad x(t_0+\theta) = -h(\theta) \qquad \theta \in [-a,0) \tag{2-27}$$

$z(t) = 0$ for $t \in [t_0,T]$ satisfies equation (2-27) and from uniqueness it must be the solution.

Hence $x_1(t) = x_2(t)$ for $t \in [t_0,T]$.

Hence the solution to (2-26) is

$$x_2(t) = \int_{t_0}^{t} \Phi^0(t,s)\bar{h}(s)ds \tag{2-28}$$

$$= \sum_{i=1}^{N} \int_{t_0}^{t} ds\, \Phi^0(t,s) \begin{cases} A_i(s)h(s-t_0+\theta_i) & s-a \le s+\theta_i \le t_0 \\ 0 & \text{otherwise} \end{cases}$$

$$+ \int_{t_0}^{t} ds\, \Phi^0(t,s) \begin{cases} \displaystyle\int_{-a}^{t_0-s} d\theta\, A_{01}(s,\theta)h(s-t_0+\theta) & t_0 \le s \le t_0+a \\ 0 & \text{otherwise} \end{cases}$$

Now $$\sum_{i=1}^{N} \int_{t_0}^{t} ds\ \Phi^0(t,s) \begin{cases} A_i(s)h(s-t_0+\theta_i) & s-a \le s+\theta_i \le t_0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \sum_{i=1}^{N} \int_{t_0}^{\min (t_0+\theta_i,t)} ds\ \Phi^0(t,s)A_i(s)h(s-t_0+\theta_i)$$

$$= \sum_{i=1}^{N} \int_{\theta_i}^{\min (0,t-t_0+\theta_i)} d\alpha\ \Phi^0(t,t_0+\alpha-\theta_i)A_i(t_0+\alpha-\theta_i)h(\alpha)$$

putting $\alpha = s - t_0 + \theta_i$

For case $t_0 < t < t_0 + a$

$$\int_{t_0}^{t} ds\ \Phi^0(t,s) \begin{cases} \int_{-a}^{t_0-s} d\theta\ A_{01}(s,\theta)h(s-t_0+\theta) & t_0 \le s \le t_0+a \\ 0 & \text{otherwise} \end{cases}$$

$$= \int_{t_0}^{t} ds \int_{-a}^{t_0-s} d\theta\ \Phi^0(t,s)A_{01}(s,\theta)h(s-t_0+\theta)$$

changing to $\alpha = s-t_0+\theta$, $\beta = \theta$ coordinates and interchanging order of integration by Fubini

$$= \int_{-a}^{0} d\alpha \int_{\max (-a,\alpha-t+t_0)}^{\alpha} d\beta\ \Phi^0(t,t_0+\alpha-\beta)A_{01}(t_0+\alpha-\beta,\beta)h(\alpha)$$

In the case $t \geq t_0 + a$, $-a \geq \alpha - t + t_0$ for all $\alpha \varepsilon [-a,0)$ and we get the same result as before. Hence the solution to (2-26) is

$$x_2(t) = \sum_{i=1}^{N} \int_{\theta_i}^{\min(0,t-t_0+\theta_i)} d\alpha\, \Phi^0(t,t_0+\alpha-\theta_i)A_i(t_0+\alpha-\theta_i)h(\alpha)$$

$$+ \int_{-a}^{0} d\alpha \int_{\max(-a,\alpha-t+t_0)}^{\alpha} d\beta\, \Phi^0(t,t_0+\alpha-\beta)A_{01}(t_0+\alpha-\beta,\beta)h(\alpha)$$

$$= \int_{-a}^{0} d\alpha\, \Phi^1(t,t_0,\alpha)h(\alpha) \tag{2-29}$$

Now

$$\begin{aligned} x(t;t_0) &= \Phi^0(t,t_0)h(0) + \int_{-a}^{0} \Phi^1(t,t_0,\alpha)h(\alpha)d\alpha + \int_{t_0}^{t} \Phi^0(t,s)f(s)ds \\ x(t_0+\theta;t_0) &= h(\theta) \qquad \theta \in [-a,0] \end{aligned} \tag{2-30}$$

satisfies R.F.D.E. (2-1) and from uniqueness, it must be the solution.

Q.E.D.

Corollary 1

The solution to the autonomous R.F.D.E. (2-10) can be written in the form

$$\begin{aligned} \phi(t;h,f) = {} & \Phi^0(t)h(0) + \sum_{i=1}^{N} \int_{\theta_i}^{\min(0,t+\theta_i)} d\alpha\, \Phi^0(t-\alpha+\theta_i)A_i h(\alpha) \\ & + \int_{-a}^{0} d\alpha \int_{\max(-a,\alpha-t)}^{\alpha} d\beta\, \Phi^0(t-\alpha+\beta)A_{01}(\beta)h(\alpha) \\ & + \int_0^t \Phi^0(t-s)f(s) \end{aligned} \tag{2-31}$$

or more compactly

$$\phi(t;h,f) = \Phi^0(t)h(0) + \int_{-a}^{0} \Phi^1(t,\alpha)h(\alpha)d\alpha + \int_0^t \Phi^0(t-s)f(s)ds \tag{2-32}$$

where $\Phi^0(t) \varepsilon \mathcal{L}(R^n)$ and satisfies the matrix R.F.D.E.

$$\begin{aligned} \frac{d\Phi^0(t)}{dt} &= A_{00}\Phi^0(t) + \sum_{i=1}^{N} A_i\Phi^0(t+\theta_i) + \int_{-a}^{0} A_{01}(\theta)\Phi^0(t+\theta)d\theta \\ \Phi^0(0) &= I, \quad \Phi^0(t) = 0 \quad t < 0 \end{aligned} \tag{2-33}$$

$\Phi^1(t,\alpha) \in \mathcal{L}(R^n)$ and

$$\Phi^1(t,\alpha) = \sum_{i=1}^{N} \begin{cases} \Phi^0(t-\alpha+\theta_i)A_i & \alpha-t<\theta_i<\alpha \\ 0 & \text{otherwise} \end{cases} + \int_{\max(-a,\alpha-t)}^{\alpha} \Phi^0(t-\alpha+\beta)A_{01}(\beta)\,d\beta \tag{2-34}$$

Proof

The autonomous R.F.D.E. on $[s,\infty)$, $s > 0$

$$\frac{dx}{dt} = A_{00}x(t) + \sum_{i=1}^{N} A_i(t)x(t+\theta_i) + \int_{-a}^{0} A_{01}(\theta)x(t+\theta)d\theta$$

$$x(s) = h(0) \quad x(s+\theta) = 0 \quad \theta \in [-a,0) \tag{2-35}$$

has unique solution $x(t;s) = \Phi^0(t,s)h(0)$.

But $x(t;s) = x(t-s;0)$, since R.F.D.E. autonomous.

Hence $\Phi^0(t,s) = \Phi^0(t-s,0)$.

Now define $\Phi^0(t-s) = \Phi^0(t,s)$.

From this point, proof of corollary proceeds as in proof of theorem.

We are now in a position to exhibit an exact and explicit closed form solution to a particular class of R.F.D.E. - in fact a differential-difference equation with one delay. To the best of the author's knowledge, this is the first time this has been done.

Corollary 2

The solution of the R.F.D.E.

$$\frac{dx}{dt} = A_{00}x(t) + A_1 x(t-a) + f(t)$$

$$x(\theta) = h(\theta) \qquad \theta \in [-a,0] \tag{2-36}$$

where A_{00}, A_1 ε $\mathcal{L}(R^n)$ and commute, is given by

$$\phi(t;h,f) = \Phi^0(t)\,h(0) + \int_{-a}^{\min(0,t-a)} \Phi^0(t-\alpha-a)A_1\,h(\alpha)d\alpha + \int_0^t \Phi^0(t-s)\,f(s)ds \tag{2-37}$$

where $\Phi^0(t) = \begin{cases} e^{A_{00}t} \displaystyle\sum_{j=0}^{p} \frac{(e^{-aA_{00}}A_1)^j (t-ja)^j}{j!} & t \in [pa,(p+1)a] \\ & p \in \mathbb{Z}^+ \\ 0 & t < 0 \end{cases}$

Proof

From corollary 1, it is sufficient to observe that $\Phi^0(t)$ satisfies the matrix R.F.D.E.

$$\frac{d\Phi^0(t)}{dt} = A_{00}\Phi^0(t) + A_1\Phi^0(t-a)$$

$$\Phi^0(0) = I, \quad \Phi^0(t) = 0 \qquad t < 0 \tag{2-38}$$

2.3 Continuity and differentiability in M^2; State evolution equation

Definition

(i) Given a map $x : [t_0-a,T] \to R^n$ such that for $t \in [t_0,T]$, $x(t) \in R^n$ is well defined and the map $x_t : [-a,0] \to R^n$ defined by $x_t(\theta) = x(t+\theta)$ is an element of $L^2(-a,0;R^n)$, define the map

$$\tilde{x} : [t_0,T] \to M^2(-a,0;R^n)$$

$$\text{by} \quad (\tilde{x}(t))(\theta) = x(t+\theta) \tag{2-38}$$

(ii) The map $\tilde{x}$ is said to be continuous at the point $t \in [t_0,T]$ if given $\varepsilon > 0$, $\exists\, \delta$ such that

$|t-s| < \delta$, $s \in [t_0,T]$

$$\Rightarrow \|\tilde{x}(t) - \tilde{x}(s)\|_{M^2} < \varepsilon \quad \text{i.e.} \quad \lim_{s \to t} \|\tilde{x}(t) - \tilde{x}(s)\| = 0 \tag{2-39}$$

The map is said to be continuous on $[t_0,T]$ if it is continuous at every $t \in [t_0,T]$.

(iii) The map $\tilde{x}$ is said to be differentiable at the point $t \in [t_0,T]$ if there exists an element denoted by

$\frac{d\tilde{x}(t)}{dt} \in M^2(-a,0;R^n)$ such that

$$\lim_{s \to t} \left|\left|\frac{\tilde{x}(t) - \tilde{x}(s)}{t - s} - \frac{d\tilde{x}(t)}{dt}\right|\right|_{M^2} = 0 \tag{2-40}$$

(iv) The M^2 state of the solution of the hereditary system (2-1) is the map

$$\begin{aligned} t &\mapsto \tilde{x}(t;t_0,h,f) \\ [t_0,T] &\to M^2(-a,0;R^n) \end{aligned} \tag{2-41}$$

defined by

$$\tilde{x}(t;t_0,h,f)(\theta) = \begin{cases} x(t+\theta;t_0,h,f) & t+\theta \geq t_0 \\ h(t-t_0+\theta) & t+\theta < t_0 \end{cases}$$

Remark

The concept of a state - that object which embodies all the necessary information to determine the future evolution of a system - is a very useful concept in systems theory. We have already seen that for hereditary systems governed by R.F.D.E. the state is not an element of R^n, but an element of some function space. The usual choice for this function space is $C(-a,0;R^n)$, but for reasons that will be elaborated upon in section 2.7, we will choose to work in $M^2(-a,0;R^n)$. We shall be able to treat a R.F.D.E. as a differential equation in the Hilbert space $M^2(-a,0;R^n)$. As we shall see, this approach has many technical and theoretical advantages.

Theorem 2C

(i) The map $t \mapsto \tilde{x}(t;t_0,h,f) : [t_0,T] \to M^2(-a,0;R^n)$ is continuous

(ii) For $h \in AC^2(-a,0;R^n)$, the subspace of $M^2(-a,0;R^n)$ of absolutely continuous maps $[-a,0] \to R^m$ with derivative in $L^2(-a,0;R^n)$, the map

$$t \mapsto \tilde{x}(t;t_0,h,f) : [t_0,T] \to M^2(-a,0;R^n)$$

is differentiable with derivative $\frac{d\tilde{x}(t)}{dt} \in M^2(-a,0;R^n)$ defined by

$$\frac{d\tilde{x}(t)}{dt}(\alpha) = \begin{cases} A_{00}(t)x(t) + \sum_{i=1}^{N} A_i(t)x(t+\theta_i) + \int_{-a}^{0} A_{01}(t,\theta)x(t+\theta)\,d\theta + f(t) & \alpha = 0 \\ \frac{dx(t+\alpha)}{d\alpha} & \alpha \in [-a,0) \end{cases}$$

where $x(s) = \begin{cases} \phi(s;t_0,h,f) & s \geq t_0 \\ h(s-t_0) & s < t_0 \end{cases}$

Proof

(i) Case a $\quad t > t_0 + a$

$$||\tilde{x}(t;t_0,h,f)-\tilde{x}(s;t_0,h,f)||^2_{M^2} = |x(t;t_0,h,f)-x(s;t_0,h,f)|^2$$

$$+ \int_{-a}^{0} |x(t+\theta;t_0,h,f)-x(s+\theta;t_0,h,f)|^2 d\theta$$

Now the solution $x(\cdot;t_0,h,f)$ will be absolutely continuous on compact interval $[t_0,T]$ and hence will be uniformly continuous on $[t_0,T]$.

Hence given $\varepsilon > 0$ there exists $\delta > 0$ such that for any $t',s' \in [t_0,T]$, $|t'-s'| < \delta$, $|x(t';t_0,h,f)-x(s';t_0,h,f)| \leq \frac{\varepsilon}{(1+a)^{\frac{1}{2}}}$. Hence $||\tilde{x}(t;t_0,h,f)-\tilde{x}(s;t_0,h,f)||^2_{M^2}$

$$\leq \frac{\varepsilon^2}{1+a} + \frac{a\varepsilon^2}{1+a} = \varepsilon^2.$$

Case (b) $\quad t = t_0 + a \qquad s > t$

As before, if $|s-t| < \delta$, we have

$$||\tilde{x}(t;t_0,h,f)-\tilde{x}(s;t_0,h,f)||^2_{M^2} \leq \varepsilon^2$$

Case (c) $\quad t = t_0 + a \qquad s < t$

$$s + \delta = t = t_0 + a \qquad \delta > 0$$

$$||\tilde{x}(t,t_0,h,f)-\tilde{x}(s,t_0,h,f)||^2_{M^2} = |x(t;t_0,h,f)-x(s;t_0,h,f)|^2$$

$$+ \int_{t_0-s}^{0} |x(t+\theta;t_0,h,f) -x(s+\theta;t_0,h,f)|^2 d\theta$$

$$+ \int_{-a}^{t_0-s} |x(t+\theta;t_0,h,f)-h(s-t_0+\theta)|^2 d\theta$$

From the uniform continuity of $x(\cdot;t_0,h,f)$ we have

$$\lim_{s \uparrow t} |x(t;t_0,h,f)-x(s;t_0,h,f)|^2 = 0$$

and $\displaystyle \lim_{s \uparrow t} \int_{t_0-s}^{0} |x(t+\theta,t_0,h,f)-x(s+\theta,t_0,h,f)|^2 d\theta = 0$

Now $\displaystyle \int_{-a}^{t_0-s} |x(t+\theta;t_0,h,f)-h(s-t_0+\theta)|^2 d\theta$

$$= \int_{-\delta}^{0} |x(t+\delta+\alpha;\ t_0,h,f) - h(\alpha)|^2 d\alpha$$

$$= \int_{-a}^{0} x(t+\delta+\alpha;\ t_0,h,f)-h(\alpha)|^2 \chi_{[-\delta,0]}(\alpha) d\alpha$$

(where χ is the characteristic function)

$\to 0$ as $\delta \to 0$ by the Lebesgue dominated convergence theorem.

Case (d) $\quad t_0 < t < t_0 + a$

Take $s < t$ (The proof for $s > t$ is similar) $t_0<s<t_0+a$

$$||\tilde{x}(t;t_0,h,f)-\tilde{x}(s;t_0,h,f)||^2_{M^2}$$

$$= |x(t;t_0,h,f)-x(s;t_0,h,f)|^2$$

$$+ \int_{t_0-s}^{0} x(t+\theta;t_0,h,f)-x(s+\theta;t_0,h,f)|^2 d\theta$$

$$+ \int_{t_0-t}^{t_0-s} |x(t+\theta;t_0,h,f)-h(s-t_0+\theta)|^2 d\theta$$

$$+ \int_{-a}^{t_0-t} |h(t-t_0+\theta)-h(s-t_0+\theta)|^2 d\theta$$

As before exploiting the uniform continuity of $x(\cdot;t_0,h,f)$ on $[t_0,T]$, we have that

$$\lim_{s \uparrow t} \Big\{ |x(t;t_0,h,f)-x(s;t_0,h,f)|^2$$

$$+ \int_{t_0-s}^{0} |x(t+\theta;t_0,h,f)-x(s+\theta;t_0,h,f)|^2 d\theta \Big\} = 0.$$

Now $\int_{t_0-t}^{t_0-s} |x(t+\theta;t_0,h,f)-h(s-t_0+\theta)|^2 d\theta$

$$= \int_{-\delta}^{0} |x(t_0+\delta+\alpha;t_0,h,f)-h(\alpha)|^2 d\alpha \qquad s + \delta = t$$

$$= \int_{-a}^{0} |x(t_0+\delta+\alpha;t_0,h,f)-h(\alpha)|^2 \chi_{[-\delta,0]}(\alpha)d\alpha$$

$\to 0$ as $\delta \to 0$ by the Lebesgue dominated convergence theorem

Now $\int_{-a}^{t_0-t} h(t-t_0+\theta)-h(s-t_0+\theta)|^2 d\theta$

$$= \int_{s-t_0-a}^{-\delta} |h(\alpha+\delta)-h(\alpha)|^2 d\theta$$

Now the set of continuous functions on $[-a,0]$ is dense in $L^2(-a,0;R^n)$ and so given any $\varepsilon > 0$, $\exists$ h_0 continuous on $[-a,0]$ and hence uniformly continuous and $\delta_0 > 0$ such that

$$||h-h_0||_{L^2} < \varepsilon \text{ and } |h_0(\alpha+\delta)-h_0(\alpha)| < \varepsilon \text{ for } \delta < \delta_0$$

and $\alpha \in [-a,0]$. Hence

$$\int_{-a}^{t_0-t} |h(t-t_0+\theta)-h(s-t_0+\theta)|^2 d\theta$$

$$= \int_{s-t_0-a}^{-\delta} |h(\alpha+\delta)-h(\alpha)|^2 d\alpha$$

$$\leq 3 \int_{s-t_0-a}^{-\delta} |h(\alpha+\delta)-h_0(\alpha+\delta)|^2 d\alpha$$

$$+ 3 \int_{s-t_0-a}^{\delta} |h_0(\alpha+\delta)-h_0(\alpha)|^2 d\alpha + 3 \int_{s-t_0-a}^{-\delta} h_0(\alpha)-h(\alpha)|^2 d\alpha$$

$$\leq (6+3a)\varepsilon^2$$

Hence $\lim\limits_{s \uparrow t} ||\tilde{x}(t;t_0,h,f)-\tilde{x}(s;t_0,h,f)||^2_{M^2} = 0$

Case (e) $t = t_0$ $s > t$

$$||\tilde{x}(t_0;t_0,h,f)-\tilde{x}(s;t_0,h,f)||^2_{M^2} = |h(0)-x(s;t_0,h,f)|^2$$

$$+ \int_{t_0-s}^{0} |h(\theta)-x(s+\theta;t_0,h,f)|^2 d\theta$$

$$+ \int_{-a}^{t_0-s} |h(\theta)-h(s-t_0+\theta)|^2 d\theta$$

and using same techniques as before, we show that

$$\lim_{s \downarrow t_0} \|\tilde{x}(t_0;t_0,h,f)-\tilde{x}(s;t_0,h,f)\|^2_{M^2} = 0$$

Hence the map $t \to \tilde{x}(t;t_0,h,f)$ is continuous on $[t_0,T]$

(ii)

$$\left\| \frac{\tilde{x}(t+\delta;t_0,h,f)-\tilde{x}(t;t_0,h,f)}{\delta} - \frac{d\tilde{x}(t)}{dt} \right\|^2_{M^2}$$

$$= \left| \frac{x(t+\delta;t_0,h,f)-x(t;t_0,h,f)}{\delta} - A_{00}x(t) - \sum_{i=1}^{N} A_i(t)x(t+\theta_i) - \int_{-a}^{0} A_{01}(t,\theta)x(t+\theta)d\theta - f(t) \right|$$

$$+ \int_{-a}^{0} \left| \frac{x(t+\delta+\theta)-x(t+\theta)}{\delta} - \frac{dx(t+\theta)}{d\theta} \right|^2 d\theta$$

Now the first term on the left hand side tends to 0 as $\delta \to 0$ since $x(t;t_0,h,f)$ satisfies R.F.D.E. (2-1)

Now $$I_\delta = \int_{-a}^{0} \left| \frac{x(t+\delta+\theta)-x(t+\theta)}{\delta} - \frac{dx(t+\theta)}{d\theta} \right|^2 d\theta$$

$$= \int_{-a}^{0} d\theta \left| \frac{1}{\delta} \int_{t}^{t+\delta} ds \left(\frac{dx(s+\theta)}{d\theta} - \frac{dx(t+\theta)}{d\theta} \right) \right|^2$$ $x(\cdot)$ is absolutely continuous

$$\leq \int_{-a}^{0} d\theta \, \frac{1}{\delta} \int_{t}^{t+\delta} ds \left| \frac{dx(s+\theta)}{d\theta} - \frac{dx(t+\theta)}{d\theta} \right|^2$$

$$= \frac{1}{\delta} \int_{t}^{t+\delta} ds \int_{-a}^{0} d\theta \left| \frac{dx(s+\theta)}{d\theta} - \frac{dx(t+\theta)}{d\theta} \right|^2$$

(interchanging order of integration by Fubini)

Now let $g(\theta) = \frac{dx(t+\theta)}{d\theta}$. $g \in L^2(-a,\delta_1;R^n)$ some $\delta_1 > 0$ $\delta_1 < a$. From the density of the continuous functions in L^2 we can find g_0 such that $||g-g_0||_{L^2(-a,\delta_1;R^n)} < \varepsilon$ and g_0 is absolutely continuous on $[-a,\delta_1]$. Also there exists $\delta_0 < \delta_1$ such that $|g_0(\theta+\delta)-g_0(\theta)| < \varepsilon$ for $\delta < \delta_0$, $\theta \in [-a,0]$.

Hence for $\delta < \delta_0$, $I_\delta \leq \frac{1}{\delta} \int_{t}^{t+\delta} ds \, (6+4a)\varepsilon^2 = (6+4a)\varepsilon^2$.

Hence $I_\delta \to 0$ as $\delta \to 0$.

Hence we have the differentiability of the map

$$t \mapsto \tilde{x}(t;t_0,h,f)$$

for $h \in AC^2(-a,0;R^n)$.

Definition

(i) Define the differential operator $\mathcal{A}(t) : M^2 \to M^2$ with domain $AC^2(-0,a;R^n)$ dense in M^2 by

$$\left[\mathcal{A}(t)h\right](\alpha) = \begin{cases} A_{00}(t)h(0) + \sum_{i=1}^{N} A_i(t)h(\theta_i) + \int_{-a}^{0} A_{01}(t,\theta)h(\theta)d\theta & \alpha = 0 \\ \frac{dh(\alpha)}{d\alpha} & \alpha \in [-a,0) \end{cases} \tag{2-42}$$

where $h \in AC^2(-a,0;R^n)$.

(ii) For $f(t) \in L^2(t_0,T;R^n)$ $f(t)$ well defined for $t \in [t_0,T]$, define the element $\tilde{f}(t) \in M^2(-a,0;R^n)$ by

$$\tilde{f}(t)(\alpha) = \begin{cases} f(t) & \alpha = 0 \\ 0 & \alpha \in [-a,0) \end{cases} \tag{2-43}$$

We can now state a corollary to theorem 2C.

Corollary

For $h \in AC^2(-a,0;R^n)$, $\tilde{x}(\cdot;t_0,h,f)$ satisfies the differential equation in $M^2(-a,0;R^n)$

$$\begin{aligned}\frac{d\tilde{x}(t)}{dt} &= \mathcal{A}(t)\tilde{x}(t) + \tilde{f}(t) \\ \tilde{x}(t_0) &= h\end{aligned} \qquad (2\text{-}44)$$

Remark

(1) Using a lifting process (see Delfour and Mitter [21]) we can for equation (2-44) extend the space of initial data from $AC^2(-a,0;R^n)$ to $M^2(-a,0;R^n)$ since $AC^2(-a,0;R^n)$ is a dense subspace of $M^2(-a,0;R^n)$.

(2) Equation (2-44) is called the M^2 state evolution equation and can be written in integral form, Delfour and Mitter [21], as

$$\tilde{x}(t) = \Phi(t,t_0)h + \int_{t_0}^{t} \Phi(t,s)\tilde{f}(s)ds \qquad (2\text{-}45)$$

where the integral is taken in the sense of Bochner.

Lemma 2.1

Suppose that x,y are absolutely continuous maps $[t_0-a,T] \to R^n$ with square integrable derivative. Then for any $t \in [t_0,T]$

$$\text{(i)} \quad \frac{d}{dt}\int_{-a}^{0} x(t+\theta)d\theta = \int_{-a}^{0} \frac{dx(t+\theta)}{d\theta}d\theta \tag{2-46}$$

$$\text{(ii)} \quad \frac{d}{dt}(\tilde{x}(t),\tilde{y}(t))_{M^2} = (\frac{d\tilde{x}(t)}{dt},\tilde{y}(t))_{M^2} + (\tilde{x}(t),\frac{d\tilde{y}(t)}{dt})_{M^2} \tag{2-47}$$

Proof

$$I_\delta = \left|\int_{-a}^{0} d\theta\frac{x(t+\delta+\theta)-x(t+\theta)}{\delta} - \int_{-a}^{0} d\theta\frac{dx(t+\theta)}{d\theta}\right|^2$$

$$= \left|\int_{-a}^{0} d\theta \ \frac{1}{\delta}\int_{t}^{t+\delta} ds\{\frac{dx(s+\theta)}{d\theta} - \frac{dx(t+\theta)}{d\theta}\}\right|^2$$

$$\leq \frac{1}{\delta}\int_{-a}^{0} d\theta \int_{t}^{t+\delta} ds\left|\frac{dx(s+\theta)}{d\theta} - \frac{dx(t+\theta)}{d\theta}\right|^2$$

$$= \frac{1}{\delta}\int_{t}^{t+\delta} ds \int_{-a}^{0} \left|\frac{dx(s+\theta)}{d\theta} - \frac{dx(t+\theta)}{d\theta}\right|^2$$

Now for some $\delta_1 > 0$, $\delta_1 < a$, $g(\theta) = \frac{dx(t+\theta)}{d\theta}$, $g \in L^2(-a,\delta_1;R^n)$.

As before given $\varepsilon > 0$, there exists g_0 such that

$$||g-g_0||_{L^2(-a,\delta_1;R^n)} < \varepsilon$$

and $\delta_0 < \delta_1$ such that $|g_0(\theta+\delta)-g_0(\theta)| < \varepsilon$ for all $\delta < \delta_0$, $\theta \in [-a,0]$. Hence

$$I_\delta \leq \frac{1}{\delta}\int_t^{t+\delta} (6+4a)\varepsilon^2 = (6+4a)\varepsilon^2 .$$ Hence

$I_\delta \to 0$ as $\delta \to 0$ and we have (i).

(ii)

$$\begin{aligned}\frac{d}{dt}\left(\tilde{x}(t), \tilde{y}(t)\right)_{M^2} &= \frac{d}{dt}\left\{ (x(t), y(t)) + \int_{-a}^{0} (x(t+\theta), y(t+\theta))\, d\theta \right\} \\ &= \left(\frac{dx(t)}{dt}, y(t)\right) + \left(x(t), \frac{dy(t)}{dt}\right) + \int_{-a}^{0}\left\{\left(\frac{dx(t+\theta)}{d\theta}, y(t+\theta)\right) + \left(x(t+\theta), \frac{dy(t+\theta)}{d\theta}\right)\right\} d\theta \\ &= \left(\frac{d\tilde{x}(t)}{dt}, \tilde{y}(t)\right)_{M^2} + \left(\tilde{x}(t), \frac{d\tilde{y}(t)}{dt}\right)_{M^2}\end{aligned}$$

2.4 Semigroup of operators

Definition

Let $t_0 < s < t < r < T$ and let $\tilde{x}(t;s,h)$ be the M^2 solution of the R.F.D.E. (2-1) with $f = 0$ and initial instant s. Define the transition operator $\Phi(t,s) : M^2 \to M^2$ by

$$\Phi(t,s)h = \tilde{x}(t;s,h)$$

Theorem 2D

$\Phi(t,s)$ is a two parameter semigroup of operators on

M^2 satisfying the following properties:

(i) for fixed t,s $t \geq s$, $\Phi(t,s)$ is strongly continuous linear M^2 operator

(ii) $\Phi(t,t) = I$, the identity operator in $\mathcal{L}(M^2)$

(iii) $\Phi(r,s) = \Phi(r,t)\Phi(t,s)$

(iv) the map $t \mapsto \Phi(t,s)h : [s,T] \to M^2$ (2-48)

is continuous for all $h \in M^2$

(v) for fixed t, the differential operator of $\{\Phi(r,t) ; r \in [t,T]\}$ defined by

$$\mathcal{A}_0(t)h = \lim_{r \downarrow t} \frac{1}{r-t}[\Phi(r,t)-\Phi(t,t)]h \qquad (2\text{-}49)$$

(when the limit exists) is also the differential generator of $\{\Phi(r,t); r \in [t,T]\}$; it has dense domain $AC^2(-a,0;R^n)$ and for $h \in AC^2(-a,0;R^n)$

$$[\mathcal{A}_0(t)h](\alpha) = \begin{cases} A_{00}(t)h(0) + \sum_{i=1}^{N} A_i(t)h(\theta_i) + \int_{-a}^{0} A_{01}(t,\theta)h(\theta)d\theta & \alpha = 0 \\ \dfrac{dh(\alpha)}{d\alpha} & \alpha \in [-a,0) \end{cases}$$

$$\text{(vi)} \quad \lim_{\delta \downarrow 0} \left[\frac{\Phi(t+\delta,s)-\Phi(t,s)}{\delta}\right] h = \mathcal{A}_0(t)\Phi(t,s)h \qquad (2\text{-}51)$$

$$\text{(vii)}\quad \lim_{\delta \downarrow 0}\left[\frac{\Phi(r,t+\delta)-\Phi(r,t)}{\delta}\right] h = -\,\Phi(r,t)\mathcal{A}_0(t)h \qquad (2\text{-}52)$$

Proof (i) $\Phi(t,s)$ is clearly linear.

Let h_1, $h_2 \in M^2(-a,0;R^n)$.

$$\Phi(t,s)h_1 - \Phi(t,s)h_2 = \Phi(t,s)(h_1-h_2) = \tilde{x}(t;s,h_1-h_2)$$

Now let $\pi_{t,s}$ denote the restriction of a solution of (2-1) with $f = 0$ and initial instant s. The restriction of the solution will be a continuous function on $[s,t]$ and Delfour and Mitter [18] have showed that

$$||\pi_{t,s}(x(\cdot;s,h_1-h_2)||_C \le 2d_1(t-s)||h_1-h_2||_{M^2}$$

where $d_1(t-s)$ is a constant for fixed t,s

$$||\Phi(t,s)h_1 - \Phi(t,s)h_2||^2_{M^2} = ||\tilde{x}(t;s,h_1-h_2)||^2_{M^2}$$

$$= |x(t;s,h_1-h_2)|^2$$

$$+ \int_{-a}^{0}\left\{\begin{matrix} |x(t+\theta;s,h_1-h_2)|^2 & t+\theta \ge s \\ |h_1(t-s+\theta)-h_2(t-s+\theta)|^2 & t+\theta < s \end{matrix}\right\} d\theta$$

$$\le (1+a)||\pi_{t,s}(x(\cdot;s,h_1-h_2)||^2_C + ||h_1-h_2||^2_{M^2}$$

$$= [4(1+a)d_1^2(t-s)+1]||h_1-h_2||^2_{M^2}$$

Hence result

(ii) $\Phi(t,t)h = \tilde{x}(t;t,h) = h$ for all $h \in M^2$

Hence result

(iii) $$\Phi(r,t)\Phi(t,s)h = \Phi(r,t)\tilde{x}(t;s,h) = \tilde{x}(r;t,\tilde{x}(t;s,h))$$

$$= \tilde{x}(r;s,h) \quad \text{from uniqueness}$$

$$= \Phi(r,s)h \quad \text{for all} \quad h \in M^2$$

Hence result

(iv) Follows directly from (i) of theorem 2C

(v) For fixed t, $\Phi(r,t)$ is a C_0 (strongly continuous) semigroup of operators from (iv), i.e. $\lim_{r \downarrow t} \Phi(r,t)h = h$, $\Phi(t,t) = I$, identity operator in $\mathcal{L}(M^2)$ (for definition, see Hille and Phillips [38] pp. 321) and thus we can define the infinitesimal operator

$$\mathcal{A}_0(t)h = \lim_{r \downarrow t} \frac{1}{r-t}[\Phi(r,t)-\Phi(t,t)]h$$

The infinitesimal generator will be the smallest closed extension of $\mathcal{A}_0(t)$. But since $\Phi(r,t)$ is a C_0 semigroup, the infinitesimal operator is closed and thus the infinitesimal generator is given by (2-49). (See Hille and Phillips [38] chapters 10,11). We now want to compute the infinitesimal operator.

For $\alpha \varepsilon [-a,0)$, $h \varepsilon AC^2(-a,0;R^n)$

$$\left\{\left[\frac{\Phi(r,t)-\Phi(t,t)}{r-t}\right] h\right\}(\alpha) = \frac{1}{r-t}[h(r-t+\alpha)-h(\alpha)]$$

for r such that $r-t+\alpha < 0$

$$\lim_{r \downarrow t} \left\{\left[\frac{\Phi(r,t)-\Phi(t,t)}{r-t}\right] h\right\}(\alpha) = \{[\mathcal{A}_0(t)]h\}(\alpha) = \frac{dh}{d\alpha}$$

$$\left\{\left[\frac{\Phi(r,t)-\Phi(t,t)}{r-t}\right] h\right\}(0) = \frac{1}{r-t}\{x(r;t,h)-h(0)\}$$

$$= \frac{1}{r-t}\Big\{\int_t^r \Big[A_{00}(u)x(u;t,h) \quad + \sum_{i=1}^{N} A_i(u)h(u-t+\theta_i)\Big]du$$

$$+ \int_t^r du \int_{-a}^{0} d\theta \; A_{01}(u,\theta) \begin{cases} x(u+\theta;t,h) & u+\theta > t \\ h(u-t+\theta) & u+\theta < t \end{cases} \Big\}$$

$$\rightarrow A_{00}(t)h(0) + \sum_{i=1}^{N} A_i(t)h(\theta_i) + \int_{-a}^{0} d\theta \; A_{01}(t,\theta)h(\theta)$$

assuming that $A_{00}(t)$, $A_i(t)$, $A_{01}(t,\cdot)$ are piecewise continuous and continuous from the right.

Hence

$$[\mathcal{A}_0(t)h](\alpha) = \begin{cases} A_{00}(t)h(0) + \sum_{i=1}^{N_1} A_i(t)h(\theta_i) + \int_{-a}^{0} A_{01}(t,\theta)h(\theta)d\theta & \alpha = 0 \\ \dfrac{dh(\alpha)}{d\alpha} & \alpha \in [-a, 0) \end{cases}$$

(vi) $$\lim_{\delta \downarrow 0} \left[\frac{\Phi(t+\delta,s) - \Phi(t,s)}{\delta}\right] h = \lim_{\delta \downarrow 0} \left[\frac{\Phi(t+\delta,t)\Phi(t,s) - \Phi(t,t)\Phi(t,s)}{\delta}\right] h$$

$$= \lim_{\delta \downarrow 0} \left[\frac{\Phi(t+\delta,t) - \Phi(t,t)}{\delta}\right] \Phi(t,s)h$$

$$= \mathcal{A}_0(t)\,\Phi(t,s)h$$

(vii) $$\lim_{\delta \downarrow 0} \left[\frac{\Phi(r,t+\delta) - \Phi(r,t)}{\delta}\right] h = \lim_{\delta \downarrow 0} \left[\frac{\Phi(r,t+\delta) - \Phi(r,t+\delta)\Phi(t+\delta,t)}{\delta}\right] h$$

$$= -\lim_{\delta \downarrow 0} \Phi(r,t+\delta)\left[\frac{\Phi(t+\delta,t) - \Phi(t,t)}{\delta}\right] h$$

$$= -\,\Phi(r,t)\,\mathcal{A}_0(t)h$$

Remarks

1. Note that $\mathcal{A}_0(t) = \mathcal{A}(t)$ where $\mathcal{A}(t)$ defined in (2-42) since $(\mathcal{A}_0(t) - \mathcal{A}(t))h = 0$ for all $h \in \mathcal{D}(\mathcal{A}_0(t)) = \mathcal{D}(\mathcal{A}(t)) = AC^2(-a,0;R^n)$.

2. (vii) states that for fixed r, the right hand derivative of $\Phi(r,t)h$ with respect to t is $-\Phi(r,t)\mathcal{A}_0(t)h$. Since $\Phi(r,t-\delta)h$ is not defined for $\delta > 0$, the left hand derivative will be meaningless. We will make more use of (vii) in chapter 3 section 5.

Let $\tilde{x}(t;h)$ be the M^2 solution of the autonomous R.F.D.E. (2-10) with $f = 0$. Defining the transition operator $\Phi(t) : M^2 \rightarrow M^2$ by

$$\Phi(t)h = \tilde{x}(t;h)$$

we have as a corollary to theorem 2D

Corollary

$\Phi(t)$ is a semigroup of operators on M^2 satisfying the following properties

(i) for fixed t, $\Phi(t)$ is a strongly continuous linear M^2 operator

(ii) $\Phi(0) = I$, the identity operator in $\mathcal{L}(M^2)$

(iii) $\Phi(t_1+t_2) = \Phi(t_1)\Phi(t_2)$

(iv) The map $t \rightarrow \Phi(t)h : [0,\infty) \rightarrow M^2$ is continuous for all $h \in M^2$ i.e. $\Phi(t)$ is a C_0 (strongly continuous) semigroup of operators on M^2

(v) The differential generator of $\{\Phi(t), t \geq 0\}$ is defined by

$$\mathcal{A} h = \lim_{t \downarrow 0} \frac{1}{t} \{\Phi(t)-\Phi(o)\}h \qquad (2\text{-}53)$$

when the limit exists. It has dense domain $AC^2(-a,0;R^n)$ in M^2 and for $h \in \mathcal{D}(\mathcal{A})$

$$[\mathcal{A}h](\alpha) = \begin{cases} A_{00}h(0) + \sum_{i=1}^{N} A_i h(\theta_i) + \int_{-a}^{0} A_{01}(\theta)h(\theta)d\theta & \alpha = 0 \\ \frac{dh}{d\alpha} & \alpha \in [-a,0) \end{cases} \qquad (2\text{-}54)$$

(vi) $$\lim_{\delta \to 0} [\frac{\Phi(t+\delta)-\Phi(t)}{\delta}]h = \mathcal{A}h \qquad (2\text{-}55)$$

2.5 Hereditary adjoint equation; Hereditary product

In the theory of linear ordinary differential equations

$$\frac{dx}{dt} = A(t)x(t) \qquad t \in [t_0,T]$$
$$x(0) = x_0 \qquad (2\text{-}56)$$

where $A \in L^1(t_0,T;\mathcal{L}(R^n)$ the adjoint differential equation

$$\frac{dp}{dt} = - A^*(t)p(t) \qquad t \; \varepsilon \; [t_0, T]$$

$$p(T) = p_T \tag{2-57}$$

plays a very useful role and even more so in the theory of optimal control of systems governed by ordinary differential equations, where, by means of the maximal principle, the optimal control depends upon the solution of the adjoint differential equation. Another of its properties is that the inner product of $x(t)$ and $p(t)$ is a constant

$$\text{i.e. } (p(t), x(t)) = \text{constant} \qquad t \; \varepsilon \; [t_0, T] \tag{2-58}$$

In the study of the optimal control of systems governed by R.F.D.E. the analogues of the adjoint differential equation and the R^n inner product play a very significant role.

Definition

Corresponding to R.F.D.E., (2-1), we define the hereditary adjoint equation for $t \; \varepsilon \; [t_0, T]$

$$\frac{dp}{dt} + A_{00}^*(t)p(t) + \sum_{i=1}^{N} A_i^*(t-\theta_i)p(t-\theta_i) + \int_{-a}^{0} A_{01}^*(t-\theta,\theta)p(t-\theta)d\theta + g(t) = 0$$

$$p(T) = p_T, \quad p(T+\beta) = 0 \qquad \beta \; \varepsilon \; (0,a] \tag{2-59}$$

where $g \in L^2(t_0,T;R^n)$

Remarks

1. Observe that $A_i^*(t-\theta_i)$ and $A_{01}^*(t-\theta,\theta)$ are not defined for $t - \theta_i > T$ and $t - \theta > T$ respectively. However, for these values $p(t-\theta_i)$ and $p(t-\theta)$ are zero and hence $A_i^*(t-\theta_i)p(t-\theta_i)$ and $A_{01}^*(t-\theta,\theta)p(t-\theta)$ will be well defined (equal zero) for arbitrary values of $A_i^*(t-\theta_i)$ and $A_{01}^*(t-\theta,\theta)$ respectively.

2. Note the restricted nature of the final data which is essentially a R^n point data. (see Delfour and Mitter [19]) In principle, we could use more general data

$$p(T+\beta) = k(\beta), \quad k \neq 0 \quad \text{on} \quad (0,a], \quad k \quad \text{final data}$$

but then $A_i^*(t-\theta_i)$ and $A_{01}^*(t-\theta,\theta)$ would have to be defined by $t - \theta_i > T$ and $t - \theta > T$ respectively. However for our purposes, that will not be necessary.

3. For the autonomous R.F.D.E. the problem discussed in the previous two remarks does not arise.

In keeping with the development in section 2.1, we can analogously to $M^2(-a,0;R^n)$ construct the space $\overline{M}^2(0,a;R^n)$ as follows: Take $\overline{\mathcal{L}}_0^2(0,a;R^n)$, the vector space of all Lebesgue measurable and square integrable maps

$$k : [0,a] \to R^n$$

with $k(0)$ well defined.

We impose the seminorm

$$||k|| = \{|k(0)|^2 + \int_0^a |k(\beta)|^2 d\beta\}^{\frac{1}{2}} \qquad (2\text{-}60)$$

and define the linear subspace $\bar{\mathcal{S}}$ of $\bar{\mathcal{L}}_0^2(0,a;R^n)$ by

$$\bar{\mathcal{S}} = \{k; ||k|| = 0\}$$

$\bar{M}^2(0,a;R^n)$ is defined to be the quotient space of $\bar{\mathcal{L}}_0^2(0,a;R^n)$ by $\bar{\mathcal{S}}$.

$\bar{M}^2(0,a;R^n)$ with the norm

$$||k||_{\bar{M}^2} = \{|k(0)|^2 + \int_0^a |k(\beta)|^2 d\beta\}^{\frac{1}{2}} \qquad (2\text{-}61)$$

is a Hilbert isometrically isomorphic to $R^n \times L^2(0,a;R^n)$.

Now by reversing time and starting out at T (the initial instant), we can regard the hereditary adjoint equation (2-59) as a R.F.D.E. We can then evoke theorem 2A to establish the uniqueness and continuity with respect to the final data p_T of a solution to (2-59). Denote this solution by $p(\cdot;T,p_T,g)$. As in section 3, we can define the $\bar{M}^2$ state of the solution of the hereditary adjoint equation (2-63) to be the map

$$t \to \tilde{p}(t;T,p_T,g) : [t_0,T] \to \overline{M}^2(0,a;R^n)$$

where
$$\tilde{p}(t;T,p_T,g)(\beta) = \begin{cases} p(t+\beta;T,p_T,g) & t+\beta \leq T \\ 0 & t+\beta > T \end{cases} \qquad (2\text{-}62)$$

Definition

The hereditary product corresponding to R.F.D.E. (2-1) and its hereditary adjoint equation (2-59) is a mapping

$$\mathcal{H}_T : [t_0,T] \times \overline{M}^2(0,a;R^n) \times M^2(-a,0;R^n) \to R$$

$$\mathcal{H}_T(t,\tilde{p}(t),\tilde{x}(t)) = (p(t),x(t)) + \sum_{i=1}^{N} \int_t^{t-\theta_i} ds(p(s),A_i(s)x(s+\theta_i))$$
$$+ \int_{-a}^{0} d\theta \int_t^{t-\theta} ds(p(s),A_{01}(s,\theta)x(s+\theta)) \qquad (2\text{-}63)$$

Remarks

1. The hereditary product introduced by De Bruijn [16] and subsequently exploited by Bellman and Cooke, [5], Halanay [34], Hale [26], Delfour and Mitter [19], [21].

2. As in the definition of the hereditary adjoint equation $A_i(s)$ and $A_{01}(s,\theta)$ will not be defined for

$s > T$. However $p(s)$ is zero for $s > T$ and as before the hereditary product will be well defined.

3. The hereditary product will be used in chapter 3 section 2 to characterize the optimal control and in chapter 5 section 1 to project from M^2 onto an eigenspace of M^2.

Theorem 2E Delfour and Mitter [19]

Let $\tilde{x}(\cdot),\tilde{p}(\cdot)$ be the M^2 state of R.F.D.E. (2-1) and the $\overline{M}^2$ state of the hereditary adjoint equation (2-59) respectively. Then

$$\mathcal{H}_T(t,\tilde{p}(t),\tilde{x}(t)) - \mathcal{H}_T(s,\tilde{p}(s),\tilde{x}(s)) = \int_s^t dr(p(r),\dot{x}(r))$$

$$- \int_r^t dr(p(r),A_{00}(r)x(r)) - \sum_{i=1}^{N} \int_s^t dr(p(r),A_i(r)x(r+\theta_i))$$

$$- \int_s^t dr \int_{-a}^{0} d\theta(p(r),A_{01}(r,\theta)x(r+\theta)) + \int_s^t dr(\dot{p}(r),x(r))$$

$$+ \int_s^t dr(A_{00}^*(r)p(r),x(r)) + \sum_{i=1}^{N} \int_s^t dr(A_i^*(r-\theta_i)p(r-\theta_i),x(r)$$

$$+ \int_s^t dr \int_{-a}^{0} d\theta(A_{01}^*(r-\theta,\theta)p(r-\theta),x(r)) \tag{2-64}$$

Proof

$$I = \int_s^t dr(p(r),\dot{x}(r)) - \int_r^t dr(p(r),A_{00}(r)x(r))$$

$$- \sum_{i=1}^{N} \int_s^t dr(p(r),A_i(r)x(r+\theta_i)) - \int_s^t dr \int_{-a}^{0} d\theta(p(r),A_{01}(r,\theta)x(r+\theta)$$

$$+ \int_s^t dr(\dot{p}(r),x(r)) + \int_s^t dr(A_{00}^*(r),p(r),x(r))$$

$$+ \sum_{i=1}^{N} \int_s^t dr(A_i^*(r-\theta_i)p(r-\theta_i),x(r))$$

$$+ \int_s^t dr \int_{-a}^{0} d\theta(A_{01}^*(r-\theta,\theta)p(r-\theta),x(r))$$

Now $\int_s^t dr(p(r),\dot{x}(s)) = (p(r),x(r))\Big|_s^t - \int_s^t dr(\dot{p}(r),x(r))$

$$= (p(t),x(t)) - (p(s),x(s))$$

$$- \int_s^t dr(\dot{p}(r),x(r))$$

(by integrating by parts which is permissible since $p(\cdot)$ and $x(\cdot)$ are absolutely continuous maps $[t_0,T] \to R^n$)

$$\sum_{i=1}^{N} \int_{s}^{t} dr(A_i^*(s-\theta_i)p(s-\theta_i),x(r)) - \sum_{i=1}^{N} \int_{s}^{t} dr(p(r),A_i(r)x(r+\theta_i))$$

$$= \sum_{i=1}^{N} \int_{s-\theta_i}^{t-\theta_i} dr(p(r),A_i(r)x(r+\theta_i))$$

$$- \sum_{i=1}^{N} \int_{s}^{t} dr(p(r),A_i(r)x(r+\theta_i))$$

(changing variables in the first expression)

$$= \sum_{i=1}^{N} \int_{t}^{t-\theta_i} dr(p(r),A_i(r)x(r+\theta_i))$$

$$- \sum_{i=1}^{N} \int_{s}^{s-\theta_i} dr(p(r),A_i(r)x(r+\theta_i))$$

$$\int_{s}^{t} dr \int_{-a}^{0} d\theta(A_{01}^*(r-\theta,\theta)p(s-\theta),x(r))$$

$$- \int_{r}^{t} dr \int_{-a}^{0} d\theta(p(r),A_{01}(r,\theta)x(r+\theta))$$

$$= \int_{-a}^{0} d\theta \left\{ \int_{s-\theta}^{t-\theta} dr \ (p(r),A_{01}(r,\theta)x(r+\theta) \right.$$

$$\left. - \int_{s}^{t} dr \ (p(r),A_{01}(r,\theta)x(r+\theta)) \right\}$$

(changing variables in first expression and interchanging the order of integration by Fubini)

$$= \int_{-a}^{0} d\theta \int_{t}^{t-\theta} dr(p(r),A_{01}(r,\theta)x(r+\theta))$$

$$- \int_{-a}^{0} d\theta \int_{s}^{s-\theta} dr(p(r),A_{01}(r,\theta)x(r+\theta))$$

Hence $$I = (p(t),x(t)) + \sum_{i=1}^{N} \int_{t}^{t-\theta_i} dr(p(r),A_i(r)x(r+\theta_i))$$

$$+ \int_{-a}^{0} d\theta \int_{t}^{t-\theta} dr(p(r),A_{01}(r,\theta)x(r+\theta))$$

$$- p(s),x(s)) - \sum_{i=1}^{N} \int_{s}^{s-\theta_i} dr(p(r),A_i(r)x(r+\theta_i))$$

$$+ \int_{-a}^{0} d\theta \int_{s}^{s-\theta} dr(p(r),A_{01}(r,\theta)x(r+\theta))$$

$$= \mathcal{H}_T(t,\tilde{p}(t),\tilde{x}(t)) - \mathcal{H}_T(s,\tilde{p}(s),\tilde{x}(s))$$

Hence result.

Corollary

If $f = 0$ in (2-1) and $g = 0$ in (2-59) then

$$\mathcal{H}_T(t,\tilde{p}(t),\tilde{x}(t)) = \text{constant} \quad t \in [t_0,T] \tag{2-65}$$

Proof From previous theorem and (2-64), for any $t,s \in [t_0,T]$ we have

$$\mathcal{H}_T(t,\tilde{p}(t),\tilde{x}(t)) - \mathcal{H}_T(s,\tilde{p}(s),\tilde{x}(s)) = 0$$

Hence result.

Remark

Equation (2-65) is the analogue of equation (2-58) for R.F.D.E.

2.6 Linear bounded operators and unbounded differential operators in M^2

Definition

Let $\Lambda : M^2 \to M^2$ be a linear operator on M^2

(i) Λ is said to be bounded if

$$||\Lambda h||_{M^2} \leq c||h||_{M^2} \quad \text{for all} \quad h \in M^2 \text{ and some } c > 0 \tag{2-66}$$

(ii) For Λ bounded we define $||\Lambda|| = \sup \dfrac{||\Lambda h||_{M^2}}{||h||_{M^2}}$

(iii) Λ is said to be symmetric if

$$(h,\Lambda k)_{M^2} = (k,\Lambda h)_{M^2} \quad \text{for all} \quad h,k \in M^2 \tag{2-67}$$

(iv) Λ is said to be positive if

$$(h,\Lambda h)_{M^2} > 0 \quad \text{for all} \quad h \in M^2 \tag{2-68}$$

Let Λ be a bounded linear operator on M^2. Exploiting the isometric isomorphism between $M^2(-a,0;R^n)$ and $R^n \times L^2(-a,0;R^n)$

$$\text{i.e.} \quad M^2(-a,0;R^n) \approx R^n \times L^2(-a,0;R^n) \tag{2-69}$$

we can decompose Λ into a matrix of bounded transformations

$$\Lambda = \begin{pmatrix} \Lambda_{00} & \Lambda_{01} \\ \Lambda_{10} & \Lambda_{11} \end{pmatrix} \tag{2-70}$$

where

(i) $\Lambda_{00} \in \mathcal{L}(R^n)$ can can be represented as an $n \times n$ matrix

(ii) $\Lambda_{01} \in \mathcal{L}(L^2(-a,0;R^n),R^n)$ and from the Riez representation theorem, we can represent Λ_{01} as

$$\Lambda_{01}x = \int_{-a}^{0} \Lambda_{01}(\alpha)x(\alpha)d\alpha \tag{2-71}$$

where $x \in L^2(-a,0;R^n)$ and $\Lambda_{01}(\cdot) \in L^2(-a,0;\mathcal{L}(R^n))$

(iii) $\Lambda_{10} \in \mathcal{L}(R^n,L^2(-a,0;R^n))$ and can be represented in the form

$$(\Lambda_{10}x)(\theta) = \Lambda_{10}(\theta)x \tag{2-72}$$

where $x \in R^n$ and $\Lambda_{10}(\cdot) \in L^2(-a,0;\mathcal{L}(R^n))$

(iv) $\Lambda_{11} \in \mathcal{L}(L^2(-a,0;R^n))$

It would be pleasant to be able to give an integral representation for $\Lambda_{11} \in \mathcal{L}(L^2 -a,0;R^n)$ in terms of a kernel $\Lambda_{11}(\theta,\alpha) \in L^2(-a,0;-a,0;\mathcal{L}(R^n))$. However this is not possible unless Λ_{11} is a Hilbert-Schmidt operator (See Dunford [25] and Schatten [71] for more detailed discussion). However we can use the Schwartz kernel theorem (Schwartz [73]) to represent Λ_{11} in the form

$$(\Lambda_{11}x)(\theta) = \int_{-a}^{0} \Lambda_{11}(\theta,\alpha)x(\alpha)d\alpha \tag{2-73}$$

where $\Lambda_{11}(\theta,\alpha)$ is a distribution on $[-a,0] \times [-a,0]$ defined uniquely by Λ_{11}.

For any bounded linear operator $\Lambda : M^2 \to M^2$, we can define $\Lambda^0 : M^2 \to R^n$ by

$$\Lambda^0 h = (\Lambda h)(0) = \Lambda_{00}h(0) + \int_{-a}^{0} \Lambda_{01}(\alpha)h(\alpha)d\alpha \tag{2-74}$$

From (2-71), (2-72) for any $h,k \in M^2(-a,0;R^n)$ we can write

$$(h,\Lambda k)_{M^2} = (h(0),\Lambda_{00}k(0) + \int_{-a}^{0} (h(0),\Lambda_{01}(\alpha)k(\alpha))d\alpha$$

$$+ \int_{-a}^{0} (h(\theta),\Lambda_{10}(\theta)k(0))d\theta + (h^1,\Lambda_{11}k^1)_{M^2}$$

where exploiting the isometric isomorphism between $M^2(-a,0;R^n)$ and $R^n \times L^2(-a,0;R^n)$ we can write

$$h = (h(0),h^1) \tag{2-75}$$

where $h(0) \in R^n$ and $h^1 \in L^2(-a,0;R^n)$.

Corresponding to the autonomous R.F.D.E.

$$\frac{dx}{dt} = A_{00}x(t) + \sum_{i=1}^{N} A_i x(t+\theta_i) + \int_{-a}^{0} A_{01}(\theta)x(t+\theta)d\theta$$

$$x(\theta) = h(\theta), \quad \theta \in [-a,0], \quad h \in M^2 \tag{2-76}$$

we have the closed differential operator

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \to M^2$$

with dense domain $\mathcal{D}(\mathcal{A}) = AC^2(-a,0;R^n)$ defined by for $h \in \mathcal{D}(\mathcal{A})$

$$[\mathcal{A}h](\alpha) = \begin{cases} A_{00}h(0) + \sum_{i=1}^{N} A_i h(\theta_i) + \int_{-a}^{0} A_{01}(\theta)h(\theta)d\theta & \alpha=0 \\ \frac{dh}{d\alpha} & \alpha \in [-a,0) \end{cases} \tag{2-77}$$

and $\mathcal{A}$ will be the differential generator of the semi-group of operators $\{\Phi(t),\ t \geq 0\}$ corresponding to (2-76). Associated with R.F.D.E. (2-76), we have the hereditary adjoint equation

$$\frac{dp}{dt} + A_{00}^{*}p(t) + \sum_{i=1}^{N} A_i^{*} p(t-\theta_i) + \int_{-a}^{0} A_{01}^{*}(\theta)p(t-\theta)d\theta = 0$$

$$p(T+\beta) = k(\beta), \quad \beta \in [0,a], \quad k \in \overline{M}^2 \tag{2-78}$$

and the closed differential operator

$$\mathcal{A}_* : \mathcal{D}(\mathcal{A}_*) \to M^2$$

with dense domain $\mathcal{D}(\mathcal{A}_*) = AC^2(0,a;R^n)$ and defined by for $k \in \mathcal{D}(\mathcal{A}_*)$

$$[\mathcal{A}_* k](\alpha) = \begin{cases} A_{00}^* k(0) + \sum_{i=1}^{N} A_i^* k(-\theta_i) + \int_{-a}^{0} A_{01}^*(\theta) k(-\theta) d\theta & \alpha = 0 \\ -\frac{dk}{d\alpha} & \alpha \in (0,a] \end{cases} \tag{2-79}$$

Equation (2-76) can be written as a differential equation in M^2

$$\frac{d\tilde{x}}{dt} = \mathcal{A}\tilde{x}(t)$$
$$\tilde{x}(0) = h \tag{2-80}$$

and equation (2-78) can be written as a differential equation in $\overline{M}^2$

$$\frac{d\tilde{p}}{dt} + \mathcal{A}_* \tilde{p}(t) = 0$$
$$\tilde{p}(T) = k \tag{2-81}$$

Let us now consider the complex extensions of $M^2(-a,0;R^n)$ and $\overline{M}^2(0,a;R^n)$, namely $M^2_{\mathbb{C}}$ and $\overline{M}^2_{\mathbb{C}}$ respectively

Definition

(i) The resolvent of $\mathcal{A}$ is the set in the complex plane

$\rho(\mathcal{A}) = \{\lambda \in \mathbb{C}$; range $(\lambda I-\mathcal{A})$ is dense M^2 and $(\lambda I-\mathcal{A})$
has a bounded inverse defined on its range$\}$

(ii) The spectrum of $\sigma(\mathcal{A})$ is the complement of $\rho(\mathcal{A})$ in $\mathbb{C}$.

(iii) The continuous spectrum of $\mathcal{A}$

$\sigma_C(\mathcal{A}) = \{\lambda \in \sigma(\mathcal{A})$; $(\lambda I-\mathcal{A})$ is (1,1), has dense range
and $(\lambda I-\mathcal{A})^{-1}$ exists on the range, but is
not bounded$\}$

(iv) The residual spectrum of

$\sigma_R(\mathcal{A}) = \{\lambda \in \sigma(\mathcal{A})$; $(\lambda I-\mathcal{A})$ is (1-1) but range $(\lambda I-\mathcal{A})$
is not dense in $M^2_{\mathbb{C}}\}$

(v) The point spectrum of

$\sigma_P(\mathcal{A}) = \{\lambda \in \sigma(\mathcal{A})$; $(\lambda I-\mathcal{A})$ is not $(1,1)\}$

(vi) The points $\lambda \in \sigma_P(\mathcal{A})$ are called the eigenvalues of $\mathcal{A}$ and any $h \neq 0$, $h \in M^2$ such that

$$(\lambda I-\mathcal{A})h = 0$$

is called the eigenfunction corresponding to the eigenvalue λ.

(vii) The generalized eigenspace $\mathcal{G}_\lambda(\mathcal{A})$ is the smallest subspace of M^2 containing all the elements that belong to the null space of $(\lambda I-\mathcal{A})^k$ $k = 1,2,\ldots$

(viii) The resolvent $\rho(\mathcal{A}_*)$, spectrum $\sigma(\mathcal{A}_*)$ etc. of $\mathcal{A}$ are defined similarly.

It is clear that the sets $\sigma_R(\mathcal{A})$, $\sigma_C(\mathcal{A})$, $\sigma_P(\mathcal{A})$ are pairwise disjoint and that

$$\sigma(\mathcal{A}) = \sigma_R(\mathcal{A}) \cup \sigma_C(\mathcal{A}) \cup \sigma_P(\mathcal{A}) \tag{2-82}$$

Theorem 2F Hale [36]

$$\sigma(\mathcal{A}) = \sigma_P(\mathcal{A}) = \{\lambda \in \mathbb{C}, \det \Lambda(\lambda) = 0\} \tag{2-83}$$

where $$\Lambda(\lambda) = \lambda I - A_{00} - \sum_{i=1}^{N} A_i e^{\lambda\theta_i} - \int_{-a}^{0} A_{01}(\theta) e^{\lambda\theta} d\theta \tag{2-84}$$

The roots of

$$\det \Delta(\lambda) = 0 \tag{2-85}$$

have real parts bounded above, and for $\lambda \in \sigma_P(\mathcal{A})$, $\mathcal{G}_\lambda(\mathcal{A})$ is finite dimensional.

There is an integer m such that

$$\mathcal{G}_\lambda(\mathcal{A}) = \text{null } (\lambda I - \mathcal{A})^m \tag{2-86}$$

$$\text{and} \quad M^2 = \text{null } (\mathcal{A}-\lambda I)^m \oplus \text{range } (\mathcal{A}-\lambda I)^m. \tag{2-87}$$

Proof To show that $\sigma(\mathcal{A}) = \sigma_P(\mathcal{A})$, we show that $\rho(\mathcal{A})$ consists of all $\lambda \in \mathbb{C}$ except those that satisfy (2-85). Now $\lambda \in \rho(\mathcal{A})$ iff the equation

$$(\mathcal{A}-\lambda I)h = k \tag{2-88}$$

has a unique solution $h \in \mathcal{D}(\mathcal{A})$ for every k in a dense subset of M^2 and the solution depends continuously on k.

From (2-88) we have

$$\frac{dh(\alpha)}{d\alpha} - \lambda h(\alpha) = k(\alpha) \qquad \alpha \in [-a,0) \tag{2-89}$$

and $$A_{00}h(0) + \sum_{i=1}^{N} A_i h(\theta_i) + \int_{-a}^{0} A_{01}(\theta)h(\theta) - \lambda h(0) = k(0) \qquad (2\text{-}90)$$

Solving (2-89) we have

$$h(\alpha) = e^{\lambda\alpha}h(0) + \int_{0}^{\alpha} e^{\lambda(\alpha-\xi)}k(\xi)d\xi \qquad (2\text{-}91)$$

Substituting (2-91) into (2-90) we have

$$-\lambda h(0) + A_{00}h(0) + \sum_{i=1}^{N} A_i e^{\lambda\theta_i} h(0) + \int_{-a}^{0} A_{01}(\theta)e^{\lambda\theta}h(0)d\theta$$

$$= k(0) - \sum_{i=1}^{N} \int_{0}^{\theta_i} e^{\lambda(\theta_i-\xi)} A_i k(\xi)d\xi - \int_{-a}^{0} d\theta \int_{0}^{\theta} d\xi \; e^{\lambda(\theta-\xi)}A_{01}(\theta)k(\xi)$$

Define $Z(\lambda) : M^2 \to R^n$ by

$$Z(\lambda)k = -k(0) + \sum_{i=1}^{N} \int_{0}^{\theta_1} e^{\lambda(\theta_i-\xi)} A_i k(\xi)d\xi$$

$$+ \int_{-a}^{0} d\theta \int_{0}^{\theta} d\xi \; e^{\lambda(\theta-\xi)}A_{01}(\theta)k(\xi) \qquad (2\text{-}92)$$

Hence $\Delta(\lambda)h(0) = Z(\lambda)k$ (2-93)

The map $Z(\lambda)$ covers R^n and hence (2-88) will have a solution for every k in M^2 iff $\det \Lambda(\lambda) \neq 0$. For $\det \Delta(\lambda) \neq 0$, the solution h will depend continuously on k.

Hence $\rho(\mathcal{A}) = \{\lambda;\ \det \Lambda(\lambda) \neq 0\}$

If $\det \Delta(\lambda) = 0$, then (2-91) and (2-92) imply that there exists a nonzero solution of (2-88)

$\{h(\alpha) = e^{\lambda\alpha}h(0)$ where $h(0) \in \text{null}\ \Delta(\lambda)\}$ for which $k = 0$, and hence $\lambda \in \sigma_P(\mathcal{A})$.

Hence $\sigma(\mathcal{A}) = \sigma_P(\mathcal{A})$.

$\det \Delta(\lambda)$ is an entire function of λ

and hence has roots of finite order. Hence the resolvent operator $(\mathcal{A}-\lambda I)^{-1}$ has a pole of order m at λ_0 if λ_0 is a zero of $\det \Lambda(\lambda)$ of order m. Since $\mathcal{A}$ is a closed operator, it follows from theorem 5.8-A Taylor [74] pp. 306 that

$\mathcal{G}_{\lambda_0}(\mathcal{A})$ is finite dimensional and

$M^2 = \text{null}\ (\mathcal{A}-\lambda I)^m \oplus \text{range}\ (\mathcal{A}-\lambda I)^m$

$\det \Delta(\lambda)$ is a polynomial in λ of degree n with leading coefficient one and the lower order terms have coefficients which depend on λ through $e^{\lambda\theta_1}$ and integrations of terms of the form $e^{\lambda\theta}$. Hence it follows that there is a $\gamma > 0$ such that no roots of $\det \Delta(\lambda)$ have real parts greater than γ

Q.E.D.

Corollary

$$\sigma(\mathcal{A}_*) = \sigma_p(\mathcal{A}_*) = \sigma_p(\mathcal{A}) \qquad (2\text{-}94)$$

For $\lambda \in \sigma_p(\mathcal{A}_*)$, $\mathcal{G}_\lambda(\mathcal{A}_*)$ is finite dimensional. (2-95)

Proof Same as for the proof of the theorem except that now we have to solve the equation

$$(\mathcal{A}_* - \lambda I)k = h \qquad (2\text{-}96)$$

for h in a dense subset of $\overline{M}^2$ and that the solution k depends continuously on h.

From (2-96) we have

$$-\frac{dk(\alpha)}{d\alpha} - \lambda k(\alpha) = h(\alpha) \qquad \alpha \in [0,a] \qquad (2\text{-}97)$$

$$A^*_{00}k(0) + \sum_{i=1}^{N} A^*_i k(-\theta_i) + \int_{-a}^{0} A^*_{01}(\theta)k(-\theta)d\theta - \lambda k(0) = k(0) \qquad (2\text{-}98)$$

Solving (2-97)

$$k(\alpha) = e^{-\lambda\alpha}k(0) - \int_0^{\alpha} e^{-\lambda(\alpha-\xi)}h(\xi)d\xi \tag{2-99}$$

Substituting (2-99) into (2-98) we have

$$A_{00}^{*}k(0) + \sum_{i=1}^{N} A_i^{*}e^{\lambda\theta_i}k(0) + \int_{-a}^{0} A_{01}^{*}(\theta)e^{\lambda\theta}k(0)d\theta - \lambda k(0)$$

$$= h(0) - \sum_{i=1}^{N} \int_0^{-\theta_i} A_i^{*}e^{\lambda(\theta_i+\xi)} h(\xi)d\xi$$

$$- \int_{-a}^{0} d\theta \int_0^{-\theta} d\xi \; e^{\lambda(\theta+\xi)}A_{01}^{*}(\theta)h(\xi)$$

Define $\overline{Z}(\lambda) : \overline{M}^2 \to R^n$ by

$$\overline{Z}(\lambda)h = - h(0) + \sum_{i=1}^{N} \int_0^{-\theta_i} e^{\lambda(\theta_i+\xi)} A_i^{*}h(\xi)$$

$$+ \int_{-a}^{0} d\theta \int_0^{-\theta} d\xi \; e^{\lambda(\theta+\xi)}A_{01}^{*}(\theta)h(\xi)$$

Hence $\Delta^{*}(\lambda)k(0) = \overline{Z}(\lambda)h.$ (2-100)

As before $\rho(\mathcal{A}_{*}) = \{\lambda;\ \det \Delta^{*}(\lambda) \neq 0\}$

But $\det \Delta^{*}(\lambda) = \det \Delta(\lambda)$.

Hence proceeding as in the theorem, we obtain the result.

Lemma 2.2

The eigenfunction h_λ corresponding to an eigenvalue λ of $\mathcal{A}$ with multiplicity m is of the form

$$h_\lambda(\alpha) = \sum_{j=0}^{m-1} v_{\lambda,j} \frac{\alpha^j}{j!} e^{\lambda\alpha} \qquad \alpha \in [-a,0] \tag{2-101}$$

Proof We first prove the lemma in the case $m = 1$.

h_λ satisfies $\mathcal{A}\ h = \lambda h$

$$\text{i.e.} \quad \frac{dh(\alpha)}{d\alpha} = \lambda h(\alpha) \qquad \alpha \in [-a,0) \tag{2-102}$$

Solving $\quad h(\alpha) = h(0)e^{\lambda\alpha}$

where $h(0)$ satisfies $\Delta(\lambda)h(0) = 0$

Since $\det \Delta(\lambda) = 0$, there is a n vector v_λ such that

$$\Delta(\lambda)v_\lambda = 0$$

Hence $h_\lambda(\alpha) = v_\lambda e^{\lambda\alpha}$

In the general case,

$$(\mathcal{A}-\lambda I)^m h = 0$$

$$\left(\frac{d}{d\alpha} - \lambda\right)^m h(\alpha) = 0 \qquad \alpha \in [-a,0) \tag{2-103}$$

(2-103) has solution

$$h_\lambda(\alpha) = \sum_{j=0}^{m-1} v_{\lambda,j} \frac{\alpha^j}{j!} e^{\lambda\alpha} \qquad \alpha \in [-a,0] \qquad (2\text{-}104)$$

where the n vectors $v_{\lambda,j}$ are chosen so that

$$h_\lambda \in \text{null } (\mathcal{A}-\lambda I)^m$$

Corollary

The eigenfunction k_λ corresponding to an eigenvalue λ of $\mathcal{A}_*$ with multiplicity m is of the form

$$k_\lambda(\alpha) = \sum_{j=0}^{m-1} v_{\lambda,j} \frac{(-1)^j\alpha^j}{j!} e^{-\lambda\alpha} \qquad \alpha \in [0,a] \qquad (2\text{-}105)$$

Proof Same as for the lemma.

For the semigroup of operators $\{\Phi(t); t \geq 0\}$ corresponding to (2-76) and $\mathcal{A}$ its differential generator, we want to list the relationships between $\sigma(\Phi(t))$ and $\sigma(\mathcal{A})$. We make use of the fact that $\Phi(t)$ is a C_0 (strongly continuous) semigroup of operators. ((iv) of corollary to theorem 2D) and that for $t \geq a$ $\Phi(t)$ is a compact operator (Delfour and Mitter [21]).

Definition

The spectral radius r_Λ of an operator Λ is the smallest disc centered at the origin of the complex plane which contains $\sigma(\Lambda)$.

The relationships between $\sigma(\Phi(t))$ and $\sigma(\mathcal{A})$ are

(i) $\sigma_P(\Phi(t)) = \exp\ (t\sigma_P(\mathcal{A}))$ plus possibly $\{0\}$ (2-106)

(Hille and Phillips [38] pp. 467

(ii) For $t \geq a$ and for any $\mu \in \sigma(\Phi(t))$, $\mu \neq 0$, $\mu \in \sigma_P(\Phi(t))$ and the only possible accumulation point is $\{0\}$.

(iii) The limit $\omega_0 = \lim_{t \to \infty} ||\Phi(t)||/t$ exists and ω_0 is finite or $-\infty$. For any $\delta > 0$, there is a constant K_δ such that

$$||\Phi(t)|| \leq K_\delta\, e^{(\omega_0+\delta)t} \quad \text{for all} \quad t \geq 0 \qquad (2\text{-}107)$$

Also $\sigma(\mathcal{A}) = \sigma_P(\mathcal{A})$ lies to the left of the line $\operatorname{Re} z = \omega_0$ in the complex plane (Dunford and Schwartz [26] pp. 619,622)

(iv) Since $\sigma_P(\mathcal{A})$ has real part bounded above, the spectral radius $r = r_{\Phi(a)}$ is finite and if β is defined by

$$\beta a = \log r_{\Phi(a)} \qquad (2\text{-}108)$$

for any $\gamma > 0$, there is a constant K_γ such that

$$||\Phi(t)h||_{M^2} \leq K_\gamma \, e^{(\beta+\gamma)t} ||h||_{M^2} \quad \text{for all} \quad t \geq 0 \qquad (2\text{-}109)$$

(Hale [36] pp. 112).

(v) From (iii) and (iv) we have that there exists $\alpha > 0$ and $K > 1$ such that $||\Phi(t)|| \leq K\, e^{-\alpha t} \Leftrightarrow \sigma_p(\mathcal{A})$ lies in the left half of the complex plane.

2.7 Advantages of M^2 over C

This is perhaps a good point to enumerate the several advantages working in the function space $M^2(-a,0;R^n)$ has over working in the function space $C(-a,0;R^n)$. They are as follows:

1) M^2 contains a larger class of functions than C and when working in the space C, we are forced to exclude discontinuous initial data. There is no good reason why discontinuous initial data should be discriminated against in such a manner and there are times when this discrimination can prove embarassing. So for instance Zverkin [82] takes as initial data

$$h(t) = 0 \quad t < a, \quad h(a) = 1 \qquad (2\text{-}110)$$

and observes that his initial data is discontinuous although he is working in the space C. He disposes of the difficulty in the peculiar (and amusing) manner of "by carrying the initial point to the right, we can consider it as a solution with continuous initial data". Incidentally, there will be many times in this thesis when a proof depends critically upon the use of a $h \in M^2$ as in (2-110).

2) Working in M^2, the representation of solutions to R.F.D.E. (2-1) (equation 2-12) is tidier and more transparent than that obtained when working in the space C (see for example Banks [3]).

3) M^2 is a Hilbert space with an inner product whereas C is a Banach space without an inner product. As such a wider range of techniques can be used in working in M^2 than in C. In particular, we can use the Lions' direct method [83] which has been successfully applied to systems governed by parabolic partial differential equations whereas in C we would be limited to using dynamic programming arguments. Ross and Flügge-Lotz [69] had speculated on the possibility of deriving the first order partial differential

equations satisfied by $\Pi_{00}(t)$, $\Pi_{01}(t,\alpha)$ and $\Pi_{11}(t,\theta,\alpha)$ from an operator Riccati differential equation, but realized that this could not be done rigorously working in a space without an inner product.

4) $\tilde{f}(t)$ as defined in (2-43) is an element of $M^2(-a,0;R^n)$, but it is not an element of $C(-a,0;R^n)$. Thus when working in M^2, we can write the R.F.D.E. (2-1) as a M^2 differential equation. It is not possible to do this when working in C, though it is possible to write the R.F.D.E. (2-1) as an integral equation in $C(-a,0;R^n)$ (see Hale [36] pp. 86).

5) Finally, experience of working in partial differential equations shows that for many problems, the choice of the function space must be exactly right to guarantee success. Our own particular problem, that of minimizing a quadratic functional, calls for a function space with an inner product.

Chapter 3

Finite Time Quadratic Criterion Optimal Control Problem

In this chapter, we shall tackle the finite time regulator and tracking quadratic criterion problems. We shall follow closely the treatment of Lions [83] for systems governed by partial differential equations and of Delfour and Mitter [21] for hereditary systems.

(3.1) Formulation of the control problem

Consider the controlled hereditary system defined on $[t_0,T]$

$$\frac{dx}{dt} = A_{00}(t)x(t) + \sum_{i=1}^{N} A_i(t)x(t+\theta_i) + \int_{-a}^{0} A_{01}(t,\theta)x(t+\theta)d\theta + B(t)v(t) + f(t)$$

$$x(t_0+\theta) = h(\theta), \qquad \theta \in [-a,0], \quad h \in M^2 \tag{3-1}$$

where $B \in L^2(t_0,T;\mathcal{L}(R^m,R^n))$, $v \in L^2(t_0,T;R^m)$, $f \in L^2(t_0,T;R^n)$

with quadratic cost criterion

$$C(v;h) = C(v) = (x(T),Fx(T)) + \int_{t_0}^{T} \{(x(t),Q(t)x(t)) + (v(t),N(t)v(t))\}dt \tag{3-2}$$

where $Q \in L^\infty(t_0,T;\mathcal{L}(R^n))$, $N \in L^\infty(t_0,T;\mathcal{L}(R^m))$, $F \in \mathcal{L}(R^n)$ $Q(t) = Q^*(t) \geq 0$, $N(t) = N^*(t) > 0$ for $t \in [t_0,T]$, $F = F^* \geq 0$ and there exists $\delta > 0$ such that

$$(v,N(t)v) \geq \delta |v|^2_{R^m} \quad \text{for all} \quad t \in [t_0,T] \tag{3-3}$$

Our class of admissible controls is

$$\mathcal{U}_{[t_0,T]} = \{v; \int_{t_0}^{T} |v(t)|^2 dt < \infty\} = L^2(t_0,T;R^m) \tag{3-4}$$

Note that $\mathcal{U}_{[t_0,T]}$ is a Hilbert space and that from (3-3)

$$C(v;h) \geq \delta ||v||^2_{\mathcal{U}[t_0,T]} \tag{3-5}$$

Unless there is any danger of confusion, we shall denote $\mathcal{U}_{[t_0,T]}$ by $\mathcal{U}$.

Our objective is to find

$$\inf_{v \,\epsilon \mathcal{U}} C(v;h) \tag{3-6}$$

which will be called the optimal cost and a $u \in \mathcal{U}$ such that

$$C(u;h) = \inf_{v \in \mathcal{U}} C(v;h) \leq C(v;h) \text{ for all } v \in \mathcal{U} \tag{3-7}$$

Such a $u \in \mathcal{U}$ will be called the optimal control.

From the representation of solutions formula (2-12) we can write the solution to (3-1) as

$$\begin{aligned} x(t;v) &= \Phi^0(t,t_0)h(0) + \int_{-a}^{0} \Phi^1(t,t_0,\alpha)h(\alpha)d\alpha \\ &\quad + \int_{t_0}^{t} \Phi^0(t,s)[B(s)v(s)+f(s)]ds \\ x(t;v) &= x_0(t) + \int_{t_0}^{t} \Phi^0(t,s)B(s)v(s)ds \end{aligned} \tag{3-8}$$

where

$$x_0(t) = \Phi^0(t,t_0)h(0) + \int_{-a}^{0} \Phi^1(t,t_0,\alpha)h(\alpha)d\alpha + \int_{t_0}^{t} \Phi^0(t,s)f(s)ds \tag{3-9}$$

Now

$$\begin{aligned} C(v) = {} & (x(T;v)-x_0(T),\ F[x(T;v)-x_0(T)]) \\ & + \int_{t_0}^{T} \{(x(t;v)-x_0(t),\ Q(t)[x(t;v)-x_0(t)]) + (v(t),N(t)v(t))\}dt \\ & + 2(x_0(T),\ F[x(t;v)-x_0(T)]) + 2\int_{t_0}^{T} (x_0(t),Q(t)[x(t;v)-x_0(t)])dt \\ & + (x_0(T),\ Fx_0(T)) + \int_{t_0}^{T} (x_0(t),\ Q(t)x_0(t)dt \end{aligned} \tag{3-10}$$

Now define bilinear form π and linear form L on $\mathcal{U}$ by

$$\pi(v_1, v_2) = (x(T; v_1) - x_0(T), F[x(T; v_2) - x_0(T)])$$
$$+ \int_{t_0}^{T} (x(t; v_1) - x_0(t), Q(t)[x(t; v_2) - x_0(t)])dt$$
$$+ \int_{t_0}^{T} (v_1(t), N(t) v_2(t))\, dt \qquad (3\text{-}11)$$

$$L(v) = -(x_0(T), F[x(T;v) - x_0(T)])$$
$$- \int_{t_0}^{T} (x_0(t), Q(t)[x(t;v) - x_0(t)])dt \qquad (3\text{-}12)$$

Note the following properties of π and L

(i) π is symmetric, i.e. $\pi(v_1, v_2) = \pi(v_2, v_1)$ for all $v_1, v_2 \in \mathcal{U}$

(ii) π is coercive, i.e. $\pi(v,v) \geq \delta \| v \|^2_{\mathcal{U}}$ for some $\delta > 0$ and all $v \in \mathcal{U}$. This follows from (3-3) and (3-5)

(iii) The map $(v_1, v_2) \rightarrow \pi(v_1, v_2) : \mathcal{U} \times \mathcal{U} \rightarrow R$ is continuous

(iv) The map $v \rightarrow L(v) : \mathcal{U} \rightarrow R$ is continuous.

Now $C(v) = \pi(v,v) - 2L(v) + (x_0(T), Fx_0(T))$

$$+ \int_{t_0}^{T} (x_0(t), Q(t)x_0(t))dt \qquad (3\text{-}13)$$

Hence to minimize $C(v)$ over $\mathcal{U}$, it is sufficient to minimize

$$C_0(v) = \pi(v,v)-2L(v) \qquad (3\text{-}14)$$

over $\mathcal{U}$.

(3.2) Existence and characterization of the optimal control

Several methods for obtaining the existence and characterization of optimal controls in hereditary control problems exist in the literature - maximal principles of Banks [3] and Kharatishvili [44], [45], Datko's [15] Frechet derivative method, and Lee and Marcus [53] set of attainability approach. However, for our purposes the most powerful method is due to Lions [83] which we will not state as a theorem.

Theorem 3A Lions [83]

Let π be a continuous symmetric bilinear form on satisfying $\pi(v,v) \geq \delta\|v\|^2_{\mathcal{U}}$ and L a continuous linear form. Then there exists a unique element $u \in \mathcal{U}$ minimizing $C_0(v) = \pi(v,v) - 2L(v)$ and characterized by

$$\pi(u,v) = L(v) \quad \text{for all} \quad v \in \mathcal{U} \qquad (3\text{-}15)$$

Proof

(i) Existence

Let $\{v_n\} \in \mathcal{U}$ be a minimizing sequence such that

$$C_0(v_n) \to \inf_{v \in \mathcal{U}} C_0(v) \tag{3-16}$$

Since L is a continuous form, there exists $\delta_1 > 0$ such that

$$L(v) \leq \frac{1}{2} \delta_1 ||v||_{\mathcal{U}}$$

and hence that

$$C_0(v) \geq \delta ||v||^2_{\mathcal{U}} - \delta_1 ||v||_{\mathcal{U}} \tag{3-17}$$

Hence $||v_n||_{\mathcal{U}} \leq$ some constant for all n. Since $\mathcal{U}$ (as a Hilbert space) is weakly compact, we may extract a subsequence $\{v_m\}$ such that

$$v_m \to u \quad \text{weakly in } \mathcal{U} \tag{3-18}$$

Now $v \to \pi(v,v)$ is lower-semi-continuous in the weak topology of $\mathcal{U}$ and

$v \to L(v)$ is continuous in the weak topology.

Thus the map $v \to C_0(v)$ is weakly lower semi-continuous and hence

$$C_0(u) \leq \lim_{m \to \infty} C_0(v_m) = \inf_{v \in \mathcal{U}} C_0(v)$$

Hence $\quad C_0(u) = \inf_{v \in \mathcal{U}} C_0(v)$

and thus we have proved existence.

(ii) Uniqueness

The map $v \to \pi(v,v)$ is strictly convex and hence the map $v \to C_0(v)$ is also strictly convex.

Let u_1 and u_2 be two distinct elements such that

$$C_0(u_1) = C_0(u_2) = \inf_{v \in \mathcal{U}} C_0(v)$$

$$C_0(\tfrac{1}{2}(u_1 + u_2)) < \tfrac{1}{2} C_0(u_1) + \tfrac{1}{2} C_0(u_2) < \inf_{v \in \mathcal{U}} C_0(v)$$

This is a contradiction and hence $u_1 = u_2 = u$.

(iii) Characterization

Let u be the minimizing element. Then for any $v \in \mathcal{U}$ and $t \in (0,1)$ we have

$$C_0(u) < C_0((1-t)u + tv) \tag{3-19}$$

Hence $\frac{1}{t}[C_0(u + t(v-u)) - C_0(u)] \geq 0.$

Letting $t \to 0$ we have

$$C_0'(u)\cdot(v-u) = 2[\pi(u,v-u) - L(v-u)] \geq 0 \tag{3-20}$$

where $C_0'(u)$ is the Frechet derivative of $C_0(u)$.

Putting $w = v - u$, we have

$$\pi(u,w) \geq L(w) \quad \text{for all} \quad w \in \mathcal{U}$$

But $\pi(u,-w) = -\pi(u,w) \geq L(-w) = -L(w)$

Hence $\pi(u,w) \leq L(w)$ for all $w \in \mathcal{U}$

$$\text{Hence} \quad \pi(u,w) = L(w) \quad \text{for all} \quad w \in \mathcal{U} \tag{3-15}$$

Conversely, suppose that

$$\pi(u,w) = L(w) \quad \text{for all} \quad w \in \mathcal{U}$$

Since $v \to C_0(v)$ is convex, we have

$$C_0((1-t)u + tv) \leq (1-t)C_0(u) + tC_0(v)$$

$$C_0(v) - C_0(u) \geq \frac{1}{t}[C_0((1-t)u + tv) - C_0(u)]$$

Taking limit $t \to 0$ we obtain

$$C_0(v) - C_0(u) \geq C_0'(u)\cdot(v-u)$$

$$= 2[\pi(u,v-u)-L(v-u)] = 0$$

Hence $C_0(u) \leq C_0(v)$ for all $v \in \mathcal{U}$.

Q.E.D.

We immediately obtain the corollary

Corollary

The control problem (3-1), (3-2), (3-4) has a unique optimal control $u \in \mathcal{U}$.

We now want to characterize this u.

From (3-15), we have $\pi(u,w) = L(w)$ for all $w \in \mathcal{U}$.

i.e.

$$0 = (x(T;u)-x_0(T),\ F[x(T;w)-x_0(T)])$$

$$+ \int_{t_0}^{T} \{(x(t;u)-x_0(t),\ Q(t)[x(t;w)-x_0(t)]) + (u(t),N(t)w(t))\}dt$$

$$+ (x_0(T),F[x(T;w)-x_0(T)]) + \int_{t_0}^{T} (x_0(t),Q(t)[x(t;w)-x_0(t)])dt$$

$$= (x(T;u),F[x(T;w)-x_0(T)]) + \int_{t_0}^{T} (x(t;u),Q(t)[x(t;w)-x_0(t)])dt$$

$$+ \int_{t_0}^{T} (u(t),N(t)w(t))dt$$

$$0 = (x(T;u),Fx_1(T;w)$$

$$+ \int_{t_0}^{T} \{(x(t;u),Q(t)x_1(t;w))+(u(t),N(t)w(t))\}dt \qquad (3\text{-}21)$$

where $x_1(t;w) = x(t;w)-x_0(t)$

and satisfies the R.F.D.E.

$$\frac{dx}{dt} = A_{00}(t)\,x(t) + \sum_{i=1}^{N} A_i(t)\,x(t+\theta_i) + \int_{-a}^{0} A_{01}(t,\theta)\,x(t+\theta)\,d\theta + B(t)\,w(t)$$

$$x(t_0+\theta) = 0 \qquad \theta \in [-a, 0] \qquad (3\text{-}22)$$

Corresponding to (3-1), we have the hereditary adjoint equation for $p(t;u)$

$$\frac{dp}{dt} + A_{00}^{*}(t)p(t) + \sum_{i=1}^{N} A_i^{*}(t-\theta_i)p(t-\theta_i) + \int_{-a}^{0} A_{01}^{*}(t-\theta,\theta)p(t-\theta)d\theta$$

$$+ Q(t)x(t;u) = 0$$

$$p(T) = Fx(T;u) \qquad p(T+\beta) = 0 \qquad \beta \in (0,a] \tag{3-23}$$

From theorem 2E equation (2-64) we have

$$\mathcal{H}_T(t_0,\tilde{p}(t_0;u),\tilde{x}_1(t_0;w)) - \mathcal{H}_T(T,\tilde{p}(T;u),\tilde{x}_1(T;w))$$

$$= \int_{t_0}^{T} (p(t;u),B(t)w(t))dt - \int_{t_0}^{T} (Q(t)x(t;u),x_1(t;w))dt \tag{3-24}$$

But $\mathcal{H}_T(t_0,\tilde{p}(t_0;u),\tilde{x}_1(t;w)) = 0$ (3-25)

and $\mathcal{H}_T(T,\tilde{p}(T;u),\tilde{x}_1(t;w)) = (x(T;u),Fx_1(T;w))$ (3-26)

Hence from (3-24), (3-25), (3-26), we have

$$\int_{t_0}^{T} (N(t)u(t) + B^{*}(t)p(t;u),w(t))dt = 0 \text{ for all } w \in \mathcal{U} \tag{3-27}$$

Hence $u(t) = -N^{-1}(t)B^{*}(t)p(t;u)$

We have thus proved the theorem

Theorem 3B

The unique optimal control of the control problem (3-1), (3-2) and (3-4) is characterized by

$$u(t) = - N^{-1}(t)B^{*}(t)p(t) \tag{3-28}$$

where $(p(\cdot),x(\cdot))$ is the unique pair of maps in $AC^2(t_0,T;R^n)$ which satisfy the following system of equations on $[t_0,T]$

$$\frac{dx}{dt} = A_{00}(t)x(t) + \sum_{i=1}^{N} A_i(t)x(t+\theta_i) + \int_{-a}^{0} A_{01}(t,\theta)x(t+\theta)d\theta - R(t)p(t) + f(t)$$

$$x(t_0+\theta) = h(\theta), \quad \theta \in [-a,0], \quad h \in M^2 \tag{3-29}$$

$$\frac{dp}{dt} + A_{00}^{*}(t)p(t) + \sum_{i=1}^{N} A_i^{*}(t-\theta_i)p(t-\theta_i) + \int_{-a}^{0} A_{01}^{*}(t-\theta,\theta)p(t-\theta)d\theta + Q(t)x(t) = 0$$

$$p(T) = F x(T) \qquad p(T+\beta) = 0 \qquad \beta \in (0,a] \tag{3-30}$$

and where $R(t) = B(t)N^{-1}(t)B^{*}(t)$ (3-31)

(3.3) Decoupling optimality pair of equations

We now set out to decouple the optimality pair of equations (3-29) and (3-30). To do so, we consider the control problem on the interval $[s,T]$ where $s \varepsilon [t_0,T]$

$\phi(t;s,v)$ satisfies for $t \varepsilon [s,T]$

$$\frac{d\phi}{dt} = A_{00}(t)\phi(t) + \sum_{i=1}^{N} A_i(t)\phi(t+\theta_i) + \int_{-a}^{0} A_{01}(t,\theta)\phi(t+\theta)d\theta + B(t)v(t) + f(t)$$

$$\phi(s+\theta) = h(\theta), \quad \theta \in [-a,0], \quad h \in M^2 \tag{3-32}$$

and we have cost functional

$$C_s(v;h) = C_s(v) = (\phi(T;s,v),F\phi(T;s,v))$$

$$+ \int_{t_0}^{T} \{(\phi(t;s,v),Q(t)\phi(t;s,v))+(v(t),N(t)v(t))\}dt \tag{3-33}$$

and with class of admissible controls

$$\mathcal{U}_{[s,T]} = \{v; \int_s^T |v(t)|^2 dt < \infty\} = L^2(s,T;R^m) \tag{3-34}$$

Unless there is any danger of confusion, we shall denote $\mathcal{U}_{[s,T]}$ by $\mathcal{U}_s$.

The hereditary adjoint solution $\psi(t;s,v)$ satisfies

$$\frac{d\psi}{dt} + A_{00}^*(t)\psi(t) + \sum_{i=1}^{N} A_i^*(t-\theta_i)\psi(t-\theta_i) + \int_{-a}^{0} A_{01}^*(t-\theta,\theta)\psi(t-\theta)d\theta$$

$$+ Q(t)\phi(t) = 0$$

$$\psi(T) = F\phi(T), \quad \psi(T+\beta) = 0 \quad \beta \in (0,a] \tag{3-35}$$

From theorem 3B, the unique optimal control $u \in \mathcal{U}_s$ is characterized by

$$u(t) = - N^{-1}(t)B^*(t)\psi(t;s) \tag{3-36}$$

where $(\phi(\cdot;s), \psi(\cdot;s))$ is the solution of the coupled system of equations

$$\frac{d\phi}{dt} = A_{00}(t)\phi(t) + \sum_{i=1}^{N} A_i(t)\phi(t+\theta_i) + \int_{-a}^{0} A_{01}(t,\theta)\phi(t+\theta)d\theta$$

$$- R(t)\psi(t) + f(t) \tag{3-37}$$

$$\phi(s+\theta) = h(\theta), \quad \theta \in [-a,0], \quad h \in M^2$$

$$\frac{d\psi}{dt} + A_{00}^*(t)\psi(t) + \sum_{i=1}^{N} A_i^*(t-\theta_i)\psi(t-\theta_i) + \int_{-a}^{0} A_{01}^*(t-\theta,\theta)\psi(t-\theta)d\theta$$

$$+ Q(t)\phi(t) = 0 \tag{3-38}$$

$$\psi(T) = F\phi(T), \quad \psi(T+\beta) = 0 \quad \beta \in (0,a]$$

Lemma 3.1

The map $(h,f) \to (\phi(\cdot;s), \psi(\cdot;s))$

$$M^2(-a,0;R^n) \times L^2(s,T;R^n) \to AC^2(s,T;R^n) \times AC^2(s,T;R^n) \tag{3-39}$$

is bilinear and continuous.

Proof The map is clearly bilinear.
To show that it is continuous, we take a sequence $\{h_n\}$ converging to h in M^2 and $\{f_n\}$ converging to f in $L^2(s,T;R^n)$.
For some $v \in \mathcal{U}_s$, let $(\phi_n(\cdot;s,v), \psi_n(\cdot;s,v))$ and $(\phi(\cdot;s,v), \psi(\cdot;s,v))$ be the solutions of (3-37) and (3-38) with initial data h_n, h and forcing terms f_n, f respectively.
From theorem 2A, $h_n \to h$ in M^2 and $f_n \to f$ in $L^2(s,T;R^n)$

$$\Rightarrow \phi_n(\cdot;s,v) \to \phi(\cdot;s,v) \text{ in } AC^2(s,T;R^n)$$

Denote the cost functional for initial data h_n,h and forcing terms f_n,f by $C_s^n(v)$ and $C_s(v)$ respectively, and denote the optimal control for $C_s^n(v)$, $C_s(v)$ by u_n and u respectively.

$$C_s^n(u_n) = \inf_{v \,\varepsilon\, \mathcal{U}_s} C_s^n(v) \leq C_s^n(u)$$

and $C_s^n(u) \to C_s(u)$ as $n \to \infty$

Hence $\overline{\lim}_{n \to \infty} C_s^n(u_n) \leq \overline{\lim}_{n \to \infty} C_s^n(u) = C_s(u)$ (3-40)

Since, from (3-3) $||u_n||^2_{\mathcal{U}_s} \leq \frac{1}{\delta} C_s^n(u_n)$, $\{u_n\}$ belongs to a bounded subset of $\mathcal{U}_s$, and since $\mathcal{U}_s$ is weakly compact, there exists a subsequence $\{u_m\}$ and an element $w \,\varepsilon\, \mathcal{U}_s$ such that

$$u_m \to w \text{ weakly in } \mathcal{U}_s$$

Thus $\phi_m(\cdot;s,u_m) \to \phi(\cdot;s,w)$ weakly in $AC^2(s,T;R^n)$.

Since $v \to C_s(v)$ is convex

$$C_s(w) \leq \underline{\lim}_{m \to \infty} C_s^m(u_m) \qquad (3\text{-}41)$$

Combining (3-40) and (3-41), we have

$$C_s(w) \leq \underline{\lim}_{m \to \infty} C_s^m(u_m) \leq \overline{\lim}_{m \to \infty} C_s^m(u_m) \leq C_s(u)$$

Hence $w = u$.

Thus we have

$$u_m \to u \quad \text{weakly}$$

$$C_s^m(u_m) \to C_s(u)$$

$$\phi_m(\cdot;s,u_m) \to \phi(\cdot;s,u) \quad \text{weakly in} \quad AC^2(s,T;R^n)$$

$$\psi_m(\cdot;s,u_m) \to \psi(\cdot;s,u) \quad \text{weakly in} \quad AC^2(s,T;R^n)$$

This proves the continuity of the map (3-39) where continuity is with respect to the strong topology of M^2 and $L^2(s,T;R^n)$ and the weak topology of $AC^2(s,T;R^n)$.

Q.E.D.

Corollary

For $s \in [t_0,T]$, $t \in [s,T]$, the map

$$\begin{aligned} &(h,f) \mapsto \psi(t;s) \\ &M^2 \times L^2(s,T;R^n) \to R^n \end{aligned} \tag{3-42}$$

is bilinear and continuous and has representation

$$\psi(t;s) = P(t,s)h + F(t,s)f$$

where $P(t,s) \in \mathcal{L}(M^2,R^n)$, $F(t,s) \in \mathcal{L}(L^2(s,T;R^n),R^n)$

Proof The map (3-42) is a composition of the maps

$$(h,f) \to (\phi(\cdot;s), \psi(\cdot;s))$$

and $(\phi(;s),\psi(\cdot;s)) \to \psi(t,s)$

both of which are continuous.

Lemma 3.2

Let $(p(\cdot),x(\cdot))$ be the solution of the coupled system of equations (3-29), (3-30). Then for all pairs $s \leq t$ in $[t_0,T]$

$$p(t) = P(t,s)\tilde{x}(s) + d(t,s) \tag{3-43}$$

where $P(t,s) \in \mathcal{L}(M^2,R^n)$ and $d(t,s) \in R^n$ are obtained in the following manner: we solve the system in $[s,T]$

$$\frac{d\phi}{dt} = A_{00}(t)\phi(t) + \sum_{i=1}^{N} A_i(t)\,\phi(t+\theta_i) + \int_{-a}^{0} A_{01}(t,\theta)\,\phi(t+\theta)\,d\theta - R(t)\Psi(t)$$

$$\phi(s+\theta) = h(\theta), \quad \theta \in [-a, 0], \quad h \in M^2 \qquad (3\text{-}44)$$

$$\frac{d\Psi}{dt} + A_{00}^{*}(t)\Psi(t) + \sum_{i=1}^{N} A_i^{*}(t-\theta_i)\,\Psi(t-\theta_i) + \int_{-a}^{0} A_{01}^{*}(t-\theta,\theta)\,\Psi(t-\theta)\,d\theta + Q(t)\,\phi(t) = 0$$

$$\Psi(T) = F\,\phi(T), \quad \Psi(T+\beta) = 0 \quad \beta \in (0, a] \qquad (3\text{-}45)$$

and then $\psi(t) = P(t,s)h$ (3-46)

and we solve on $[s,T]$

$$\frac{d\eta}{dt} = A_{00}(t)\,\eta(t) + \sum_{i=1}^{N} A_i(t)\,\eta(t+\theta_i) + \int_{-a}^{0} A_{01}(t,\theta)\,\eta(t+\theta)\,d\theta - R(t)\,\xi(t) + f(t)$$

$$\eta(s+\theta) = 0 \qquad \theta \in [-a, 0] \qquad (3\text{-}47)$$

$$\frac{d\xi}{dt} + A_{00}^{*}(t)\,\xi(t) + \sum_{i=1}^{N} A_i^{*}(t-\theta_i)\,\xi(t-\theta_i) + \int_{-a}^{0} A_{01}^{*}(t-\theta,\theta)\,\xi(t-\theta)\,d\theta + Q(t)\,\eta(t) = 0$$

$$\xi(T) = F\,\eta(T) \qquad \xi(T+\beta) = 0 \qquad \beta \in (0, a] \qquad 93\text{-}48)$$

and then $d(t,s) = \xi(t)$ (3-49)

Proof P(t,s) and d(t,s) are clearly defined from (3-46) and (3-49).

We want to prove the identity (3-43).

Let $p(\cdot)$, $x(\cdot)$ be the solution to (3-29) and (3-30) and consider the system of equations (3-37) and (3-38) with initial data $\tilde{x}(s)$, and with solutions $\phi(\cdot)$, $\psi(\cdot)$.

Let $\phi_s(\cdot)$, $\psi_s(\cdot)$ be the restriction of $x(\cdot)$ and $p(\cdot)$ respectively on $[s,T]$.

ϕ_s and ψ_s satisfy (3-37) and (3-38) with some initial data as $\phi(\cdot)$, $\psi(\cdot)$.

Hence from uniqueness $\phi_s = \phi$, $\psi_s = \psi$.

Hence $\psi_s(t) = p(t) = \psi(t) = P(t,s)\tilde{x}(s) + d(t,s)$

Q.E.D.

Corollary 1

The map $t \mapsto P(t,s)h + d(t,s)$ (3-50)

is in $AC^2(s,T;R^n)$ for fixed $s \in [t_0,T]$.

Corollary 2

$$p(t) = P(t,t)\tilde{x}(t) + d(t,t) = P(t)\tilde{x}(t) + d(t) \tag{3-51}$$

where $P(t) = P(t,t) \in \mathcal{L}(M^2,R^n)$ (3-52)

$$d(t) = d(t,t) \in R^n \tag{3-53}$$

The map $t \mapsto d(t)$ is absolutely continuous in $[t_0,T]$.

The proof of corollaries 1, 2 follow immediately from lemma 3.2.

From the Riez representation formula, since $P(t,s) \in \mathcal{L}(M^2,R^n)$, we can write

$$P(t,s)h = P_0(t,s)h(0) + \int_{-a}^{0} P_1(t,s,\alpha)h(\alpha)d\alpha \qquad (3\text{-}54)$$

$$\text{where} \quad P_0(t,s) \in \mathcal{L}(R^n),\ P_1(t,s,\cdot) \in L^2(-a,0;\mathcal{L}(R^n)) \qquad (3\text{-}55)$$

Defining $P_0(t) = P_0(t,t)$, $P_1(t,\alpha) = P_1(t,t,\alpha)$ we have from (3-51)

$$p(t) = P_0(t)x(t) + \int_{-a}^{0} P_1(t,\alpha)x(t+\alpha)d\alpha + d(t) \qquad (3\text{-}56)$$

Hence we can express the optimal control to (3-1), (3-2), (3-4) in the feedback form

$$u(t) = -N^{-1}(t)B^{*}(t)\{P_o(t)x(t) + \int_{-a}^{0} P_1(t,\alpha)x(t+\alpha)d\alpha + d(t)\} \qquad (3\text{-}57)$$

(3.4) The operator $\Pi(t)$ and the optimal cost and optimal control

In this section and section 3.5, we shall study the regulator problem for which

$$f(t) = 0 \qquad t \in [t_0,T]$$

We shall introduce and study a family $\{\Pi(t);\ t \in [t_0,T]\}$ of M^2 operators.

Let us denote by $\phi_0(\cdot;s,h)$ and $\psi_0(\cdot;s,h)$ the solution of (3-37) and (3-38) with initial data h and forcing term $f = 0$.

Lemma 3.3

(i)
$$\mathcal{H}_T(s, \tilde{\psi}_0(s;s,k), h) = (\phi_0(T;s,h), F\phi_0(T;s,k)) + \int_s^T \{(\psi_0(t;s,h), R(t)\psi_0(t;s,k)) + (\phi_0(t;s,h), Q(t)\phi_0(t;s,k))\}\,dt \tag{3-58}$$

(ii) the map $(h,k) \mapsto \mathcal{H}_T(s,\tilde{\psi}_0(s;s,k),h)$

$$M^2 \times M^2 \to R \tag{3-59}$$

is a continuous, positive, symmetric, bilinear form.

(iii) $\exists$ operator $\Pi(s) : M^2 \to M^2$ defined by

$$(h,\Pi(s)k)_{M^2} = \mathcal{H}_T(s,\tilde{\psi}_0(s;s,k),h) \tag{3-60}$$

(iv) $$(h,\Pi(s)h)_{M^2} = \inf_{v \in \mathcal{U}_s} C_s(v;h) \tag{3-61}$$

(v) $c > 0$ such that

$$||\Pi(s)h||_{M^2} \leq c||h||_{M^2} \text{ for } s \in [t_o,T],\ h \in M^2 \tag{3-62}$$

Proof

(i) From theorem 2E equation (2-64), we have

$$\mathcal{H}_T(T,\tilde{\psi}_0(T;s,k),\tilde{\phi}_0(T;s,h)-\mathcal{H}_T(s,\tilde{\psi}_0(s;s,k),\tilde{\phi}_0(s;s,h))$$

$$= -\int_s^T\{(\psi_0(t;s,k),R(t)\psi_0(t;s,h))+(Q(t)\phi_0(t;s,k),\phi_0(t;s,h))\}$$

Now $\mathcal{H}_T(s,\tilde{\psi}_0(s;s,k),\tilde{\phi}(s;s,h)) = \mathcal{H}_T(s,\tilde{\psi}_0(s;s,k),h)$

and $\mathcal{H}_T(T,\tilde{\psi}_0(T,s,k),\tilde{\phi}_0(T;s,h)) = (F\phi_0(T;s,h),\phi_0(T;s,k))$

Hence result

(ii) Now the map $h \to (\phi_0(\cdot;s,h),\psi_0(\cdot;s,h))$ is linear and continuous from lemma 3.1 and hence the map (3-59) is bilinear and continuous. The symmetry and positivity of the map follow from the symmetry and positivity of $R(t)$, $Q(t)$ and F.

(iii) Since the map $(h,k) \mapsto \mathcal{H}_T(s,\tilde{\psi}_0(s;s,k),h)$ is continuous, it follows from Horvath [39] pp. 44 that there exists a continuous operator

$$\Pi(s) : M^2 \to M^2$$

such that $(h,\Pi(s)k)_{M^2} = \mathcal{H}_T(s,\tilde{\psi}_0(s;s,k),h)$

$$\begin{aligned}
\text{(iv)}\quad (h,\Pi(s)h)_{M^2} &= \mathcal{H}_T(s,\tilde{\psi}_0(s;s,h),h) \\
&= (\phi_0(T;s,h), F\,\phi_0(T;s,h)) \\
&\quad + \int_s^T \{\phi_0(t;s,h), Q(t)\,\phi_0(t;s,h)\}\,dt \\
&\quad + \int_s^T \{\psi_0(t;s,h), R(t)\,\psi_0(t;s,h)\}\,dt \\
&= \inf_{v\in\mathcal{U}_s} C_s(v;h)
\end{aligned}$$

since the optimal control is given by

$$u(t) = -N^{-1}(t)B^*(t)\psi_0(t;s,h)$$

$$\text{(v)}\quad (h,\Pi(s)h)_{M^2} \leq C_s(0;h) \leq C_{t_0}(0;h) \leq c\,||h||^2_{M^2}$$

Hence result. Q.E.D.

From the results of chapter 2 section 6, we can decompose the operator $\Pi(t)$ into a matrix of bounded transformations

$$\Pi(t) = \begin{pmatrix} \Pi_{00}(t) & \Pi_{01}(t) \\ \Pi_{10}(t) & \Pi_{11}(t) \end{pmatrix}$$

where

(i) $\Pi_{00}(t) \in \mathcal{L}(R^n)$ and can be represented at a matrix.

(ii) $\Pi_{01}(t) \in \mathcal{L}(L^2(-a,0;R^n),R^n)$ and has representation

$$\Pi_{01}(t)x = \int_{-a}^{0} \Pi_{01}(t,\alpha)x(\alpha)d\alpha \tag{3-63}$$

where $x \in L^2(-a,0;R^n)$ and $\Pi_{01}(t,\cdot) \in L^2(-a,0;\mathcal{L}(R^n))$

(iii) $\Pi_{10}(t) \in \mathcal{L}(R^n,L^2(-a,0;R^n))$ and has representation

$$(\Pi_{10}(t)x)(\theta) = \Pi_{10}(t,\theta)x$$

where $x \in R^n$ and $\Pi_{10}(t,\cdot) \in L^2(-a,0;\mathcal{L}(R^n))$ (3-64)

(iv) $\Pi_{11}(t) \in \mathcal{L}(L^2(-a,0;R^n))$

Also we have

$$(h,\Pi(t)k)_{M^2} = (h(0),\Pi_{00}(t)k(0)) + \int_{-a}^{0} (h(0),\Pi_{01}(t,\alpha)k(\alpha)d\alpha$$

$$+ \int_{-a}^{0} (h(\theta),\Pi_{10}(t,\theta)k(0)d\theta + (h^1,\Lambda_{11}k^1)_{M^2} \qquad (3\text{-}65)$$

Lemma 3.4

$$\text{(i)} \quad \Pi_{00}(t) = P_0(t) \qquad (3\text{-}66)$$

$$\text{(ii)} \quad \Pi_{01}(t,\alpha) = P_1(t,\alpha) \quad \text{a.e.} \quad \alpha \in [-a,0) \qquad (3\text{-}67)$$

$$\text{(iii)} \quad (h,\Pi(t)k)_{M^2} = (h(0),\Pi_{00}(t)k(0))$$

$$+ \int_{-a}^{0} (h(0),\Pi_{01}(t,\alpha)k(\alpha))d\alpha$$

$$+ \int_{-a}^{0} (h(\theta),\Pi_{10}(t,\theta)k(0)) \qquad (3\text{-}68)$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha (h(\theta),\Pi_{11}(t,\theta,\alpha)k(\alpha))$$

$$\text{where } \Pi_{11}(t,\theta,\alpha) = \sum_{i=1}^{N} \begin{cases} A_i^*(t+\theta-\theta_i)P_1(t+\theta-\theta_i,t,\alpha) & \theta_i<\theta<0 \\ 0 & \text{otherwise} \end{cases} \qquad (3\text{-}69)$$

$$+ \int_{\theta}^{0} d\delta \; A_{01}^*(t-\delta,\theta+\delta)P_1(t-\delta,t,\alpha)$$

and $\Pi_{11}(t,\cdot,\cdot) \in L^2(-a,0;-a,0;\mathcal{L}(R^n))$

$$\text{(iv)}\quad \Pi_{00}(t) = \Pi_{00}^*(t);\ \Pi_{01}(t,\alpha) = \Pi_{10}^*(t,\alpha) \quad \text{a.e. } \alpha \in [-a,0]$$

$$\Pi_{11}(t,\theta,\alpha) = \Pi_{11}^*(t,\alpha,\theta) \quad \text{a.e.} \quad (\theta,\alpha) \in [-a,0]\times[-a,0] \tag{3-70}$$

Proof We exploit the relationship

$$(h,\Pi(t)k)_{M^2} = \mathcal{H}_T(t,\tilde{\psi}_0(t;t,k),h)$$

$$\begin{aligned}(h(0),\Pi_{00}(t)k(0)) &+ \int_{-a}^{0}(h(0),\Pi_{01}(t,\alpha)k(\alpha))d\alpha \\ &+ \int_{-a}^{0}(h(\theta),\Pi_{10}(t,\theta)k(0))d\theta + (h^1,\Lambda_{11}k^1)_{M^2} \\ &= (\psi_0(t;t,k),h(0)) \\ &+ \sum_{i=1}^{N}\int_{t}^{t-\theta_i} ds(\psi_0(s;t,k),A_i(s)h(s-t+\theta_i)) \\ &+ \int_{-a}^{0} d\theta \int_{t}^{t-\theta} ds(\psi_0(s;t,k),A_{01}(s,\theta)h(s-t+\theta)) \\ &= (h(0),P_0(t)k(0)) + \int_{-a}^{0}(h(0),P_1(t,\alpha)k(\alpha))d\alpha +\end{aligned}$$

$$+ \sum_{i=1}^{N} \int_{t}^{t-\theta_i} ds(A_i(s)h(s-t+\theta_i),P_0(s,t)k(0))$$

$$+ \sum_{i=1}^{N} \int_{t}^{t-\theta_i} ds \int_{-a}^{0} d\alpha(A_i(s)h(s-t+\theta_i),P_1(s,t,\alpha)k(\alpha))$$

$$+ \int_{-a}^{0} d\theta \int_{t}^{t-\theta} ds(A_{01}(s,\theta)h(s-t+\theta),P_0(s,t)k(0))$$

$$+ \int_{-a}^{0} d\theta \int_{t}^{t-\theta} ds \int_{-a}^{0} d\alpha(A_{01}(s,\theta)h(s-t+\theta),P_1(s,t,\alpha)k(\alpha))$$

$$= (h(0),P_0(t)k(0)) + \int_{-a}^{0} (h(0),P_1(t,\alpha)k(\alpha))d\alpha$$

$$+ \sum_{i=1}^{N} \int_{\theta_i}^{0} d\theta(A_i(t+\theta-\theta_i)h(\theta),P_0(t+\theta-\theta_i)k(0))$$

$$+ \sum_{i=1}^{N} \int_{\theta_i}^{0} d\theta \int_{-a}^{0} d\alpha\ A_i(t+\theta-\theta_i)h(\theta),P_0(t+\theta-\theta_i,t,\alpha)k(\alpha)$$

$$+ \int_{-a}^{0} d\theta \int_{\theta}^{0} d\delta(A_{01}(t-\delta,\theta+\delta)h(\theta),P_0(t-\delta,t)k(0))$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \int_{\theta}^{0} d\delta(A_{01}(t-\delta,\theta+\delta)h(\theta),P_1(t-\delta,t,\alpha)k(\alpha)) \quad (3\text{-}71)$$

(changing variables and interchanging order of integration in the 6th expression on left hand side)

(i) Now putting $h^1 = k^1 = 0$, $h(0), k(0) \neq 0$ in (3-71) we have

$$(h(0), \Pi_{00}(t)k(0)) = (h(0), P_0(t)k(0)) \qquad (3\text{-}72)$$

for all $h(0)$, $k(0)$ in R^n. Hence result

(ii) Putting $k(0) = 0$, $k^1 \neq 0$, $h^1 = 0$, $h(0) \neq 0$ in (3.71) we have

$$\int_{-a}^{0} (h(0), \Pi_{01}(t,\alpha)k(\alpha))d\alpha = \int_{-a}^{0} (h(0), P_1(t,\alpha)k(\alpha))d\alpha \qquad (3\text{-}73)$$

for all $h(0) \in R^n$, $k^1 \in L^2(-a,0;R^n)$.

Hence result.

(iii) Putting $h(0) = k(0) = 0$, $h^1, k^1 \neq 0$ in (3-71), we have

$$(h^1, \Lambda_{11} k^1)_{L^2} = \sum_{i=1}^{N} \int_{\theta_i}^{0} d\theta \int_{-a}^{0} d\alpha \, (h(\theta), A_i^*(t+\theta-\theta_i) P_1(t+\theta-\theta_i, t, \alpha) k(\alpha))$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \int_{\theta}^{0} d\delta \;\; h(\theta), A_{01}^*(t-\delta, \theta+\delta) P_1(t-\delta, t, \alpha) k(\alpha)$$

$$= \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \, (h(\theta), \Pi_{11}(t, \theta, \alpha) k(\alpha))$$

where $\Pi_{11}(t,\theta,\alpha)$ is given in (3-69)

(iv) From symmetry, we have

$$(h,\Pi(t)k)_{M^2} = (k,\Pi(t)h)_{M^2}$$

Hence

$$\begin{aligned}
&(h(0),\Pi_{00}(t)k(0)) + \int_{-a}^{0}(h(0),\Pi_{01}(t,\alpha)k(\alpha))d\alpha \\
&+ \int_{-a}^{0}(h(\theta),\Pi_{10}(t,\theta)k(0))d\theta + \int_{-a}^{0}d\theta \int_{-a}^{0}d\alpha(h(\theta),\Pi_{11}(t,\theta,\alpha)k(\alpha)) \\
&= (k(0),\Pi_{00}(t)h(0)) + \int_{-a}^{0}(k(0),\Pi_{01}(t,\alpha)h(\alpha))d\alpha \qquad (3\text{-}74) \\
&+ \int_{-a}^{0}(k(\theta),\Pi_{10}(t,\theta)h(0))d\alpha + \int_{-a}^{0}d\theta \int_{-a}^{0}d\alpha(k(\theta),\Pi_{11}(t,\theta,\alpha)h(\alpha))
\end{aligned}$$

Considering in turn $h^1 = k^1 = 0$, $h(0)$, $k(0) \neq 0$; $k(0) = 0$, $k^1 \neq 0$, $h(0) \neq 0$, $h^1 = 0$; $h(0) = k(0) = 0$, $h^1, k^1 \neq 0$, the result follows.

Q.E.D.

Corollary

$$p(t) = \Pi^0(t)\tilde{x}(t) \qquad (3\text{-}75)$$

Proof Follows from (3-56) ($f = 0 \Rightarrow d(t) = 0$), (i) and (ii) of lemma 3.4 and definition (2-74) of $\Pi^0(t)$.

Q.E.D.

With $f = 0$, we can express the optimal control of (3-1), (3-2) and (3-4) in the feedback form

$$u(t) = - N^{-1}(t)B^*(t)\{\Pi_{00}(t)x(t) + \int_{-a}^{0} \Pi_{01}(t,\alpha)x(t+\alpha)d\alpha\} \qquad (3\text{-}76)$$

and the optimal cost to go at time $t \in [t_0,T]$

$$\begin{aligned} \inf_{v \in \mathcal{U}_t} C_t(v;h) = {} & (h(0),\Pi_{00}(t)k(0)) \\ & + 2 \int_{-a}^{0} (h(0),\Pi_{01}(t,\alpha)h(\alpha))d\alpha \\ & \int_{-a}^{0} d\theta \int_{-a}^{0} d\theta(h(\theta),\Pi_{11}(t,\theta,\alpha)h(\ )) \end{aligned} \qquad (3\text{-}77)$$

Either or both of the expressions (3-76), (3-77) have been obtained by Krasovskii [46], Ross and Flügge-Lotz [69], Eller, Aggarwal and Banks [27], Kushner and Barnea [50], Alekal, Brunovsky, Chyung and Lee [1], Mueller [62] and Delfour and Mitter [21].

3.5 Operator Riccati differential equation for $\Pi(t)$

Definition

Define M^2 operators $\mathcal{Q}(t)$, $\mathcal{R}(t)$, $\mathcal{F}$ by

$$[\mathcal{Q}(t)h](\alpha) = \begin{cases} Q(t)h(0) & \alpha = 0 \\ 0 & \alpha \in [-a,0) \end{cases} \tag{3-78}$$

$$[\mathcal{R}(t)h](\alpha) = \begin{cases} R(t)h(0) & \alpha = 0 \\ 0 & \alpha \in [-a,0) \end{cases} \tag{3-79}$$

$$[\mathcal{F}h](\alpha) = \begin{cases} Fh(0) & \alpha = 0 \\ 0 & \alpha \in [-a,0) \end{cases} \tag{3-80}$$

Note that $\mathcal{Q}(t)$, $\mathcal{R}(t)$, $\mathcal{F}$ are symmetric positive operators.

Now let $x(\cdot;t,h)$ be the solution of (3-1) with $f = 0$, with initial data h at initial instant $t \in [t_0,T]$ and corresponding to the optimal control u.

Then for $s \in (t,T)$, $x(s;t,h)$ satisfies

$$\begin{aligned} \frac{dx}{ds} &= D_{00}(s)x(s) + \sum_{i=1}^{N} A_i(s)x(s+\theta_i) + \int_{-a}^{0} D_{01}(s,\theta)x(s+\theta)d\theta \\ x(t+\theta) &= h(\theta) \quad \theta \in [-a,0] \quad h \in M^2 \end{aligned} \tag{3-81}$$

$$\text{where} \quad D_{00}(s) = A_{00}(s) - R(s)\Pi_{00}(s) \tag{3-82}$$

$$D_{01}(s,\theta) = A_{01}(s,\theta) - R(s)\Pi_{01}(s,\theta) \tag{3-83}$$

From theorem 2B and (2-12), the solution can be written in the form

$$
\begin{aligned}
x(s;t,h) &= \Phi^{0}(s,t)h(0) \\
&\quad + \sum_{i=1}^{N} \int_{\theta_i}^{\min(0,s-t+\theta_i)} d\alpha \;\; \Phi^{0}(s,t+\alpha-\theta_i)A_i(t+\alpha-\theta_i)h(\alpha) \\
&\quad + \int_{-a}^{0} d\alpha \int_{\max(-a,\alpha-s+t)}^{\alpha} d\beta \;\; \Phi^{0}(s,t+\alpha-\beta)D_{01}(t+\alpha-\beta,\beta)h(\alpha) \qquad (3\text{-}84)
\end{aligned}
$$

or more compactly

$$x(s;t,h) = \Phi^{0}(s,t)h(0) + \int_{-a}^{0} \Phi^{1}(s,t,\alpha)h(\alpha)d\alpha \qquad (3\text{-}85)$$

where
$$
\begin{aligned}
\frac{\partial\Phi^{0}(s,t)}{\partial s} &= D_{00}(s)\Phi^{0}(s,t) + \sum_{i=1}^{N} A_i(s)\Phi^{0}(s+\theta_i,t) \\
&\quad + \int_{-a}^{0} D_{01}(s,\theta)\Phi^{0}(s+\theta,t)d\theta \qquad (3\text{-}86)
\end{aligned}
$$

$$\Phi^{0}(t,t) = I, \quad \Phi^{0}(t+\theta,t) = 0 \qquad \theta \in [-a,0)$$

$$
\begin{aligned}
\Phi^{1}(s,t,\alpha) &= \sum_{i=1}^{N} \begin{cases} \Phi^{0}(s,t+\alpha-\theta_i)A_i(t+\alpha-\theta_i) & \alpha+t-s<\theta_i<\alpha \\ 0 & \text{otherwise} \end{cases} \qquad (3\text{-}87) \\
&\quad + \int_{\max(-a,\alpha-s+t)}^{\alpha} d\beta \;\; \Phi^{0}(s,t+\alpha-\beta)D_{01}(t+\alpha-\beta,\beta)
\end{aligned}
$$

We denote the M^2 solution of (3-81) by $\tilde{x}(s;t,h)$ and we define the two parameter semigroup of operators corresponding to (3-81) by

$$\Phi(s,t)h = \tilde{x}(s;t,h) \tag{3-88}$$

The differential generator of the semigroup of operators $\{\Phi(s,t) \; ; \; s \geq t\}$ (see chapter 2, section 4) is

$$\mathcal{A}(t) - \mathcal{R}(t)\Pi(t) \tag{3-89}$$

and recalling (vii) of theorem 2D, we have

$$\lim_{\delta \downarrow 0} \left[\frac{\Phi(s,t+\delta) - \Phi(s,t)}{\delta}\right]h = - \Phi(s,t)[\mathcal{A}(t) - \mathcal{R}(t)\Pi(t)]h \tag{3-90}$$

where $h \in \mathcal{D}(\mathcal{A}(t) - \mathcal{R}(t)\Pi(t)) = \mathcal{D}(\mathcal{A}(t)) = AC^2(-a,0;R^n)$ i.e. for fixed $s > t$, the right hand derivative in M^2 of $\Phi(s,t)h$ is $- \Phi(s,t)[\mathcal{A}(t) - \mathcal{R}(t)\Pi(t)]h$. Recall that the left hand derivative is meaningless since $\Phi(s,t+\delta)h$ is not defined for $\delta < 0$

$\tilde{x}(s;t,h)$ will satisfy the M^2 differential equation on $[t,T]$

$$\begin{aligned} \frac{d\tilde{x}(s)}{ds} &= [\mathcal{A}(s) - \mathcal{R}(s)\Pi(s)]\tilde{x}(s) \\ \tilde{x}(t) &= h \end{aligned} \tag{3-91}$$

From (i) (3-58) and (iii) (3-60) of lemma 3.3, we have

$$(h,\Pi(t)k)_{M^2} = (x(T;t,h), Fx(T;t,k)) + \int_t^T (x(s;t,h), Q(s)\,x(s;t,k))ds + \int_t^T (\Pi^0(s)\,\tilde{x}(s;t,h), R(s)\Pi^0(s)\,\tilde{x}(s;t,k))ds$$

$$(h,\Pi(t)k)_{M^2} = (\Phi(T,t)h, \mathcal{F}\Phi(T,t)k)_{M^2} + \int_t^T \{(\Phi(s,t)h, \mathcal{Q}(s)\Phi(s,t)k)_{M^2} + (\Pi(s)\Phi(s,t)h, \mathcal{R}(s)\Pi(s)\Phi(s,t)k)_{M^2}\}ds \tag{3-92}$$

Note that (3-92) is an integral operator equation for $\Pi(t)$. By writing it out in full, we obtain the theorem:

Theorem 3C

(i) $\Pi_{00}(T) = F$ (3-93)

The map $t \mapsto \Pi_{00}(t) : [t_0,T] \to \mathcal{L}(R^n)$ (3-94)

is absolutely continuous

(ii) $\Pi_{01}(T,\alpha) = 0$ a.e. $\alpha \in [-a,0]$ (3-95)

For fixed $t \in [t_0,T]$, the map

$\alpha \mapsto \Pi_{01}(t,\alpha) : [-a,0] \to \mathcal{L}(R^n)$ (3-96)

is piecewise absolutely continuous with jumps at

$\alpha = \theta_i \quad i = 1 \ldots N-1$ of magnitude $\Pi_{00}(t)A_i(t)$. For fixed $\alpha \in [-a,0]$, $\alpha \neq \theta_i$ for any $i = 1 \ldots N$

the map $t \mapsto \Pi_{01}(t,\alpha) : [t_0,T] \to \mathcal{L}(R^n)$ (3-97)

is absolutely continuous

(iii) $\Pi_{11}(T,\theta,\alpha) = 0$ a.e. $(\theta,\alpha) \in [-a,0]\times[-a,0]$ (3-98)

For fixed $t \in [t_0,T]$, the map

$$(\theta,\alpha) \mapsto \Pi_{11}(t,\theta,\alpha) : [-a,0] \times [-a,0] \to \mathcal{L}(R^n) \tag{3-99}$$

is piecewise absolutely continuous with jumps at $\alpha = \theta_j$, $j = 1,\ldots N-1$ of magnitude $\Pi^*_{01}(t,\theta)A_j(t)$ and at $\theta = \theta_i \quad i = 1 \ldots N-1$ of magnitude $A^*_i(t)\Pi_{01}(t,\alpha)$. For fixed $(\theta,\alpha) \in [-a,0] \times [-a,0]$, $\theta \neq \theta_i$, $\alpha \neq \theta_j$ for any $i,j = 1 \ldots N$

the map $t \mapsto \Pi_{11}(t,\theta,\alpha) : [t_0,T] \to \mathcal{L}(R^n)$ (3-100)

is absolutely continuous.

Proof See **appendix 1**

Remarks

1) In case $F \neq 0$, we assume, without any loss in generality that $A_i(T) = 0$, $A_i(\cdot)$ **absolutely continuous**, $i=1\ldots N$

2) Note that for $\alpha = -a$, (or $\alpha = \theta_i$ for any $i=1\ldots N-1$).

the map $t \mapsto \Pi_{01}(t,-a) = \Pi_{00}(t)A_N(t)$

will not necessarily be continuous, since $A_N(t)$ does not have to be continuous.

A similar remark holds for the map (3-100).

Theorem 3D

$\Pi(t)$ satisfies the operator Riccati differential equation

$$\begin{aligned}&\frac{d}{dt}(h,\Pi(t)k)_{M^2} + (\mathcal{A}(t)h,\Pi(t)k)_{M^2} + (h,\Pi(t)\mathcal{A}(t)k)_{M^2}\\ &-(h,\Pi(t)\mathcal{R}(t)\Pi(t)k)_{M^2} + (h,\mathcal{Q}(t)k)_{M^2} = 0\end{aligned} \qquad (3\text{-}101)$$

$$\Pi(T) = \mathcal{F}$$

for all $h, k \in \mathcal{D}(\mathcal{A})$ and where the derivative $\frac{d}{dt}$ is taken to be the right hand derivative. Also equation (3-101) has a unique solution.

Proof Taking the right hand derivative with respect to t of both sides of (3-92) and using lemma 2.1 we obtain

$$\begin{aligned}\frac{d}{dt}(h,\Pi(t)k)_{M^2} = & -(\Phi(T,t)[\mathcal{A}(t)-\mathcal{R}(t)\Pi(t)]h,\ \mathcal{F}\,\Phi(T,t)k)_{M^2}\\ & -(\Phi(T,t)h,\ \mathcal{F}\,\Phi(T,t)[\mathcal{A}(t)-\mathcal{R}(t)\Pi(t)]k)_{M^2}\\ & -(h,\ Q(t)k)_{M^2}-(\Pi(t)h,\ \mathcal{R}(t)\Pi(t)k)_{M^2}\\ & -\int_t^T(\Phi(s,t)[\mathcal{A}(t)-\mathcal{R}(t)\Pi(t)]h,\ Q(s)\,\Phi(s,t)k)_{M^2}\,ds\\ & -\int_t^T(\Phi(s,t)h,\ Q(s)\,\Phi(s,t)[\mathcal{A}(t)-\mathcal{R}(t)\Pi(t)]k)_{M^2}\,ds\\ & -\int_t^T(\Pi(s)\,\Phi(s,t)[\mathcal{A}(t)-\mathcal{R}(t)\Pi(t)]h,\ \mathcal{R}(s)\Pi(s)\Phi(s,t)k)_{M^2}\,ds\\ & -\int_t^T(\Pi(s)\,\Phi(s,t)h,\ \mathcal{R}(s)\Pi(s)\,\Phi(s,t)[\mathcal{A}(t)-\mathcal{R}(t)\Pi(t)]k)_{M^2}\,ds\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}(h,\Pi(t)k)_{M^2} = & -(h,\mathcal{Q}(t)k)_{M^2} - (\Pi(t)h,\mathcal{R}(t)\Pi(t)k)_{M^2}\\ & - (h,\Pi(t)[\mathcal{A}(t)-\mathcal{R}(t)\Pi(t)]k)_{M^2}\\ & - ([\mathcal{A}(t)-\mathcal{R}(t)\Pi(t)]h,\Pi(t)k)_{M^2}\\ = & - (h,\mathcal{Q}(t)k)_{M^2} - (\Pi(t)h,\mathcal{R}(t)\Pi(t)k)_{M^2}\\ & - (h,\Pi(t)\mathcal{A}(t)k)_{M^2} + (h,\Pi(t)\mathcal{R}(t)\Pi(t)k)_{M^2}\\ & - (\mathcal{A}(t)h,\Pi(t)k)_{M^2} + (\mathcal{R}(t)\Pi(t)h,\Pi(t)k)_{M^2}\end{aligned}$$

Hence we have (3-101), and $\Pi(T) = \mathcal{F}$ follows immediately from (3-92).

To show that (3-101) has a unique solution, suppose that $\Pi_1(t)$ and $\Pi_2(t)$ are solutions of (3-101), $\Pi_1(t)$, $\Pi_2(t)$ bounded symmetric operators for all $t \in [t_0,T]$.

Let $\Pi_0(t) = \Pi_1(t) - \Pi_2(t)$ (3-102)

Then $\Pi_0(T) = 0$ (3-103)

and $\Pi_0(t)$ will satisfy the operator differential equation

$$\frac{d}{dt}(h,\Pi_0(t)k)_{M^2} + (\mathcal{A}(t)h,\Pi_0(t)k)_{M^2} + (h,\Pi_0(t)\mathcal{A}(t)k)_{M^2}$$

$$- (\Pi_1(t)h,\mathcal{R}(t)\Pi_1(t)k)_{M^2} + (\Pi_2(t)h,\mathcal{R}(t)\Pi_2(t)k)_{M^2} \qquad (3\text{-}104)$$

$$= 0$$

Now let $\Phi_1(s,t)$ and $\Phi_2(s,t)$ be the **semigroups** generated by $\mathcal{A}(t) - \mathcal{R}(t)\Pi_1(t)$ and $\mathcal{A}(t) - \mathcal{R}(t)\Pi_2(t)$ respectively.

Now let us consider

$$\frac{d}{ds}(\Phi_1(s,t)h,\ \Pi_0(s)\Phi_2(s,t)k)_{M^2}$$

where the derivative $\frac{d}{ds}$ is taken from the right and where $s \in [t,T]$

$$
\begin{aligned}
&\frac{d}{ds}(\Phi_1(s,t)h,\Pi_0(s)\Phi_2(s,t)k)_{M^2} \\
&= ([\mathcal{A}(s) - \mathcal{R}(s)\Pi_1(s)],\Phi_1(s,t)h,\Pi_0(s)\Phi_2(s,t)k)_{M^2} \\
&\quad + (\Phi_1(s,t)h,\Pi_0(s)[\mathcal{A}(s)-\mathcal{R}(s)\Pi_2(s)]\Phi_2(s,t)k)_{M^2} \\
&\quad - (\mathcal{A}(s)\Phi_1(s,t)h,\Pi_0(s)\Phi_2(s,t)k)_{M^2} \\
&\quad - (\Phi_1(s,t)h,\Pi_0(s)\mathcal{A}(s)\Phi_2(s,t)k)_{M^2} \\
&\quad + (\Pi_1(s)\Phi_1(s,t)h,\mathcal{R}(s)\Pi_1(s)\Phi_2(s,t)k)_{M^2} \\
&\quad - (\Pi_2(s)\Phi_1(s,t)h,\mathcal{R}(s)\Pi_2(t)\Phi_2(s,t)k)_{M^2} \\
&= - (\mathcal{R}(s)\Pi_1(s)\Phi_1(s,t)h,\Pi_1(s)\Phi_2(s,t)k)_{M^2} \\
&\quad + (\mathcal{R}(s)\Pi_1(s)\Phi_1(s,t)h,\Pi_2(s)\Phi_2(s,t)k)_{M^2} \\
&\quad - (\Phi_1(s,t)h,\Pi_1(s)\mathcal{R}(s)\Pi_2(s)\Phi_2(s,t)k)_{M^2} \\
&\quad + (\Phi_1(s,t)h,\Pi_2(s)\mathcal{R}(s)\Pi_2(s)\Phi_2(s,t)k)_{M^2} \\
&\quad + (\Pi_1(s)\Phi_1(s,t)h,\mathcal{R}(s)\Pi_1(s)\Phi_2(s,t)k)_{M^2} \\
&\quad - (\Pi_2(s)\Phi_1(s,t)h,\mathcal{R}(s)\Pi_2(t)\Phi_2(s,t)k)_{M^2} \\
&= 0
\end{aligned}
$$

Hence $(\Phi_1(s,t)h,\Pi_0(s)\Phi_2(s,t)k)_{M^2}$ is constant for $s \in [t,T]$

Hence $(\Phi_1(t,t)h,\Pi_0(t)\Phi_2(t,t)k)_{M^2} = (h,\Pi_0(t)k)_{M^2}$

$$= (\Phi_1(T,t)h,\Pi_0(T)\Phi_2(T,t)k)_{M^2}$$

$$= 0 \ , \quad \text{since} \quad \Pi_0(T) = 0$$

Hence $(h,\Pi_0(t)k)_{M^2} = 0$ for all $h,k \in AC^2(-a,0;R^n)$ a dense subset of $M^2(-a,0;R^n)$.

Hence $\Pi_0(t) = 0$

But $t \in [t_0,T]$ is arbitrary

Hence $\Pi_0(t) = 0$ for all $t \in [t_0,T]$ and hence we have uniqueness of a solution to (3-101).

Q.E.D.

Writing out (3-101) in full, we can establish

Theorem 3E

(i) $\Pi_{00}(t)$ satisfies the differential equation

$$\frac{d\Pi_{00}(t)}{dt} + A_{00}^*(t)\Pi_{00}(t) + \Pi_{00}(t)A_{00}(t) - \Pi_{00}(t)R(t)\Pi_{00}(t) + \Pi_{01}^*(t,0) + \Pi_{01}(t,0) + Q(t) = 0 \qquad (3\text{-}105)$$

a.e. in $[t_0,T]$

$\Pi_{00}(T) = F$

(ii) $\Pi_{01}(t,\alpha)$ satisfies the differential equation

$$[\frac{\partial}{\partial t} - \frac{\partial}{\partial \alpha}]\Pi_{01}(t,\alpha) + A_{00}^{*}(t)\Pi_{01}(t,\alpha) - \Pi_{00}(t)R(t)\Pi_{01}(t,\alpha)$$

$$+ \Pi_{00}(t)A_{01}(t,\alpha) + \Pi_{11}(t,0,\alpha) = 0$$

$$\text{a.e. in } [t_0,T] \times \bigcup_{i=0}^{N-1} (\theta_{i+1},\theta_i) \qquad (3\text{-}106)$$

$\Pi_{01}(T,\alpha) = 0$ a.e. $\alpha \varepsilon [-a,0]$, $\Pi_{00}(t,-a) = \Pi_{00}(t)A_N(t)$

and $\Pi_{01}(t,\alpha)$ has jumps at $\alpha = \theta_i$ $i=1 \ldots N-1$ of magnitude $\Pi_{00}(t)A_i(t)$

(iii) $\Pi_{11}(t,\theta,\alpha)$ satisfies the differential equation

$$[\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \alpha}]\Pi_{11}(t,\theta,\alpha) + \Pi_{01}^{*}(t,\theta)A_{01}(t,\alpha)$$

$$+ A_{01}^{*}(t,\theta)\Pi_{01}(t,\alpha) - \Pi_{01}^{*}(t,\theta)R(t)\Pi_{01}(t,\alpha) = 0$$

$$\text{a.e. in } [t_0,T] \times \bigcup_{i=0}^{N-1} (\theta_{i+1},\theta_i) \times \bigcup_{i=0}^{N-1} (\theta_{i+1},\theta_i) \qquad (3\text{-}107)$$

$\Pi_{11}(T,\theta,\alpha) = 0$ a.e. $(\theta,\alpha) \varepsilon [-a,0] \times [-a,0]$

$\Pi_{11}(t,-a,\alpha) = A_N^{*}(t)\Pi_{01}(t,\alpha)$, $\Pi_{11}(t,\theta,-a) = \Pi_{01}^{*}(t,\theta)A_N(t)$

$\Pi_{11}(t,\theta,\alpha)$ has jumps at $\alpha = \theta_j$, $j = 1 \ldots N-1$ of magnitude $\Pi_{01}^{*}(t,\theta)A_j(t)$ and at $\theta = \theta_i$ $i = 1 \ldots N-1$ of magnitude $A_i^{*}(t)\Pi_{01}(t,\alpha)$.

Proof See appendix 2

Remarks

1. Equations (3-105), (3-106), (3-107) give a coupled set of Riccati type first order partial differential equation for the entities $\Pi_{00}(t)$, $\Pi_{01}(t,\alpha)$, $\Pi_{11}(t,\theta,\alpha)$ which appeared in the feedback form of the optimal control (3-76) and in an expression for the optimal cost to go.

2. In the dynamic programming approach, the existence and uniqueness of an optimal control is made to depend on the existence and uniqueness of the equations (3-105), (3-106), (3-107). Our approach is different. We first prove the existence and uniqueness of an optimal control. We then exhibit an M^2 operator $\Pi(t)$ (thereby disposing of the problem of existence) and the related matrix functions $\Pi_{00}(t)$, $\Pi_{01}(t,\alpha)$, $\Pi_{11}(t,\theta,\alpha)$. We show that $\Pi(t)$ satisfies an operator Riccati differential equation (3-101) and that it is the unique solution. From (3-101), we deduce the coupled set of Riccati type partial differential equations (3-105), (3-106), (3-107) satisfied by $\Pi_{00}(t)$, $\Pi_{01}(t,\alpha)$, $\Pi_{11}(t,\theta,\alpha)$ and since (3-101) has the unique solution $\Pi(t)$, equations (3-105), (3-106), (3-107) must have the unique solution $\Pi_{00}(t)$, $\Pi_{01}(t,\alpha)$, $\Pi_{11}(t,\theta,\alpha)$.

3. The optimal control problem (3-1), (3-2), (3-4) will be completely solved once we have the solutions to equations (3-105), (3-106) and (3-107). However solving those equations is no easy business and we postpone further discussion until next chapter when we obtain a slightly simpler version of (3-105), (3-106), (3-107) which however is still very difficult - if not impossible - to solve.

3.6 Finite time tracking problem

Denote by $\phi(\cdot;s,h)$ and $\psi(\cdot;s,h)$ the solution of (3-37) and (3-38) with forcing term $f \neq 0$. From corollary 2 to lemma 3.2 and corollary to lemma 3.4

$$\psi(t;s,k) = \Pi^0(t)\tilde{\phi}(t;s,k) + d(t) \tag{3-108}$$

Hence $\phi(\cdot;s,h)$ satisfies the R.F.D.E.

$$\frac{d\phi}{dt} = D_{00}(t)\phi(t) + \sum_{i=1}^{N} A_i(t)\phi(t+\theta_i) + \int_{-a}^{0} d\theta\, D_{01}(t,\theta)\phi(t+\theta)$$

$$- R(t)d(t) + f(t) \tag{3-109}$$

$$\phi(s+\theta) = h(\theta) \quad \theta \in [-a,0],\ h \in M^2$$

where $D_{00}(t)$, $D_{01}(t,\theta)$ are as defined in (3-82), (3-83) respectively.

Hence (3-109) will have solution

$$\phi(t;s,h) = \phi_0(t;s,h) + \int_s^t \Phi^0(t,t_1)f_0(t_1)dt_1 \tag{3-110}$$

where $f_0(t_1) = f(t_1) - R(t_1)d(t_1)$ (3-111)

$\Phi^0(t,t_1)$ is as defined in (3-86)

and $\phi_0(t;s,h)$ is the solution of (3-37) with $f = 0$.

Now define $\tilde{d}(t)$, $\tilde{f}_0(t) \in M^2$ by

$$[\tilde{d}(t)](\alpha) = \begin{cases} d(t) & \alpha = 0 \\ 0 & \alpha \in [-a,0) \end{cases} \tag{3-112}$$

$$[\tilde{f}_0(t)](\alpha) = \begin{cases} f_0(t) & \alpha = 0 \\ 0 & \alpha \in [-a,0) \end{cases} \tag{3-113}$$

It is clear that $\tilde{f}_0(t) = \tilde{f}(t) - \mathcal{R}(t)\tilde{d}(t)$ (3-114)

We can write (3-110) in the M^2 form

$$\tilde{\phi}(t;s,h) = \tilde{\phi}_0(t;s,h) + \int_s^t \Phi(t,t_1)\tilde{f}_0(t_1)dt_1 \tag{3-115}$$

where $\Phi(t,t_1)$ is as defined in (3-88)
(3-108) can be rewritten

$$\psi(t;s,k) = \Pi^0(t)\{\tilde{\phi}_0(t;s,k) + \int_s^t \Phi(t,t_1)\tilde{f}_0(t_1)dt_1\} + d(t) \qquad (3\text{-}116)$$

Lemma 3.5

(i)
$$\begin{aligned}\mathcal{H}_T(s, \tilde{\psi}(s;s,k),h) = {} & (\phi(T;s,h), F\phi(T;s,k)) \\ & + \int_s^T (\psi(t;s,h), R(t)\,\psi(t;s,k))dt \\ & + \int_s^T (\phi(t;s,h), Q(t)\,\phi(t;s,k))dt \\ & - \int_s^T (\psi(t;s,k), f(t))\,dt \qquad (3\text{-}116)\end{aligned}$$

(ii)
$$\begin{aligned}\mathcal{H}_T(s,\tilde{\psi}(s;s,h),h) = {} & \inf_{v\,\varepsilon\,\mathcal{U}_s} C_s(v;h) \\ & - \int_s^T (\psi(t;s,h),f(t)dt \qquad (3\text{-}117)\end{aligned}$$

Proof

(i) From theorem 2E, equation (2-64) we have

$$\mathcal{H}_T(T,\tilde{\psi}(T;s,k),\tilde{\phi}(T;s,h))-\mathcal{H}_T(s,\tilde{\psi}(s,s,k),h)$$

$$= -\int_s^T \{(\psi(t;s,h),R(t)\psi(t;s,k))+(\phi(t;s,h),Q(t)\phi(t;s,k))\}dt$$

$$+ \int_s^T (\psi(t;s,k),f(t))dt$$

from which the proof follows.

(ii) follows from observing that

$$\inf_{v \in \mathcal{U}_s} C_s(v;h) = (\phi(T;s,h),F\phi(T;s,h))$$

$$+ \int_s^T \{(\psi(t;s,h),R(t)\psi(t;s,h))+(\phi(t;s,h),Q(t)\phi(t;s,h))\}dt$$

Q.E.D.

Hence

$$\mathcal{H}_T(s,\tilde{\psi}(s;s,k),h)$$

$$= (\tilde{\phi}(T;s,h),\mathcal{F}\tilde{\phi}(T;s,k))_{M^2}$$

$$+ \int_s^T \{(\tilde{\psi}(t;s,h),\mathcal{R}(t)\tilde{\psi}(t;s,k))_{M^2}+(\tilde{\phi}(t;s,h),\mathcal{Q}(t)\tilde{\phi}(t;s,k))_{M^2}\}dt$$

$$- \int_s^T (\psi(t;s,k),f(t)dt$$

$$= (\tilde{\phi}_0(T;s,h), \mathcal{F}\tilde{\phi}_0(T;s,k))_{M^2}$$

$$+ \int_s^T (\tilde{\psi}_0(t;s,h), \mathcal{R}(t)\tilde{\psi}_0(t;s,k)_{M^2}dt$$

$$+ \int_s^T (\tilde{\phi}_0(t;s,h), \mathcal{Q}(t)\tilde{\phi}_0(t;s,k))_{M^2}dt$$

$$+ \int_s^T (\tilde{\phi}_0(T;s,h), \mathcal{F}\Phi(T,t_1)\tilde{f}_0(t_1))_{M^2} dt_1$$

$$+ \int_s^T (\tilde{\psi}_0(t;s,h), \mathcal{R}(t)\tilde{d}(t))_{M^2} dt$$

$$+ \int_s^T dt \int_s^t dt_1 (\tilde{\psi}_0(t;s,h), \mathcal{R}(t)\Pi(t)\Phi(t,t_1)\tilde{f}_0(t_1))_{M^2}$$

$$+ \int_s^T dt \int_s^t dt_1 (\tilde{\phi}_0(t;s,h), \mathcal{Q}(t)\Phi(t,t_1)\tilde{f}_0(t_1))_{M^2}$$

plus four expressions as above with h replaced by k

$$+ \int_s^T dt_1 \int_s^T dt_2 (\Phi(T,t_1)\tilde{f}_0(t_1), \mathcal{F}\Phi(T,t_2)\tilde{f}_0(t_2))_{M^2}$$

$$+ 2 \int_s^T dt \int_s^t dt_1 (\tilde{d}(t), \mathcal{R}(t)\Pi(t)\Phi(t,t_1)\tilde{f}_0(t_1))_{M^2}$$

$$+ \int_s^T dt \int_s^t dt_1 \int_s^t dt_2 (\Pi(t)\Phi(t,t_1)\tilde{f}_0(t_1), \mathcal{R}(t)\Pi(t)\Phi(t,t_2)\tilde{f}_0(t_2))_{M^2}$$

$$+ \int_s^T dt \int_s^t dt_1 \int_s^t dt_2 (\Phi(t,t_1)\tilde{f}_0(t_1), \mathcal{Q}(t)\Phi(t,t_2)\tilde{f}_0(t_2))_{M^2}$$

$$- \int_s^T (\psi(t;s,k), f(t)) dt$$

Definition

Define an element $\tilde{g}(s) \varepsilon M^2$ by

$$
\begin{aligned}
(\tilde{g}(s),h)_{M^2} = {} & \int_s^T (\tilde{\phi}_0(T;s,h), \mathfrak{F}\Phi(T,t\ )f_0(t\ ))_{M^2}dt \\
& + \int_s^T (\tilde{\psi}_0(t;s,h), \mathcal{R}(t)\tilde{d}(t))_{M^2}dt \\
& + \int_s^T dt \int_s^t dt_1(\tilde{\psi}_0(t;s,h), \mathcal{R}(t)\Pi(t)\Phi(t,t_1)f_0(t_1))_{M^2} \\
& + \int_s^T dt \int_s^t dt_1(\tilde{\phi}_0(t;s,h), \mathcal{Q}(t)\Phi(t,t_1)\tilde{f}_0(t_1))_{M^2} \\
= {} & \int_s^T (\Phi(T,s)h, \mathfrak{F}\Phi(T,t\ )\tilde{f}_0(t\ ))_{M^2}dt \\
& + \int_s^T (\Pi(t)\Phi(t,s)h, \mathcal{R}(t)\tilde{d}(t))_{M^2}dt \\
& + \int_s^T dt \int_s^t dt_1(\Pi(t)\Phi(t,s)h, \mathcal{R}(t)\Pi(t)\Phi(t,t_1)\tilde{f}_0(t_1))_{M^2} \\
& + \int_s^T dt \int_s^t dt_1(\Phi(t,s)h, \mathcal{Q}(t)\Phi(t,t_1)\tilde{f}_0(t_1))_{M^2} \qquad (3\text{-}118)
\end{aligned}
$$

$\tilde{g}(s)$ is well defined from the Riez representation theorem, since the left hand side(s) of (3-118) is a continuous linear functional on M^2.

Using the isometric isomorphic between M^2 and $R^n \times L^2$ we can write

$$\tilde{g}(s) = (g_0(s), g_1(s,\cdot)) \tag{3-119}$$

where $g_0(s) \varepsilon R^n$ and $g_1(s,\cdot) \varepsilon L^2(-a,0;R^n)$.

Similarly, we can define $(\tilde{g}(s),k)_{M^2}$

Define the scalar function

$$\begin{aligned} c(s) = {} & \int_s^T dt_1 \int_s^T dt_2 (\Phi(T,t_1)\tilde{f}_0(t_1), \mathcal{F}\Phi(T,t_2)\tilde{f}_0(t_2))_{M^2} \\ & + 2\int_s^T dt \int_s^T dt_1 (\tilde{d}(t), \mathcal{R}(t)\Pi(t)\Phi(t,t_1)\tilde{f}_0(t_1))_{M^2} \\ & + \int_s^T dt \int_s^t dt_1 \int_s^t dt_2 (\Pi(t)\Phi(t,t_1)\tilde{f}_0(t_1), \mathcal{R}(t)\Pi(t)\Phi(t,t_2)\tilde{f}_0(t_2))_{M^2} \\ & + \int_s^T dt \int_s^t dt_1 \int_s^t dt_2 (\Phi(t,t_1)\tilde{f}_0(t_1), \mathcal{Q}(t)\Phi(t,t_2)f_0(t_2))_{M^2} \end{aligned} \tag{3-120}$$

Lemma 3.6

(i) $$\min_{v \,\varepsilon\, \mathcal{U}_t} C_t(v;h) = (h,\Pi(t)h)_{M^2} + 2(\tilde{g}(t),h)_{M^2} + c(t) \tag{3-121}$$

(ii) $$g_0(t) = d(t) \tag{3-122}$$

for $t \,\varepsilon\, [t_0,T]$

Proof

(i) This follows immediately from (ii) of lemma 3.5 equation (3-117) and the definitions of $\tilde{g}(t)$ and $c(t)$

(ii) We exploit the relationship

$$\mathcal{H}_T(t,\tilde{\psi}(t;t,k),h) = (h,\Pi(t)k)_{M^2} + (\tilde{g}(t),h)_{M^2} + (\tilde{g}(t),k)_{M^2} + c(t) - \int_t^T (\psi(s;t,k),f(s))ds \qquad (3\text{-}123)$$

which follows from the definitions.

Hence

$$(h(0),\Pi_{00}(t)k(0) + \int_{-a}^{0}(h(0),\Pi_{01}(t,\alpha)k(\alpha)) + \int_{-a}^{0}(h(\theta),\Pi_{10}(t,\theta)k(0))d\theta$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha(h(\theta),\Pi_{11}(t,\theta,\alpha)k(\alpha)) + (g_0(t),h(0))$$

$$+ \int_{-a}^{0}(g_1(t,\theta),h(\theta)d\theta + (g_0(t),k(0)) + \int_{-a}^{0}(g_1(t,\alpha),k(\alpha))d\alpha + c(t)$$

$$- \int_{t}^{T}(\psi(s;t;k),f(s))ds = (P_0(t)k(0),h(0) + (d(t),h(0))$$

$$+ \int_{-a}^{0} (P_1(s,\alpha)k(\alpha),h(0))d\alpha +$$

$$+ \sum_{i=1}^{N} \int_{t}^{t-\theta_i} ds(P_0(s,t)k(0), A_{01}(s,\theta)h(s-t+\theta))$$

$$+ \int_{-a}^{0} d\theta \int_{t}^{t-\theta} ds(d(s,t), A_{01}(s,\theta)h(s-t+\theta))$$

$$+ \int_{-a}^{0} d\theta \int_{t}^{t-\theta} ds \int_{-a}^{0} d\alpha(P_1(s,t,\alpha)k(\alpha), A_{01}(s,\theta)h(s-t+\theta))$$

Put $k(0) = 0$, $h^1 = k^1 = 0$, $f = 0$. $f = 0 \Rightarrow d(t) = 0$

$d(s,t) = 0$ and $c(t) = 0$

Hence $(d(t),h(0) = (g_0(t),h(0))$ for all $h(0) \varepsilon R^n$.

Hence result.

Hence we have proved

Theorem 3F

The optimal control to control problem (3-1), (3-2), (3-4) can be expressed in feedback form

$$u(t) = -N^{-1}(t)B^*(t)\{\Pi_{00}(t)x(t) + \int_{-a}^{0} \Pi_{01}(t,\alpha)x(t+\alpha)d\alpha + g_0(t)\} \tag{3-124}$$

and the optimal cost to go is expressed in the form

$$\inf_{v \in \mathcal{U}_t} \mathcal{C}_t(v;h) = (h(0),\Pi_{00}(t)h(0))$$

$$+ 2 \int_{-a}^{0} (h(0),\Pi_{01}(t,\alpha)h(\alpha))d\alpha$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha(h(\theta),\Pi_{11}(t,\theta,\alpha)k(\alpha)) + 2(g_0(t),h) \tag{3-125}$$

$$+ 2 \int_{-a}^{0} (g_1(t,\alpha),h(\alpha))d\alpha + c(t)$$

We have already given a full description for $\Pi(t)$. We now want to do the same for $\tilde{g}(t)$.

From the definition (3-119) of $\tilde{g}(t)$ and (ii) of lemma (3.6) equation (3-122) we have that

$$\mathcal{R}(t)\tilde{d}(t) = \mathcal{R}(t)\tilde{g}(t) \tag{3-126}$$

and hence we have an integral equation for $\tilde{g}(t)$:

$$(\tilde{g}(t),h)_{M^2} = \int_t^T ds(\Phi(T,t)h, \mathcal{F}\Phi(T,s)[\tilde{f}(s) - \mathcal{R}(s)\tilde{g}(s)])_{M^2}$$

$$+ \int_s^T ds(\Pi(s)\Phi(s,t)h, \mathcal{R}(s)\tilde{g}(s))_{M^2} +$$

$$+ \int_s^T ds \int_t^s ds_1 (\Pi(s)\Phi(s,t)h, \mathcal{R}(s)\Pi(s)\Phi(s,s_1)[\tilde{f}(s_1) - \mathcal{R}(s_1)\tilde{g}(s_1)])_{M^2}$$

$$+ \int_t^T ds \int_t^s ds_1 (\Phi(s,t)h, \mathcal{Q}(s)\Phi(s,s_1)[\tilde{f}(s_1) - \mathcal{R}(s_1)\tilde{g}(s_1)])_{M^2}$$

(3-127)

Writing out equation (3-127) in full, we have

Theorem 3G

(i) The map $t \mapsto g_0(t) : [t_0,T] \to R^n$ (3-128)

is absolutely continuous

and $g(T) = 0$ (3-129)

(ii) For fixed $t \in [t_0,T]$,

the map $\theta \mapsto g_1(t,\theta) : [-a,0] \to R^n$ (3-130)

is piecewise absolutely continuous,

with jumps at $\theta = \theta_i$ $i = 1 \dots N-1$ of magnitude $A_i^*(t)g_0(t)$

For fixed $\theta \in [-a,0]$, $\theta \neq \theta_i$ $i = 1 \dots N$

the map $t \mapsto g_1(t,\theta) : [t_0,T] \to R^n$ (3-131)

is absolutely continuous.

Proof See appendix 3

Theorem 3H

$\tilde{g}(t)$ satisfies the differential equation

$$\frac{d}{dt}(\tilde{g}(t),h)_{M^2} + (\tilde{g}(t),[\mathcal{A}(t)-\mathcal{R}(t)\Pi(t)]h)_{M^2} + (\Pi(t)\tilde{f}(t),h)_{M^2} = 0 \tag{3-132}$$

for all $h \in \mathcal{D}(\mathcal{A})$ and where the derivative $\frac{d}{dt}$ is taken to be the right hand derivative.

Proof Taking the right hand derivative with respect to t of both sides of equation (3-127) we obtain

$$\begin{aligned}
\frac{d}{dt}(\tilde{g}(t),h)_{M^2} = &- (\Phi(T,t)h, \mathcal{F}\Phi(T,t)[\tilde{f}(t)-\mathcal{R}(t)\tilde{g}(t)])_{M^2} \\
&- \int_t^T ds\,(\Phi(T,t)[\mathcal{A}(t)-\mathcal{R}(t)\Pi(t)]h, \mathcal{F}\Phi(T,s)[\tilde{f}(s)-\mathcal{R}(s)g(s)])_{M^2} \\
&- (\Pi(t)h, \mathcal{R}(t)\tilde{g}(t))_{M^2} \\
&- \int_t^T ds\,(\Pi(s)\Phi(s,t)[\mathcal{A}(t)-\mathcal{R}(t)\Pi(t)]h, \mathcal{R}(s)\tilde{g}(s))_{M^2} \\
&- \int_t^T ds\,(\Pi(s)\Phi(s,t)h, \mathcal{R}(s)\Pi(s)\Phi(s,t)[\tilde{f}(t)-\mathcal{R}(t)\tilde{g}(t)])_{M^2} \\
&- \int_t^T ds \int_t^s ds_1\,(\Pi(s)\Phi(s,t)[\mathcal{A}(t)-\mathcal{R}(t)\Pi(t)]h, \mathcal{R}(s)\Pi(s)\Phi(s,s_1)[\tilde{f}(s_1)-\mathcal{R}(s_1)\tilde{g}(s_1)])_{M^2} \\
&- \int_t^T ds\,(\Phi(s,t)h, \mathcal{Q}(s)\Phi(s,t)[\tilde{f}(t)-\mathcal{R}(t)\tilde{g}(t)])_{M^2}
\end{aligned}$$

$$- \int_t^T ds \int_t^s ds_1 \left(\Phi(s,t) \mathcal{A}(t) - \mathcal{R}(t) \Pi(t) h,\ \mathcal{Q}(s) \Phi(s,s_1) [\tilde{f}(s_1) - \mathcal{R}(s_1) \tilde{g}(s_1)] \right)_{M^2}$$

$$= - \left(h, \Pi(t) [f(t) - \mathcal{R}(t) \tilde{g}(t)]\right)_{M^2} - \left(\Pi(t) h,\ \mathcal{R}(t) \tilde{g}(t)\right)_{M^2}$$

$$- \left(\tilde{g}(t), [\mathcal{A}(t) - \mathcal{R}(t) \Pi(t)] h\right)_{M^2}$$

$$= - \left(\Pi(t) \tilde{f}(t), h\right)_{M^2} - \left(\tilde{g}(t), [\mathcal{A}(t) - \mathcal{R}(t) \Pi(t)] h\right)_{M^2}$$

Q.E.D.

Hence result.

Writing out (3-132) in full, we establish

Theorem 31

(i) $g_0(t)$ satisfies the differential equation

$$\frac{dg_0(t)}{dt} + A^*_{00}(t) g_0(t) - \Pi_{00}(t) R(t) g_0(t) + \Pi_{00}(t) f(t) + g_1(t,0)$$

$$= 0 \tag{3-133}$$

a.e. in $[t_0, T]$

$g(T) = 0$

(ii) $g_1(t,\theta)$ satisfies the differential equation

$$[\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta}]g_1(t,\theta) + A_{01}^*(t,\theta)g_0(t) - \Pi_{01}^*(t,\theta)R(t)g_1(t,\theta)$$

$$+ \Pi_{01}^*(t,\theta)f(t) = 0 \qquad (3\text{-}134)$$

a.e. in $[t_0,T] \times \bigcup_{i=0}^{N} (\theta_{i+1},\theta_i)$

$g_1(T,\theta) = 0$ a.e. $\theta \varepsilon [-a,0]$; $g_1(t,-a) = A_N^*(t)g_0(t)$

and $g_1(t,\theta)$ has jumps at $\theta = \theta_i$, $i = 1 \ldots N-1$ of magnitude $A_i^*(t)g_0(t)$

Proof See appendix 4

Remarks

1. Equations (3-133), (3-134) along with equations (3-105), (3-106), (3-107) gives a complete characterization of the entities $\Pi_{00}(t)$, $\Pi_{01}(t,\alpha)$, $\Pi_{11}(t,\theta,\alpha)$, $g_0(t)$, $g_1(t,\theta)$ appearing in the optimal control feedback form (3-124) and the optimal cost to go (3-125).

2. Notice the resemblance between the solution of the optimal control problem for R.F.D.E. and that for

ordinary linear differential equations. Equations (3-101) and (3-132) would be exactly the same as in the ordinary differential equation solution where instead of the M^2 operator function $\Pi(t)$ and the M^2 function $\tilde{g}(t)$, we would have a R^n matrix function and a R^n function. This resemblance will become stronger in chapter 5.

VOLUME 2

Clement A. W. McCalla

Chapter 4

Infinite Time Regulator Problem for Autonomous R.F.D.E.

The solution of the infinite time regulator problem for linear autonomous ordinary differential equations is well known. (See, for example, Kalman [41] and Lee and Markus [53]). One of the conditions that ensures that the problem is well posed is that the control system is stabilizable, i.e. there is a constant feedback matrix such that the resulting closed loop system has all its eigenvalues strictly in the left half of the complex plane. The stabilizability of the system guarantees that an optimal control exists. This can be expressed in feedback form as a constant matrix operating on the state of the system, and it can be shown that this matrix satisfies a matrix Riccati equation.

The infinite time regulator problem for certain classes of infinite dimensional systems has been studied in recent times: Lions [83] has examined the problem for partial differential equations of parabolic type and Lukes and Russell [85] have tackled the problem for linear differential equations in Hilbert space. The first attempts at the quadratic criterion optimal control problem for hereditary systems, Krasovskii [46], Ross and Flügge-Lotz [69], dealt with the infinite time regulator problem. Recent attempts include Delfour, McCalla and Mitter [22] whose approach is closely analogous to that for linear autonomous ordinary differential equations outlined above and whose treatment is as complete. We shall stick closely to the treatment of Delfour, McCalla and Mitter [22] and in so doing, we shall interweave results, concepts and techniques from the work of Lions [83], Delfour and Mitter [18], [21] and Datko [14].

(4.1) Formulation of the control problem

Consider the controlled hereditary system on $[0,\infty)$

$$\frac{dx}{dt} = A_{00}x(t) + \sum_{i=1}^{N} A_i x(t+\theta_i) + \int_{-a}^{0} A_{01}(\theta)x(t+\theta)d\theta + Bv(t)$$

$$x(\theta) = h(\theta), \quad \theta \in [-a,0], \quad h \in M^2 \tag{4-1}$$

where A_{00}, A_i, $(i = 1 \dots N) \in \mathcal{L}(R^n)$, $B \in \mathcal{L}(R^m, R^n)$, $A_{01}(\cdot) \in L^2(-a,0;\mathcal{L}(R^n))$ with quadratic cost functional

$$C(v;h) = C(v) = \int_0^\infty \{(x(t),Qx(t)) + (v(t),Nv(t))\}dt \tag{4-2}$$

where $Q = Q^* \geq 0$, $N = N^* > 0$

with admissible class of controls

$$\mathcal{U} = \{v;\ \int_0^\infty |v(t)|^2 dt < \infty,\quad C(v) < \infty\} \tag{4-3}$$

Our objective is to find

$$\inf_{v \in \mathcal{U}} C(v;h)$$

which will be called the optimal cost and a $u \in \mathcal{U}$ such that

$$C(u;h) = \inf_{v \in \mathcal{U}} C(v;h) \leq C(v;h) \quad \text{for all} \quad v \in \mathcal{U}$$

Such a u will be called the optimal control.

For the problem to be well posed, $\mathcal{U}$ has to be nonempty. We shall show that if the controlled hereditary system is stabilizable in a sense to be defined later, then $\mathcal{U}$ is nonempty, and an optimal control $u \in \mathcal{U}$ exists.

4.2 L^2-stability; Stabilizability

The uncontrolled hereditary system

$$\frac{dx}{dt} = A_{00}x(t) + \sum_{i=1}^{N} A_i x(t+\theta_i) + \int_{-a}^{0} A_{01}(\theta)x(t+\theta)d\theta$$

$$x(\theta) = h(\theta), \quad \theta \in [-a,0], \quad h \in M^2 \tag{4-4}$$

will have R^n solution denoted by $x(\cdot;h)$

M^2 solution denoted by $\tilde{x}(\cdot;h)$

and will rise to a C_0 semigroup of M^2 operators $\{\Phi(t), t \geq 0\}$ where

$$\Phi(t)h = \tilde{x}(t;h) \tag{4-5}$$

with differential generator $\mathcal{A}$ defined by

$$[\mathcal{A}h](\alpha) = \begin{cases} A_{00}h(0) + \sum_{i=1}^{N} A_i h(\theta_i) + \int_{-a}^{0} A_{01}(\theta)h(\theta)d\theta & \alpha = 0 \\ \frac{dh}{d\alpha} & \alpha \in [-a,0) \end{cases}$$

where $h \in \mathcal{D}(\mathcal{A}) = AC^2(-a,0;R^n)$

Lemma 4.1 Datko [14]

$$\int_0^\infty ||\Phi(t)h||^2_{M^2}dt < \infty \Rightarrow \lim_{t\to\infty} ||\Phi(t)h||_{M^2} = 0 \tag{4-7}$$

Proof $\int_0^\infty ||\Phi(t)h||^2_{M^2}dt < \infty \Rightarrow \lim_{t \to \infty} \inf ||\Phi(t)h||_{M^2} = 0$

We want to show that $\lim_{t \to \infty} \sup ||\Phi(t)h||_{M^2} = 0$

So assume that $\lim_{t \to \infty} \sup ||\Phi(t)h||_{M^2} > 0$

Since $\Phi(t)$ is a C_0 semigroup, there exists constants $K > 1$ and $\omega > 0$ such that

$$||\Phi(t)|| \leq Ke^{\omega t} \quad \text{for all} \quad t \geq 0 \tag{4-7}$$

(see Dunford and Schwartz [26] pp. 619)

Since $||\Phi(t)h||_{M^2}$ is continuous, and

$\lim_{t \to \infty} \inf ||\Phi(t)h||_{M^2} = 0$ we can find a constant $c > 0$ and a sequence of disjoint closed intervals $[a_i, b_i]$ $i = 1,2\cdots$ such that for each i

(i) $||\Phi(a_i)h||_{M^2} = Kc||h||_{M^2}$

(ii) $Kc||h||_{M^2} \leq ||\Phi(t)h||_{M^2} \leq 2K^2c||h||_{M^2}$

for $t \in (a_i, b_i)$

(iii) $||\Phi(b_i)h||_{M^2} = 2K^2c||h||_{M^2}$

Hence for each i,

$$2K^2c||h||_{M^2} = ||\Phi(b_i)h||_{M^2} = ||\Phi(b_i-a_i)\Phi(a_i)h||_{M^2}$$

$$\leq Ke^{\omega(b_i-a_i)}Kc||h||_{M^2}$$

Hence $0 < \frac{1}{\omega}\log 2 < (b_i - a_i)$ for each i

Hence $\infty = K^2c^2||h||^2_{M^2}\frac{1}{\omega}\log 2 \lim_{i\to\infty} i$

$$\leq K^2c^2||h||_{M^2}\sum_{i=1}^{\infty}(b_i-a_i)$$

$$\leq \int_0^\infty ||\Phi(t)h||^2_{M^2}dt < \infty$$

Contradiction

Hence

$$\limsup_{t\to\infty} ||\Phi(t)h||_{M^2} = \liminf_{t\to\infty} ||\Phi(t)h||_{M^2}$$

$$= \lim_{t\to\infty} ||\Phi(t)h||_{M^2} = 0 \qquad \text{Q.E.D.}$$

Lemma 4.2

$$\int_0^\infty ||\tilde{x}(t;h)||^2_{M^2}dt = \int_0^\infty ||\Phi(t)h||^2_{M^2}dt < \infty \quad \text{for all} \quad h \in M^2 \qquad (4\text{-}8)$$

$$<=> \int_0^\infty |x(t;h)|^2 dt < \infty \quad \text{for all} \quad h \in M^2 \qquad (4\text{-}9)$$

Proof $\quad ||\tilde{x}(t;h)||_{M^2} = |x(t;h)|^2 + \int_{-a}^0 |x(t+\theta;h)|^2 d\theta$

Hence (4-8) => (4-9), since

$$\int_0^\infty |x(t;h)|^2 dt < \int_0^\infty ||\tilde{x}(t;h)||_{M^2}dt < \infty$$

Now $\int_a^T ||\tilde{x}(t;h)||^2_{M^2}dt = \int_a^T |x(t;h)|^2 dt + \int_a^T dt \int_{-a}^0 d\theta |x(t+\theta;h)|^2$

$$= \int_a^T |x(t;h)|^2 dt + \int_{-a}^0 d\theta \int_{a+\theta}^{T+\theta} ds |x(s;h)|^2$$

interchanging order of integration

by Fubini

$$\leq (1+a) \int_0^T |x(t;h)|^2 dt$$

$$\leq (1+a) \int_0^\infty |x(t;h)|^2 dt < \infty$$

Hence (4-9) => (4-8).

Definition

(i) The uncontrolled hereditary system (4-4) is said to be L^2-stable if (4-8) holds.

(ii) If the uncontrolled hereditary system (4-4) is L^2-stable, then $\mathcal{A}$ is said to be a stable differential operator.

(iii) A sequence of M^2 operators $\{\Lambda_n\}$ is said to be monotonic increasing if

$$(h,\Lambda_n h)_{M^2} \leq (h,\Lambda_m h)_{M^2} \qquad (4\text{-}10)$$

for all $h \in M^2$ and $n \leq m$.

Lemma 4.3

Let $\{\Lambda_n\}$ be a bounded monotonic increasing sequence of symmetric M^2 operators. Then Λ_n converges in the strong operator topology to a symmetric bounded operator.

Proof Since $\{\Lambda_n\}$ is bounded, we have

$$\sup ||\Lambda_n|| = A < \infty \qquad (4\text{-}11)$$

For fixed h, and $m \geq n$,

$$(h,\Lambda_n h)_{M^2} \leq (h,\Lambda_m h)_{M^2}$$

$(h,\Lambda_n h)_{M^2}$ is a monotonic increasing numerical sequence which is bounded above and hence

$$\lim_{n \to \infty} (h,\Lambda_n h)_{M^2} \text{ exists and is finite.} \tag{4-12}$$

Using the generalized Schwartz inequality

$$|(h,\Lambda k)_{M^2}|^2 \leq (h,\Lambda h)_{M^2}(k,\Lambda k)_{M^2} \tag{4-13}$$

Defining $\Lambda_{mn} = \Lambda_m - \Lambda_n$, we have

$$||\Lambda_{mn}h||^4_{M^2} \leq (\Lambda_{mn}h,\Lambda^2_{mn}h)_{M^2}(h,\Lambda_{mn}h)_{M^2}$$

$$\leq (2A)^3||h||^2(h,\Lambda_{mn}h)_{M^2}$$

$$\to 0 \text{ as } m,n \to \infty$$

since $(h,\Lambda_{mn}h) \to 0$ from (4-12)

Hence $\lim_{n \to \infty} \Lambda_n h$ exists.

Define Λ by

$$\Lambda h = \lim_{n \to \infty} \Lambda_n h \qquad (4\text{-}14)$$

Λ is obviously symmetric and bounded.

Hence proof. Q.E.D.

Theorem 4A Datko [14]

Let Γ be a bounded, symmetric positive M^2 operator.

Then $\int_0^\infty (\Phi(t)h, \Gamma\Phi(t)k)_{M^2} dt < \infty$ for all $h,k \in M^2$ (4-15)

<=> there exists a bounded positive symmetric operator Λ which is a solution of the equation

$$(\mathcal{A}h, \Lambda k)_{M^2} + (h, \Lambda\mathcal{A}k)_{M^2} + (h, \Gamma k)_{M^2} = 0 \qquad (4\text{-}16)$$

for all $h, k \in \mathcal{D}(\mathcal{A})$.

Proof (4-15) => (4-16)

Define a positive symmetric operator $\Lambda(t)$ by

$$(h, \Lambda(t)k)_{M^2} = \int_0^t (\Phi(s)h, \Gamma\Phi(s)k)_{M^2} ds \qquad (4\text{-}17)$$

Note that $(h,\Lambda(t_1)h)_{M^2} \leq (h,\Lambda(t_2)h)_{M^2}$ for $h \in M^2$ and $t_1 \leq t_2$.

Hence the family of operators $\{\Lambda(t), t \geq 0\}$ is monotonic increasing.

Now $(h,\Lambda(t)h)_{M^2} < \infty$ for all $t \in [0,\infty)$ and $h \in M^2$ and hence by the uniform boundedness principle, Horvath [39] pp. 62, there is a constant A such that

$$||\Lambda(t)|| < A \quad \text{for all} \quad t \in [0,\infty) \tag{4-18}$$

Hence from lemma 4.3 it follows that

$$\Lambda(t) \to \Lambda$$

in the strong operator topology where Λ is given by

$$(h,\Lambda k)_{M^2} = \int_0^\infty (\Phi(s)h,\Gamma\Phi(s)k)_{M^2}ds \tag{4-19}$$

For $h, k \in \mathcal{D}(\mathcal{A})$, we have

$$(\mathcal{A}h,\Lambda k)_{M^2} + (h,\Lambda\mathcal{A}k)_{M^2}$$

$$= \int_0^\infty \{(\Phi(s)\mathcal{A}h,\Gamma\Phi(s)k)_{M^2} + (\Phi(s)h,\Gamma\Phi(s)\mathcal{A}k)_{M^2}\}ds$$

$$= \int_0^\infty \frac{d}{ds}(\Phi(s)h,\Gamma\Phi(s)k)_{M^2} = \lim_{s \to \infty} (\Phi(s)h,\Gamma\Phi(s)k)_{M^2}\Big|_0^s$$

$$= -\,(h,\Gamma k)_{M^2}$$

(4-16) => (4-15)

Suppose now that Λ is a bounded positive symmetric solution of

$$(\mathcal{A}h,\Lambda k)_{M^2} + (h,\Lambda\mathcal{A}k)_{M^2} + (h,\Gamma k) = 0$$

for all $h, k \in \mathcal{D}(\mathcal{A})$.

Define a symmetric positive operator $V(t)$ by

$$(h,V(t)k)_{M^2} = (\Phi(t)h,\Lambda\Phi(t)k)_{M^2} \qquad (4\text{-}20)$$

$$\frac{d}{dt}(h,V(t)k)_{M^2} = (\mathcal{A}\Phi(t)h,\Lambda\Phi(t)k)_{M^2} + (\Phi(t)h,\Lambda\Phi(t)\mathcal{A}k)_{M^2}$$

$$= -\,(\Phi(t)h,\Gamma\Phi(t)k)_{M^2}$$

taking $h, k \in \mathcal{D}(\mathcal{A})$.

Integrating

$$(h,V(t)k)_{M^2} - (h,\Lambda k)_{M^2} = -\int_0^t (\Phi(s)h,\Gamma\Phi(s)k)_{M^2}ds$$

Hence $\int_0^t ds(\Phi(s)h,\Gamma\Phi(s)k)_{M^2} = (h,\Lambda k) - (h,V(t)k)$

$$\leq (h,\Lambda k) < \infty$$

Hence $\int_0^\infty (\Phi(s)h,\Gamma\Phi(s)k)_{M^2}ds < \infty$

for all $h, k \in \mathcal{D}(\mathcal{A})$.

But $\mathcal{D}(\mathcal{A})$ is dense in M^2 and hence the result follows for all $h, k \in M^2$.

Corollary 1

$\int_0^\infty ||\Phi(t)h||^2_{M^2}dt < \infty$ for all $h \in M^2$ <=> there is $K_0 > 1$

and $\mu > 0$ such that $||\Phi(t)h||_{M^2} \leq K_0e^{-\mu t}||h||_{M^2}$

for $t \geq 0$ (4-21)

Proof It is clear that if $||\Phi(t)h||_{M^2} \leq K_0e^{-\mu t}||h||_{M^2}$

for $t \geq 0$ that $\int_0^\infty ||\Phi(t)h||^2_{M^2}dt < \infty$

So suppose that $\int_0^\infty ||\Phi(t)h||^2_{M^2}dt < \infty$ for all $h \in M^2$

From lemma 4.1 $\lim_{t \to \infty} ||\Phi(t)h||_{M^2} = 0$

Hence $||\Phi(t)h||_{M^2}$ is bounded for every $h \in M^2$.

From the uniform boundedness principle, we have that

$$||\Phi(t)|| < K_1 \text{ for some } K_1 > 1$$

Also since $\int_0^\infty ||\Phi(t)h||^2_{M^2}dt < \infty$ for all $h \in M^2$, from the previous theorem, there is a bounded symmetric M^2 operator Λ which satisfies

$$(\mathcal{A}h,\Lambda k)_{M^2} + (h,\Lambda\mathcal{A}k)_{M^2} + (h,k)_{M^2} = 0$$

for all $h, k \in \mathcal{D}(\mathcal{A})$

and $(h,\Lambda k)_{M^2} = \int_0^\infty (\Phi(t)h,\Phi(t)k)_{M^2}dt$

Since Λ is bounded, there is a K_2 such that

$$\int_0^\infty ||\Phi(t)h||^2_{M^2}dt < K_2||h||^2_{M^2}$$

Taking $K = \max(K_1, K_2)$ we have

$$||\Phi(t)h||_{M^2} \leq K||h||_{M^2} \quad \text{for all} \quad t \geq 0 \tag{4-23}$$

$$\int_0^\infty ||\Phi(t)h||^2_{M^2}dt \leq K||h||^2_{M^2} \tag{4-24}$$

Let $0 < \varepsilon < K^{-1}$ and let $\tau(h,\varepsilon)$ be such that

$$||\Phi(t)h||_{M^2} \geq (K\varepsilon)^{\frac{1}{2}}||h||_{M^2} \quad \text{on} \quad [0,\tau(h,\varepsilon)] \tag{4-25}$$

$$\tau(h,\varepsilon) = \sup_t \{||\Phi(s)h||_{M^2} > (K\varepsilon)^{\frac{1}{2}}||h||_{M^2} \text{ on } s \in [0,t]\} \tag{4-26}$$

$\tau(h,\varepsilon)$ exists and is finite, since

$$||\Phi(t)h||_{M^2} \to 0 \quad \text{as} \quad t \to \infty$$

Using semigroup property and (4-23) we have

$$||\Phi(t)h||_{M^2} \leq K^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}||h||_{M^2} \quad \text{for} \quad t \geq \tau(h,\varepsilon) \tag{4-27}$$

Hence $K\,\varepsilon\,||h||^2_{M^2}\tau(h,\varepsilon) \leq \int_0^{\tau(h,\varepsilon)} ||\Phi(t)h||^2_{M^2}dt \leq \int_0^\infty ||\Phi(t)h||^2_{M^2}dt$

$$\leq K||h||^2_{M^2}$$

Hence $\tau(t,\varepsilon) \le \frac{1}{\varepsilon}$ for all $h \ne 0$

and hence $\tau(h,\varepsilon) = \tau(\varepsilon)$ independent of h.

Hence if $t \ge \frac{1}{\varepsilon}$, $||\Phi(t)h||_{M^2} \le K^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}||h||_{M^2}$

and hence $||\Phi(t)|| \le K^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}$

Let $\varepsilon = e^{-2}K^{-3}$.

Then $||\Phi(t)|| \le e^{-1}$ if $t \ge K^3e^2$.

In particular $||\Phi(K^3e^2)|| \le e^{-1}$.

It is well known (Dunford and Schwartz [26] pp. 619) the limit

$$\omega_0 = \lim_{t \to \infty} \frac{\log ||\Phi(t)||}{t} \text{ exists and is finite or } -\infty.$$

Now

$$\omega_0 = \lim_{t \to \infty} \frac{\log ||\Phi(t)||}{t} = \lim_{n \to \infty} \frac{\log ||\Phi(nK^3e^2)||}{nK^3e^2}$$

$$\le \lim_{n \to \infty} \frac{\log ||\Phi(K^3e^2)||^n}{nK^3e^2}$$

$$= -\frac{1}{K^3e^2}$$

Hence taking $\mu = \frac{1}{2}K^3e^2$, we can find a K_0 such that

$$||\Phi(t)|| \leq K_0 e^{-\mu t}$$

since from Dunford and Schwartz [26] pp. 619, given any $\delta > 0$, we can find a K_δ such that

$$||\Phi(t)|| \leq K_\delta e^{(\omega_0+\delta)t} \quad \text{for all} \quad t \geq 0$$

Corollary 2

The uncontrolled hereditary system is L^2 stable $\Leftrightarrow \sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$ lies entirely in the left half of the complex plane.

Proof Follows immediately from corollary 1, since from the discussion at the end of section 6 chapter 2 we know that

$||\Phi(t)|| \leq K_0 e^{-\mu t} \Leftrightarrow \sigma_p(\mathcal{A})$ lie entirely in left half plane

Definition

(i) Define a mapping $\mathcal{B} : R^m \to M^2(-a,0;R^n)$ by

$$[\mathcal{B}v](\alpha) = \begin{cases} Bv & \alpha = 0 \\ \\ 0 & \alpha \in [-a,0) \end{cases} \qquad (4\text{-}28)$$

(ii) The controlled hereditary system (4-1) is said to be stabilizable if there exists a $G \in \mathcal{L}(M^2, R^m)$ such that $\mathcal{A} + \mathcal{B}G$ defines a L^2-stable hereditary system i.e. $(\mathcal{A} + \mathcal{B}G)$ is a stable differential operator and $\sigma(\mathcal{A} + \mathcal{B}G)$ lies entirely in the left half plane.

Remarks

1. The importance of the concept of stabilizability is that it provides us with a least one $v \in \mathcal{U}$ and hence $\mathcal{U}$ is nonempty.

2. In section 4 of chapter 5, we will be able to give a necessary and sufficient condition for the stabilizability of the controlled hereditary system (4-1) in terms of the spectrum of $\mathcal{A}$.

4.3 Asymptotic behavior of $\Pi_T(t)$

Consider the controlled hereditary system restricted to the interval $[0,T]$

$$\frac{dx}{dt} = A_{00}x(t) + \sum_{i=1}^{N} A_i x(t+\theta_i) + \int_{-a}^{0} A_{01}(\theta)x(t+\theta)d\theta + Bv(t)$$

$$x(\theta) = h(\theta), \quad \theta \in [-a,0], \quad h \in M^2 \qquad (4\text{-}29)$$

with quadratic cost functional

$$C_T(v) = C_T(v;h) = \int_0^T \{(x(t),Qx(t)) + (v(t),Nv(t))\}dt \qquad (4\text{-}30)$$

and with admissible class of controls

$$\mathcal{U}_T = \{v; \int_0^T |v(t)|^2 dt < \infty\} \qquad (4\text{-}31)$$

From the results of the previous chapter we know that the control problem (4-29), (4-30) and (4-31) has a unique optimal control $u_T \in \mathcal{U}_T$ given by

$$u_T(T) = - N^{-1}B^*\Pi_T^0(t)\tilde{x}(t;u_T,h) \qquad (4\text{-}32)$$

and optimal cost

$$C_T(u_T;h) = \inf_{v \in \mathcal{U}_T} C_T(v;h) = (h,\Pi_T(0)h) \qquad (4\text{-}33)$$

and that the M^2 optimal solution satisfies the differential equation

$$\left.\begin{aligned} \frac{d\tilde{x}}{dt} &= (\mathcal{A} - \mathcal{R}\Pi_T(t))\tilde{x}(t) \\ \tilde{x}(0) &= h \end{aligned}\right. \qquad (4\text{-}34)$$

Theorem 4B

Assume that the controlled hereditary system (4-1) is stabilizable. Then

(i) $\mathcal{U}$ is nonempty

(ii) for fixed $t \in [0,\infty)$, $t < T$

$$\lim_{T \to \infty} \Pi_T(t) = \Pi$$

the limit being taken in the strong operator topology and Π is a bounded, positive symmetric operator

(iii) $$(h,\Pi h) = \int_0^\infty (\tilde{x}(t;h),[\mathcal{Q} + \Pi\mathcal{R}\Pi]\tilde{x}(t;h))_{M^2}dt \qquad (4\text{-}35)$$

$$= C(u;h)$$

where $$u = -N^{-1}B^*\Pi^0\tilde{x}(t;h) \qquad (4\text{-}36)$$

and $\tilde{x}(t;h)$ satisfies the M^2 differential equation

$$\frac{d\tilde{x}}{dt} = (\mathcal{A} - \mathcal{R}\Pi)\tilde{x}(t)$$

$$\tilde{x}(0) = h \qquad (4\text{-}37)$$

Proof

(i) Since (4-1) is stabilizable, there exists a

$G \in \mathcal{L}(M^2, R^m)$ such that

$$\mathcal{A} + \mathcal{B}G$$

is a stable differential operator.

Let $\tilde{x}(t;h)$ be the solution of the M^2 differential equation

$$\frac{d\tilde{x}(t)}{dt} = (\mathcal{A} + \mathcal{B}G)\tilde{x}(t) \qquad (4\text{-}38)$$

$$\tilde{x}(0) = h$$

with the control $v(t) = G\tilde{x}(t;h)$, it is clear that $\tilde{x}(t;h)$ is the M^2 solution of (4-1).

$$\text{Now} \quad \int_0^\infty ||\tilde{x}(t;h)||^2_{M^2}dt < \infty \qquad (4\text{-}39)$$

since $\mathcal{A} + \mathcal{B}G$ is a stable differential operator

$$|v(t)| \leq K_1||\tilde{x}(t;h)||_{M^2} \quad \text{some constant } K_1$$

since $G \in \mathcal{L}(M^2, R^m)$ and is a bounded transformation.

Hence $(v(t), Nv(t)) \leq K_2||\tilde{x}(t;h)||^2_{M^2}$ for some constant K_2

$$\text{Hence}\quad C(v) = \int_0^\infty \{(x(t,h),Qx(t;h)) + (v(t),N \quad v(t))\}dt$$

$$\leq K_3 \int_0^\infty ||\tilde{x}(t;h)||^2_{M^2}dt \quad \text{for some constant } K_3$$

$$< \infty$$

$$\text{Also}\quad \int_0^\infty |v(t)|^2 dt \leq K_1^2 \int_0^\infty ||\tilde{x}(t;h)||^2_{M^2}dt < \infty$$

Hence $v \in \mathcal{U}$ which is nonempty.

(ii) Fix $t \in (0,\infty)$

Then for $T > t$,

$$(h,\Pi_T(t)h)_{M^2} = \min_{v \in \mathcal{U}_{[t,T]}} C_{[t,T]}(v;h)$$

Now let $T_2 > T_1 > t$

$$C_{[t,T_1]}(v;h) \leq C_{[t,T_2]}(v;h) \quad \text{for all} \quad v \in \mathcal{U}_{[t,T_2]}$$

Let u_1,u_2 respectively be the optimal controls corresponding to the intervals $[t,T_1]$, $[t,T_2]$ respectively.

$$C_{[t,T_2]}(u_2;h) = \inf_{v \in \mathcal{U}_{[t,T_2]}} C_{[t,T_2]}(v;h) \geq C_{[t,T_1]}(u_2;h) \geq$$

$$\geq \inf_{v \in \mathcal{U}_{[t,T_1]}} C_{[t,T_1]}(v;h) = C_{[t,T_1]}(u_1;h)$$

Hence $(h,\Pi_{T_2}(t)h)_{M^2} \geq (h,\Pi_{T_1}(t)h)_{M^2}$ (4-40)

for $T_2 > T_1$ and all $h \in M^2$.

Hence the family $\{\Pi_T(t);T \geq t\}$ is a monotonic increasing sequence of positive symmetric operators. Also, from the stabilizability hypothesis,

$$||\Pi_T(t)|| < A \tag{4-41}$$

A some constant and for all $T \geq t$.

Hence applying lemma 4.3, we have that there exists a positive symmetric operator $\Pi(t)$ such that

$$\Pi_T(t) \to \Pi(t)$$

in the strong operator topology.

Now choose $s_2 > s_1 \geq 0$ such that

$$T_1 - s_1 = T_2 - s_2 > 0$$

$$(h,\Pi_{T_1}(s_1)h)_{M^2} = (h,\Pi_{T_2}(s_2)h)_{M^2} \quad \text{for all } h \in M^2$$

and hence $\Pi_{T_1}(s_1) = \Pi_{T_2}(s_2)$

$$\Pi(s_1) = \lim_{T_1 \to \infty} \Pi_{T_1}(s_1) = \lim_{T_1 \to \infty} \Pi_{T_1+s_2-s_1}(s_2) = \Pi(s_2)$$

Hence $\Pi(s_1) = \Pi(s_2) = \Pi$

(iii) Let $\tilde{x}_T(\cdot)$ be the solution of

$$\frac{d\tilde{x}}{dt} = (\mathcal{A} - \mathcal{R}\,\Pi_T(t))\tilde{x}(t)$$

$$\tilde{x}(0) = h \tag{4-41}$$

i.e. $x_T(\cdot)$ satisfies

$$\frac{dx}{dt} = (A_{00} - R\Pi_{00T}(t))x(t) + \sum_{i=1}^{N} A_i x(t+\theta_i)$$

$$+ \int_{-a}^{0} [A_{01}(\theta) - R\Pi_{01T}(t,\theta)]x(t+\theta)d\theta$$

$$x(\theta) = h(\theta),\ \theta \in [-a,0],\quad h \in M^2 \tag{4-42}$$

and $\tilde{x}(\cdot)$ be the solution of

$$\frac{d\tilde{x}}{dt} = (\mathcal{A} - \mathcal{R}\Pi)\tilde{x}(t)$$

(4-43)

$$\tilde{x}(0) = h$$

i.e. $x(\cdot)$ satisfies

$$\frac{dx}{dt} = (A_{00} - R\Pi_{00})\,x(t) + \sum_{i=1}^{N} A_i\, x(t+\theta_i) + \int_{-a}^{0} [A_{01}(\theta) - R\Pi_{01}(\theta)]\, x(t+\theta)\, d\theta$$

(4-44)

$$x(\theta) = h(\theta) \qquad \theta \in [-a,0] \qquad h \in M^2$$

Let $\tilde{y}_T(t) = \tilde{x}_T(t) - \tilde{x}(t)$; $y_T(t) = x_T(t) - x(t)$ (4-45)

Then $\tilde{y}_T(\cdot)$ satisfies

$$\frac{d\tilde{y}}{dt} = (\mathcal{A} - \mathcal{R}\Pi)y(t) - \mathcal{R}(\Pi_T(t) - \Pi)\tilde{x}_T(t) \tag{4-46}$$

$$\tilde{y}_T(0) = 0$$

i.e. $y_T(\cdot)$ satisfies

$$\frac{dy}{dt} = (A_{00} - \Pi_{00})y(t) + \sum_{i=1}^{N} A_i y(t+\theta_i)$$

$$+ \int_{-a}^{0} [A_{01}(\theta) - R\Pi_{01}(\theta)]y(t+\theta)d\theta - R(\Pi_{00T}(t) - \Pi_{00})x_T(t) -$$

$$- \int_{-a}^{0} R[\Pi_{01T}(t,\theta)-\Pi_{01}(\theta)]x_T(t+\theta)d\theta$$

$$y(\theta) = 0 \qquad \theta \in [-a,0] \tag{4-47}$$

Equation (4-47) has solution

$$y_T(t) = \int_0^t \Phi^0(t-s)R(\Pi_{00T}(s)-\Pi_{00})x_T(s)ds$$

$$- \int_0^t ds \int_{-a}^0 d\theta \; \Phi^0(t-s)R(\Pi_{01T}(s,\theta)-\Pi_{01}(\theta))x_T(s+\theta)$$

$$= - \int_0^T \Phi^0(t-s)R(\Pi_{00T}(s)-\Pi_{00})y_T(s)ds$$

$$- \int_0^t ds \int_{-a}^0 d\theta \; \Phi^0(t-s)R(\Pi_{01T}(s,\theta)-\Pi_{01}(\theta))y_T(s+\theta)$$

$$- \int_0^t \Phi^0(t-s)R(\Pi_{00T}(s)-\Pi_{00})x(s)ds$$

$$- \int_0^t ds \int_{-a}^0 d\theta \; \Phi^0(t-s)R(\Pi_{01T}(s,\theta)-\Pi_{01}(\theta))x(s+\theta)$$

Given any $t \in [0,\infty)$, we can find t_1, T, $T \geq t_1 > t$ and constants c_1 and c_2 such that

$$|\Phi^0(t-s)| \leq c_1, \; ||\Pi_T^0|| \leq ||\Pi^0|| \leq c_2 \qquad 0 < s < t_1$$

Hence, there exist a positive continuous function $a(s)$ and a constant $c_3 > 0$ such that

$$|y_T(t)| \leq \int_0^t a(s)|\tilde{y}_T(s)|ds + c_3 \max_{[0,t_1]} ||\Pi_T^0(s)-\Pi^0||$$

Now for $0 < \alpha < 1$, the continuous function

$$g_\alpha(s) = \exp\ (\alpha^{-1} \int_0^s a(t)dt) \tag{4-48}$$

satisfies the inequality

$$\int_0^t a(s)g_\alpha(s)ds \leq \alpha g(t) \tag{4-49}$$

$$||\tilde{y}_T(s)||_{M^2} = \{|y_T(s)|^2 + \int_{-a}^0 |y_T(s+\theta)|^2 d\theta\}^{\frac{1}{2}}$$

$$\leq (1+a)^{\frac{1}{2}} \max_{\gamma\ \varepsilon\ [0,s]} |y_T(\gamma)|^2 \tag{4-50}$$

Hence $|y_T(t)| \leq \alpha c_4 g_\alpha(t)\ |y_T|_{C_\alpha(0,t;R^n)}$

$$+ c_3 \max_{s\ \varepsilon\ [0,t_1]} ||\Pi_T^0(s)-\Pi^0||$$

i.e. $(1-\alpha c_4)|y_T|_{\mathcal{C}_\alpha(0,t;R^n)} \leq c_3 \max_{s \in [0,t_1]} ||\Pi_T^0(s)-\Pi^0||$ (4-51)

where $|y_T|_{\mathcal{C}_\alpha(0,t;R^n)} = \max_{s \in [0,t]} \{|y_T(s)/g_\alpha(s)|\}$ (4-52)

Choose $0 < \alpha < \min(1, \frac{1}{2}c_4)$

Thus $1 - \alpha c_4 > 0$ and as $T \to \infty$

$$||\Pi_T^0(s)-\Pi^0|| \to 0$$

Hence $y_T(s) \to 0$ uniformly on $(0,t_1)$

Hence $\tilde{y}_T(s) \to 0$ uniformly on $(0,t_1)$

Now define $f_T(s) = \begin{cases} (\tilde{x}_T(s),[\mathcal{Q}+\Pi_T(s)\mathcal{R}\Pi_T(s)]\tilde{x}_T(s))_{M^2} & s \in [0,T] \\ 0 & \text{otherwise} \end{cases}$

and $f(s) = (\tilde{x}(s),[\mathcal{Q}+\Pi\mathcal{R}\Pi]\tilde{x}(s))_{M^2}$

We know that

$$(h,\Pi h)_{M^2} = \lim_{T \to \infty} (h,\Pi_T(0)h)_{M^2}$$

$$= \lim_{T \to \infty} \int_0^\infty f_T(s)ds$$

Now since $\lim\limits_{T \to \infty} \Pi_T(s) = \Pi$ for all s

and $\lim\limits_{T \to \infty} \tilde{x}_T(s) = \tilde{x}(s) \qquad s \in [0,\infty)$

we conclude that $f_T(s) \to f(s)$ as $T \to \infty$ for all $s \in [0,\infty)$.

Hence by the Lebesgue dominated convergence theorem, we can show that for fixed $t_1 \in (0,\infty)$

$$\lim_{T \to \infty} \int_0^{t_1} (x_T(s)[\mathcal{Q}+\Pi_T(s)\mathcal{R}\Pi_T(s)]\tilde{x}_T(s))_{M^2}$$

$$= \int_0^{t_1} (\tilde{x}(s),[\mathcal{Q}+\Pi\mathcal{R}\Pi]\tilde{x}(s))_{M^2} ds$$

Hence result Q.E.D.

Theorem 4C

Assume that the controlled hereditary system (4-1) is stabilizable. Then there exists a unique optimal control $u \in \mathcal{U}$ and

$$C(u;h) = \inf_{v \in \mathcal{U}} C(v;h) = (h,\Pi h)_{M^2} \tag{4-53}$$

$$u(t) = - N^{-1}B^*\Pi^0\tilde{x}(t;h) \tag{4-54}$$

where $\tilde{x}(t;h)$ satisfies the differential equation

$$\frac{d\tilde{x}(t)}{dt} = (\mathcal{A}-\mathcal{R}\Pi)\tilde{x}(t)$$

$$\tilde{x}(0) = h \tag{4-55}$$

Proof From (iii) of theorem 4B, we have that $u \in \mathcal{U}$.
Consider any other $v \in \mathcal{U}$.
For all $T > 0$

$$(h,\Pi_T(0)h)_{M^2} = \min_{w \in \mathcal{U}_T} C_T(w;h)$$

$$\leq \int_0^T \{(x(s;v),Qx(s;v)) + (v(s),Nv(s))\}ds$$

Taking the limit as $T \to \infty$, we have

$$(h,\Pi h)_{M^2} \leq \int_0^\infty \{(x(s;v),Qx(s;v)) + (v(s),Nv(s))\}ds$$

and the result follows from theorem 4B.

Q.E.D.

(4.4) Operator Riccati equation for Π

Theorem 4D

Π is the unique solution of the operator equation

$$(\mathcal{A}h,\Pi k)_{M^2} + (h,\Pi\mathcal{A}k)_{M^2} - (h,\Pi\mathcal{R}\Pi k)_{M^2} + (h,\mathcal{Q}k)_{M^2} = 0 \qquad (4\text{-}56)$$

for all $h, k \in \mathcal{D}(\mathcal{A})$

Proof Let $\Phi(t)$ be the C_0 semigroup with $\mathcal{A} - \mathcal{R}\Pi$ as differential generator.

We have

$$(h,\Pi k)_{M^2} = \int_0^\infty (\Phi(t)h,[\mathcal{Q}+\Pi\mathcal{R}\Pi]\Phi(t)k)_{M^2}dt$$

For $h, k \in \mathcal{D}(\mathcal{A})$, we have

$$((\mathcal{A}-\mathcal{R}\Pi)h,\Pi k)_{M^2} + (h,\Pi(\mathcal{A}-\mathcal{R}\Pi)k)_{M^2}$$

$$= \int_0^\infty (\Phi(t)(\mathcal{A}-\mathcal{R}\Pi)h,[\mathcal{Q}+\Pi\mathcal{R}\Pi]\Phi(t)k)_{M^2}dt$$

$$+ \int_0^\infty (\Phi(t)h,[\mathcal{Q}+\Pi\mathcal{R}\Pi]\Phi(t)(\mathcal{A}-\mathcal{R}\Pi)k)_{M^2}dt =$$

$$= \int_0^\infty \frac{d}{dt}\,(\Phi(t)h, [\mathcal{Q}+\Pi\mathcal{R}\Pi]\Phi(t)k)_{M^2}dt$$

$$= \lim_{t \to 0}\ (\Phi(t)h, [\mathcal{Q}+\Pi\mathcal{R}\Pi]\Phi(t)k)_{M^2}\Big|_0^t$$

$$= -\,(h, [\mathcal{Q}+\Pi\mathcal{R}\Pi]k)_{M^2}$$

Hence (4-56).

Now suppose that they are two solutions Π_1 and Π_2 to (4-56), Π_1, Π_2 bounded M^2 operators.

Let $\Pi_0 = \Pi_1 - \Pi_2$.

Then we have

$$(\mathcal{A}h, \Pi_0 k)_{M^2} + (h, \Pi_0\mathcal{A}k)_{M^2} + (h, \Pi_2\mathcal{R}\Pi_2 k)_{M^2} - (h, \Pi_1\mathcal{R}\Pi_1 k)_{M^2} = 0$$

for all h, $k \in \mathcal{D}(\mathcal{A})$

$$\text{or}\quad ([\mathcal{A}-\mathcal{R}\Pi_2]h, \Pi_0 k)_{M^2} + (h, \Pi_0[\mathcal{A}-\mathcal{R}\Pi_1]k)_{M^2} = 0 \qquad (4\text{-}57)$$

Let $\Phi_1(t)$, $\Phi_2(t)$ be the C_0 semigroups generated by $\mathcal{A}-\mathcal{R}\Pi_1$ and $\mathcal{A}-\mathcal{R}\Pi_2$ respectively.

$$\text{Then}\quad \frac{d}{dt}\,(\Phi_2(t)h,\Pi_0\Phi_1(t)k)_{M^2} = ([\mathcal{A}-\mathcal{R}\Pi_2]\Phi_2(t)h,\Pi_0\Phi_1(t)k)_{M^2}$$

$$+\ (\Phi_1(t)h,\Pi_0(\mathcal{A}-\mathcal{R}\Pi_1)\Phi_1(t)k)_{M^2}$$

$$= 0 \qquad (4\text{-}58)$$

for all $h,\ k\ \varepsilon\,\mathcal{D}(\mathcal{A})$.

Hence $(\Phi_2(t)h,\Pi_0\Phi_1(t)k) = \text{constant} = (h,\Pi_0k)$.

But $(\Phi_2(t)h,\Pi_0\Phi_1(t))_{M^2} \to 0$ as $t \to \infty$

Hence $(h,\Pi_0k) = 0$ for all $h,\ k\ \varepsilon\,\mathcal{D}(\mathcal{A})$ and since $\mathcal{D}(\mathcal{A})$ is dense in M^2, it follows that $\Pi_0 = 0$.

Hence (4-56) has a unique solution. Q.E.D.

Theorem 4E

Π can be decomposed into a matrix of transformations

$$\Pi = \begin{pmatrix} \Pi_{00} & \Pi_{01} \\ \\ \Pi_{10} & \Pi_{11} \end{pmatrix}$$

where

$$\text{(i)}\quad \Pi_{00}\ \varepsilon\,\mathcal{L}(R^n),\quad \Pi_{00} = \Pi_{00}^{*} \qquad (4\text{-}60)$$

(ii) $\Pi_{01} \in \mathcal{L}(L^2(-a,0;R^n),R^n)$

$$\Pi_{01}x = \int_{-a}^{0} \Pi_{01}(\alpha)x(\alpha)d\alpha \qquad x \in L^2(-a,0;R^n) \qquad (4\text{-}61)$$

where the map $\alpha \to \Pi_{01}(\alpha) : [-a,0] \to \mathcal{L}(R^n)$ (4-62)

is piecewise absolutely continuous with jumps at $\alpha = \theta_i \quad i = 1 \ldots N-1$ of magnitude $\Pi_{00}A_i$

(iii) $\Pi_{10} \in \mathcal{L}(R^n,L^2(-a,0;R^n))$

$$(\Pi_{10}x)(\alpha) = \Pi_{10}(\alpha)x \qquad x \in R^n$$

$$\Pi_{10}(\alpha) = \Pi_{01}^{*}(\alpha) \qquad (4\text{-}63)$$

where the map $\alpha \to \Pi_{10}(\alpha): [-a,0] \to \mathcal{L}(R^n)$ (4-64)

is piecewise absolutely continuous with jumps at $\alpha = \theta_i \quad i = 1,\ldots N-1$ of magnitude $A_i^{*}\Pi_{00}$

(iv) $\Pi_{11} \in \mathcal{L}(L^2(-a,0;R^n))$

$$(\Pi_{11}x)(\theta) = \int_{-a}^{0} \Pi_{11}(\theta,\alpha)x(\alpha)d\alpha \qquad x \in L^2(-a,0;R^n) \qquad (4\text{-}65)$$

$$\Pi_{11}(\theta,\alpha) = \Pi_{11}^{*}(\alpha,\theta)$$

where the map $(\theta,\alpha) \to \Pi_{11}(\theta,\alpha): [-a,0] \times [-a,0] \to \mathcal{L}(R^n)$

(4-66)

is piecewise absolutely continuous in each variable with jumps at $\theta = \theta_i$, $i = 1 \ldots N-1$ of magnitude $A_i^*\Pi_{01}(\alpha)$ and at $\alpha = \theta_j$, $j = 1,\ldots N-1$ of magnitude $\Pi_{01}^*(\theta)A_j$

Moreover, Π_{00}, $\Pi_{01}(\alpha)$, $\Pi_{11}(\theta,\alpha)$ satisfy a set of coupled differential equations of Riccati type

$$\text{(v)} \quad \Pi_{00}A_{00} + A_{00}^*\Pi_{00} - \Pi_{00}R\Pi_{00} + \Pi_{01}(0) + \Pi_{01}^*(0) + Q = 0 \qquad (4\text{-}66)$$

$$\text{(vi)} \quad \frac{d\Pi_{01}(\alpha)}{d\alpha} = A_{00}^*\Pi_{01}(\alpha) - \Pi_{00}R\Pi_{01}(\alpha) + \Pi_{00}A_{01}(\alpha) + \Pi_{11}(0,\alpha) \qquad (4\text{-}67)$$

$$\text{a.e. in } \bigcup_{i=0}^{N-1} (\theta_{i+1},\theta_i)$$

$\Pi_{01}(-a) = \Pi_{00}A_N$ and jumps at $\alpha = \theta_i$ $i = 1 \ldots N-1$ of magnitude $A_i^*\Pi_{00}$

$$\text{(vii)} \quad \left[\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\alpha}\right]\Pi_{11}(\theta,\alpha) = A_{01}^{*}(\theta)\Pi_{01}(\alpha) + \Pi_{01}^{*}(\theta)A_{01}(\alpha) - \Pi_{01}^{*}(\theta)R\Pi_{01}(\alpha) \qquad (4\text{-}68)$$

$$\text{a.e. in } \bigcup_{i=0}^{N-1} (\theta_{i+1},\theta_i) \times \bigcup_{i=0}^{N-1} (\theta_{i+1},\theta_i)$$

$\Pi_{11}(-a,\alpha) = A_N^{*}\Pi_{01}(\alpha)$, $\Pi_{11}(\theta,-a) = \Pi_{01}^{*}(\theta)A_N$

and jumps at $\theta = \theta_i$ $i = 1 \ldots N-1$ of magnitude $A_i^{*}\Pi_{01}(\alpha)$

and at $\alpha = \theta_j$ $j = 1 \ldots N-1$ of magnitude $\Pi_{01}^{*}(\theta)A_j$

Proof See appendix 5.

Remarks

1. Equation (4-68) can be integrated to express $\Pi_{11}(\cdot,\cdot)$ in terms of Π_{00} and $\Pi_{01}(\cdot)$. Further simplification of (4-66), (4-67), (4-68) along those lines does not seem possible.

2. We can define

$$V(\theta,\alpha) = \Pi_{00} + \int_{\theta}^{0} \Pi_{01}^{*}(\delta)d\delta + \int_{\alpha}^{0} \Pi_{01}(\beta)d\beta$$

$$+ \int_{\theta}^{0} d\delta \int_{\alpha}^{0} d\beta\, \Pi_{11}(\delta,\beta) \qquad (4\text{-}69)$$

and $$V(0,0) = \Pi_{00} \qquad (4\text{-}70)$$

$$\frac{\partial V(0,\alpha)}{\partial \alpha} = - \Pi_{01}(\alpha) \qquad (4\text{-}71)$$

$$\frac{\partial V(\theta,\alpha)}{\partial\theta\partial\alpha} = \Pi_{11}(\theta,\alpha) \qquad (4\text{-}72)$$

and thus (4-66), (4-67) and (4-68) becomes differential equations for a single quantity V. However, we have achieved no essential simplification of (4-66), (4-67), (4-68) by this procedure.

3. Equations (4-66), (4-67), (4-68) is a simplified version of equations (3-105), (3-106), (3-107) and they both have the same structure.

We summarize with the theorem

Theorem 4F

Assuming that the controlled hereditary system (4-1) is stabilizable, the control problem (4-1), (4-2), (4-3) will have a unique optimal control u which can be expressed in feedback form as

$$u(t) = -N^{-1}B^{*}\{\Pi_{00}x(t) + \int_{-a}^{0} \Pi_{01}(\alpha)x(t+\alpha)d\alpha\} \tag{4-73}$$

and with optimal cost

$$\inf_{v \in \mathcal{U}} C(v;h) = (h(0),\Pi_{00}h(0)) + 2\int_{-a}^{0}(h(0),\Pi_{01}(\alpha)h(\alpha))d\alpha$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha\ (h(\theta),\Pi_{11}(\theta,\alpha)h(\alpha)) \tag{4-74}$$

where Π_{00}, $\Pi_{01}(\alpha)$, $\Pi_{11}(\theta,\alpha)$ satisfy the coupled set of Riccati type differential equations (4-66), (4-67), (4-68).

(4.5) Example

Consider the scalar controlled hereditary system

$$\frac{dx}{dt} = -x(t-1) + v(t) \tag{4-75}$$

$$x(\theta) = h(\theta),\ \theta \in [-1,0],\quad h \in M^{2}(-1,0;R)$$

$$v(t) \in R$$

with quadratic cost functional

$$C(v;h) = \int_0^\infty \{|x(t)|^2 + |v(t)|^2\}dt \tag{4-76}$$

$$\lambda \in \sigma(\mathcal{A}) \iff \lambda + e^{-\lambda} = 0 \tag{4-77}$$

and all the roots of (4-77) lie in the left half of the complex plane.

Hence (4-75) is stabilizable (take $v(t) = 0$, i.e. $G = 0$) and hence an optimal control exists and we have the existence of an operator Π

Equations (4-66), (4-67), (4-68) reduce to

$$-\Pi_{00}^2 + 2\Pi_{01}(0) + 1 = 0 \tag{4-78}$$

$$\begin{gathered}\frac{d\Pi_{01}(\alpha)}{d\alpha} = -\Pi_{00}\Pi_{01}(\alpha) + \Pi_{11}(0,\alpha)\\ \Pi_{01}(-1) = -\Pi_{00}\end{gathered} \tag{4-79}$$

$$\begin{gathered}\left[\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\alpha}\right]\Pi_{11}(\theta,\alpha) = -\Pi_{01}(\theta)\Pi_{01}(\alpha)\\ \Pi_{11}(-1,\alpha) = -\Pi_{01}(\alpha),\ \Pi_{11}(\theta,-1) = -\Pi_{01}(\theta)\end{gathered} \tag{4-80}$$

(4-78), (4-79), (4-80) can be transformed to

$$\Pi_{01}(\alpha) + e^{-\Pi_{00}(\alpha+1)}\Pi_{00} + \int_{-1}^{\alpha} e^{-\Pi_{00}(\alpha-\xi)}\Pi_{01}(-\xi-1)d\xi$$

$$+ \int_{-1}^{\alpha} d\xi \int_{0}^{\xi-1} du \; e^{-\Pi_{00}(\alpha-\xi)}\Pi_{01}(-1+u)\Pi_{01}(-\xi-1+u)$$

$$= 0 \tag{4-81}$$

which is still very difficult to solve. The moral of this example is that even in the simplest possible case, the Riccati type equations for Π_{00}, $\Pi_{01}(\alpha)$, $\Pi_{11}(\theta,\alpha)$ is very hairy.

Chapter 5

Approximate Optimal Control for Autonomous R.F.D.E.

There seems very little hope of obtaining exact solutions to equations (3-105), (3-106), (3-107), (3-133), (3-134) of chapter 3 and equations (4-66), (4-67), (4-68) of chapter 4. It is possible to obtain a numerical solution to those equations and one such attempt can be found in Eller, Aggarwal and Banks [27]. Rather than trying to find an approximation to the optimal control, we shall find the optimal control to a finite dimensional approximation of the control problem. This approach has the flavour of the Ritz-Galerkin method and bears many resemblances to the theory of modal control of systems governed by partial differential equations.

Stated briefly, the approach goes as follows: Following Lions [83] pp. 142, we take a basis $\phi_1 \ldots \phi_j \ldots$ of M^2 and Y_j the finite dimensional subspace spanned by $\{\phi_1 \ldots \phi_j\}$. The j^{th} order approximation of the M^2 state will be the projection of the state into Y_j. We can solve the j^{th} order optimal control problem to obtain the j^{th} order approximate optimal control u_j and we can show that $u_j \to u$ the optimal control as $j \to \infty$.

So far, we have said nothing about the choice of the basis. Here we can exploit Hale's observation [36] pp. 94 that on an eigenspace of $\mathcal{A}$, the M^2 solution can be viewed as the solution of an ordinary differential equation. Thus by taking the eigenfunctions of $\mathcal{A}$ as a basis of M^2, the j^{th} order approximate control problem reduces to a control problem for a system governed by an ordinary differential equation. The solution to the later problem is well known. This approach thus focuses attention on the eigenfunctions of $\mathcal{A}$ and thus does for hereditary systems what is already standard engineering techniques for systems governed by partial differential equations.

Crucial for the applicability of this approach is that the eigenfunctions of $\mathcal{A}$ form a basis in M^2. This is proved for a scalar R.F.D.E. in section 5. Our method of proof does not extend (in an obvious manner) to the general autonomous R.F.D.E. So at the end of section 5, we make a conjecture as to the conditions under which the eigenfunctions of $\mathcal{A}$ will form a basis in M^2.

5.1 Decomposing M^2 into eigenspace and complementary subspace

Let $\lambda_1 \ldots \lambda_j \ldots$ be the eigenvalues of $\mathcal{A}$ (and $\mathcal{A}_*$) ordered in some manner, say $\text{Re } \lambda_i \geq \text{Re } \lambda_{i+1}$. We take into account the multiplicity of the eigenvalues in the ordering i.e. if λ has multiplicity m, it is included m times. It is well known, Pinney [66], Bellman and Cooke [5] that $\text{Re } \lambda_i \to -\infty$ as $i \to \infty$ and that to the right of any line $\text{Re } z = \alpha$ that there is at most a finite number of eigenvalues.

This sets up an ordering $\{\phi_1 \ldots \phi_j \ldots\}$ and $\{\psi_1 \ldots \psi_j \ldots\}$ of the eigenfunctions of $\mathcal{A}$ and $\mathcal{A}_*$ respectively. Expressions for the eigenfunctions of $\mathcal{A}$ and $\mathcal{A}_*$ are given in equations (2-101) and (2-105) respectively.

Let Y_j, $\overline{Y}_j$ respectively be the closed finite dimensional subspaces of M^2 and $\overline{M}^2$ spanned by $\{\phi_1 \ldots \phi_j\}$ and $\{\psi_1 \ldots \psi_j\}$. Since M^2 and $\overline{M}^2$ are

Hilbert spaces, there are complementary subspaces Z_j, $\bar{Z}_j$ such that

$$M^2 = Y_j \oplus Z_j \tag{5-1}$$

$$\bar{M}^2 = \bar{Y}_j \oplus \bar{Z}_j \tag{5-2}$$

i.e. any $\phi \in M^2$ can be written

$$\phi = \tilde{y}_j + \tilde{z}_j \qquad \tilde{y}_j \in Y_j,\ \tilde{z}_j \in Z_j \tag{5-3}$$

and any $\psi \in \bar{M}^2$ can be written

$$\psi = \tilde{\tilde{y}}_j + \tilde{\tilde{z}}_j, \qquad \tilde{\tilde{y}}_j \in \bar{Y}_j,\ \tilde{\tilde{z}} \in \bar{Z}_j \tag{5-4}$$

Definition

The hereditary product for the autonomous R.F.D.E. (2-10) is the map

$$\mathcal{H} : \bar{M}^2 \times M^2 \to R$$

$$\mathcal{H}(k,h) = (k(0),h(0)) + \sum_{i=1}^{N} \int_0^{-\theta_i} d\alpha(k(\alpha),A_i h(\alpha+\theta_i)) + \int_{-a}^{0} d\theta \int_0^{-\theta} d\alpha(k(\alpha),A_{01}(\theta)h(\alpha+\theta)) \tag{5-5}$$

Lemma 5.1

$$\mathcal{H}(k,\mathcal{A}h) = \mathcal{H}(\mathcal{A}_{*}k,h) \tag{5-6}$$

for all h absolutely continuous $[-a,0] \to R^n$, $h \in \mathcal{D}(\mathcal{A})$

k absolutely continuous $[0,a] \to R^n$, $k \in \mathcal{D}(\mathcal{A}_{*})$

Proof h,k are absolutely continuous and we can integrate by parts

$$\mathcal{H}(k,\mathcal{A}h) = (k(0),A_{00}h(0)) + \sum_{i=1}^{N} (k(0),A_i h(\theta_i))$$

$$+ \int_{-a}^{0} d\theta(k(0),A_{01}(\theta)h(\theta)) + \sum_{i=1}^{N} \int_{0}^{-\theta_i} d\alpha(k(\alpha),\frac{dh(\alpha+\theta_i)}{d\alpha})$$

$$+ \int_{-a}^{0} d\theta \int_{0}^{-\theta} d\alpha(k(\alpha),A_{01}(\theta)\frac{dh(\alpha+\theta)}{d\alpha})$$

$$= (A^{*}_{00}k(0),h(0) + \sum_{i=1}^{N} (k(0),A_i h(\theta_i))$$

$$+ \int_{-a}^{0} d\theta(k(0),A_{01}(\theta)h(\theta)) + \sum_{i=1}^{N} (k(\alpha),A_i h(\alpha+\theta_i)) \Big|_{\alpha=0}^{-\theta_i}$$

$$- \sum_{i=1}^{N} \int_{0}^{-\theta_i} d\alpha(\frac{dk}{d\alpha}, A_i h(\alpha+\theta_i))$$

$$+ \int_{-a}^{0} d\theta(k(\alpha),A_{01}(\theta)h(\alpha+\theta)) \Big|_{\alpha=0}^{-\theta} -$$

$$- \int_{-a}^{0} d\theta \int_{0}^{-\theta} d\alpha\left(\frac{dk}{d\alpha}, A_{01}(\theta)h(\alpha+\theta)\right)$$

$$= (A_{00}^{*}k(0),h(0)) + \sum_{i=1}^{N} (k(0),A_{i}h(\theta_{i}))$$

$$+ \int_{-a}^{0} d\theta(k(0),A_{01}(\theta)h(\theta)) + \sum_{i=1}^{N} (k(-\theta_{i}),A_{i}h(0))$$

$$- \sum_{i=1}^{N} (k(0),A_{i}h(\theta_{i})) - \sum_{i=1}^{N} \int_{0}^{-\theta_{i}} d\alpha\left(\frac{dk}{d\alpha}, A_{i}h(\alpha+\theta_{i})\right)$$

$$+ \int_{-a}^{0} d\theta(k(-\theta),A_{01}(\theta)h(0)) - \int_{-a}^{0} d\theta(k(0),A_{01}(\theta)h(\theta))$$

$$- \int_{-a}^{0} d\theta \int_{0}^{-\theta} d\alpha\left(\frac{dk}{d\alpha},A_{01}(\theta)h(\alpha+\theta)\right)$$

$$= \mathcal{H}(\mathcal{A}_{*}k,h) \qquad \text{Q.E.D.}$$

Remark

From the previous lemma, it follows that $\mathcal{A}_{*}$ is adjoint to $\mathcal{A}$ relative to the hereditary product. This is the justification for calling $\mathcal{A}_{*}$ the hereditary adjoint of $\mathcal{A}$.

Corollary

$$\mathcal{H}(\psi_i,\phi_j) = 0 \quad \text{for} \quad \lambda_i \neq \lambda_j \tag{5-7}$$

Proof

$$\mathcal{H}(\psi_i,\mathcal{A}\phi_j) = \mathcal{H}(\psi_i,\lambda_j\phi_j) = \lambda_j\mathcal{H}(\psi_i,\phi_j)$$

Also $\mathcal{H}(\psi_i,\mathcal{A}\phi_j) = \mathcal{H}(\mathcal{A}_*\psi_i,\phi_j) = \lambda_i\mathcal{H}(\psi_i,\phi_j)$

Hence $(\lambda_i-\lambda_j)\mathcal{H}(\psi_i,\phi_j) = 0$

But $\lambda_i - \lambda_j \neq 0$ and result follows Q.E.D.

Definition

For $\theta \in [-a,0]$, $\alpha \in [0,a]$ define the $n\times j$ matrix

$$\Omega_j(\theta) = \text{column}\,\{\phi_1(\theta)\ldots\phi_j(\theta)\} \tag{5-8}$$

and for $\alpha \in [0,a]$ define the $j \times n$ matrix

$$\Psi_j(\alpha) = \text{row}\,\{\psi_1(\alpha)\ldots\psi_j(\alpha)\} \tag{5-9}$$

the $j \times j$ matrix $(\Psi_j,\Omega_j) = \mathcal{H}(\psi_i,\phi_k)$ $i=1\ldots j,\ k=1\ldots j$ (5-10)

the j column vector $$[\Psi_j,h] = \begin{pmatrix} \mathcal{H}(\psi_1,h) \\ \cdots \\ \mathcal{H}(\psi_j,h) \end{pmatrix} \qquad (5\text{-}11)$$

the j row vector $[k,\Omega_j] = (\mathcal{H}(k,\phi_1)\ldots\mathcal{H}(k,\phi_j))$ (5-12)

Lemma 5.2

(Ψ_j,Ω_j) is nonsingular

Proof

Suppose that there is a j-vector b such that

$$(\Psi_j,\Omega_j)b = 0$$

Then $[\Psi_j,\Omega_j b] = 0$

Hence $\Omega_j b$ is in the range of $(\mathcal{A} - \lambda I)^m$ and in null $(\mathcal{A} - \lambda I)^m$ for some $\lambda \in (\lambda_1\ldots\lambda_j)$ which has multiplicity m.

But from theorem 2F,

$$\text{range } (\mathcal{A} - \lambda I)^m \cap \text{null } (\mathcal{A} - \lambda I)^m = 0$$

Hence $b = 0$ Q.E.D.

Since (Ψ_j,Ω_j) is nonsingular, we can change the basis elements (for instance premultiply Ψ_j by $(\Psi_j,\Omega_j)^{-1}$) so as to obtain

$$(\Psi_j,\Omega_j) = I_j, \text{ the } j \times j \text{ identity matrix} \qquad (5\text{-}13)$$

Now since the columns of Ω_j are eigenfunctions of $\mathcal{A}$,

$$\mathcal{A}\Omega_j = \Omega_j \mathcal{L}_j$$

where $\mathcal{L}_j$ is a $j \times j$ matrix with eigenvalues $\{\lambda_1 \ldots \lambda_j\}$.

Similarly $\mathcal{A}_* \Psi_j = \mathcal{L}_{*j} \Psi_j$

where $\mathcal{L}_{*j}$ is a $j \times j$ matrix with eigenvalues $\{\lambda_1 \ldots \lambda_j\}$

Lemma 5.3

$$\text{(i)} \quad \mathcal{L}_j = \mathcal{L}_{*j} \tag{5-14}$$

$$\text{(ii)} \quad \Omega_j(\theta) = \Omega_j(0)e^{\mathcal{L}_j\theta} \qquad \theta \in [-a,0] \tag{5-15}$$

$$\text{(iii)} \quad \Psi_j(\alpha) = e^{-\mathcal{L}_j\alpha}\Psi_j(0) \qquad \alpha \in [0,a] \tag{5-16}$$

Proof

(i) $(\Psi_j, \mathcal{A}\Omega_j) = (\Psi_j, \Omega_j\mathcal{L}_j) = (\Psi_j, \Omega_j)\mathcal{L}_j = \mathcal{L}_j$

Also $(\Psi_j, \mathcal{A}\Omega_j) = (\mathcal{A}_*\Psi_j, \Omega_j) = (\mathcal{L}_{*j}\Psi_j, \Omega_j) = \mathcal{L}_{*j}$

Hence result

(ii) and (iii) follow immediately from the solution of

$$\mathcal{A}\Omega_j = \Omega_j\mathcal{L}_j \quad \text{and} \quad \mathcal{A}_*\Psi_j = \mathcal{L}_j\Psi_j$$

Q.E.D.

Now any $\phi \in Y_j$ can be written in the form

$$\phi = b_1\phi_1 + \ldots + b_j\phi_j \qquad b_1 \ldots b_j \text{ scalars}$$

$$= \Omega_j b, \qquad b \text{ a } j\text{-vector}$$

$[\Psi_j,\phi] = [\Psi_j,\Omega_j b] = (\Psi_j,\Omega_j)b = b$

Hence $Y_j = \{\phi;\ \phi \varepsilon M^2,\ \phi = \Omega_j b$ for some j vector $b\}$ (5-17)

From the corollary to lemma 5.1, we have

$$Z_j = \{\phi;\ \phi \varepsilon M^2, \mathscr{H}(\psi_i,\phi) = 0 \text{ for } i = 1\ldots j\} \qquad (5\text{-}18)$$

Now if $\phi = \tilde{y}_j + \tilde{z}_j = \Omega_j b + \tilde{z}_j$

$$[\Psi_j,\phi] = [\Psi_j,\Omega_j b + \tilde{z}_j] = b$$

Hence $\phi = \Omega_j[\Psi_j,\phi] + \tilde{z}_j$ (5-19)

Hence we can define the Y_j projection operator

$$E_j : M^2 \to Y_j \qquad E_j\phi = \Omega_j[\Psi_j,\phi] \qquad (5\text{-}20)$$

Similarly we can define the $\overline{Y}_j$ projection operator

$$\overline{E}_j : \overline{M}^2 \to \overline{Y}_j \qquad \overline{E}_j\psi = [\psi,\Omega_j]\Psi_j \qquad (5\text{-}21)$$

Now let $x(\cdot)$ be the R^n solution of

$$\frac{dx}{dt} = A_{00}x(t) + \sum_{i=1}^{N} A_i x(t+\theta_i) + \int_{-a}^{0} A_{01}(\theta)x(t+\theta)d\theta + f(t)$$

$$x(\theta) = h(\theta) \qquad \theta \in [-a,0] \qquad h \in M^2 \tag{5-22}$$

on the interval $[0,T]$

or equivalently $\tilde{x}(\cdot)$ the M^2 solution of

$$\frac{d\tilde{x}}{dt} = \mathcal{A}\tilde{x}(t) + \tilde{f}(t)$$

$$\tilde{x}(0) = h \tag{5-23}$$

and the corresponding hereditary adjoint $p(\cdot)$ the R^n solution of

$$\frac{dp}{dt} + A_{00}^{*}p(t) + \sum_{i=1}^{N} A_i^{*}p(t-\theta_i) + \int_{-a}^{0} A_{01}^{*}(\theta)p(t-\theta)d\theta = 0$$

$$p(T+\beta) = k(\beta) \qquad \beta \in [0,a], \qquad k \in \overline{M}^2 \tag{5-24}$$

or equivalently $\tilde{p}(\cdot)$ the $\overline{M}^2$ solution of

$$\frac{d\tilde{p}}{dt} + \mathcal{A}_{*}\tilde{p}(t) = 0$$

$$\tilde{p}(T) = k \tag{5-25}$$

Theorem 5A

The M^2 solution of (5-23) is

$$\tilde{x}(t) = \tilde{y}_j(t) + \tilde{z}_j(t)$$

$$\tilde{x}(t) = \Omega_j y_j(t) + \tilde{z}_j(t) \qquad (5\text{-}26)$$

where $y_j(t)$ satisfies the differential equation

$$\dot{y}_j(t) = \mathcal{L}_j y_j(t) + \Psi_j(0)f(t)$$

$$y_j(0) = [\Psi_j, h] \qquad (5\text{-}27)$$

Proof For any $t \geq t_0 \in [0,T]$, from theorem 2E (2-64) we have

$$\mathcal{H}(\tilde{p}(t), \tilde{x}(t)) - \mathcal{H}(\tilde{p}(t_0), \tilde{x}(t_0)) = \int_{t_0}^{t} (p(s), f(s))ds \quad (5\text{-}28)$$

Now each row of $e^{-\mathcal{L}_j t}\Psi_j$

where $[e^{-\mathcal{L}_j t}\Psi_j](\theta) = e^{-\mathcal{L}_j(t+\theta)}\Psi_j(0)$

is a solution of (5-24), (5-25) with appropriate final data.

Hence from (5-28)

$$[e^{-\mathcal{L}_j t}\Psi_j,\tilde{x}(t)] - [e^{-\mathcal{L}_j t_0}\Psi_j,\tilde{x}(t_0)] = \int_{t_0}^{t} e^{-\mathcal{L}_j s}\Psi_j(0)f(s)ds \quad (5\text{-}29)$$

$$\text{i.e.}\quad [\Psi_j,\tilde{x}(t)] - e^{\mathcal{L}_j(t-t_0)}[\Psi_j,\tilde{x}(t_0)] = \int_{t_0}^{t} e^{\mathcal{L}_j(t-s)}\Psi_j(0)f(s)ds \quad (5\text{-}30)$$

$$\text{i.e.}\quad y_j(t) = e^{\mathcal{L}_j(t-t_0)}y_j(t_0) + \int_{t_0}^{t} e^{\mathcal{L}_j(t-s)}\Psi_j(0)f(s)ds$$

Hence $y_j(t)$ satisfies for $t \varepsilon [0,T]$ the ordinary differential equation

$$\dot{y}_j(t) = \mathcal{L}_j y_j(t) + \Psi_j(0)f(t)$$

Hence result Q.E.D.

Remark

1. The previous theorem is a precise version of the more loosely worded phrase that the projection onto Y_j of the solution of (5-22), (5-23) behaves like a solution

to an ordinary differential equation in R^j.

2. Note the important role that the hereditary product plays in giving us an explicit representation of the Y_j projection operator E_j in (5-20) and in establishing the differential equation satisfied by $y_j(t)$ in the proof of theorem 5A.

Proposition 5.4

(i) $\Phi(t)Y_j \subset Y_j \qquad \mathcal{A}\, Y_j \subset Y_j$ (5-31)

(ii) $\Phi(t)Z_j \subset Z_j, \quad \mathcal{A}\, Z_j \subset Z_j$ (5-32)

(iii) Denote by $\Phi^{Z_j}(t)$ the restriction of $\Phi(t)$ to Z_j. Then for all j, there is a $K > 1$ such that

$$||\Phi^{Z_j}(t)|| \leq Ke^{\{1+\mathrm{Re}\ \lambda_j\}t} \qquad t \geq 0 \tag{5-33}$$

Proof (i) and (ii) are obvious.

(iii) follows from the fact that the spectrum of restricted to Z_j will have eigenvalues lying to the left of the line $\operatorname{Re} z = \operatorname{Re} \lambda_j$ and from the relationships between $\sigma(\Phi(t))$ and $\sigma(\mathcal{A})$ stated at the end of section 6 chapter 2.

5.2 Finite time regulator problem

Consider the control problem on $[0,T]$

$$\frac{dx}{dt} = A_{00}x(t) + \sum_{i=1}^{N} A_i x(t+\theta_i) + \int_{-a}^{0} A_{01}(\theta)x(t+\theta)d\theta + B(t)v(t) \tag{5-34}$$

$$x(\theta) = h(\theta) \qquad \theta \in [-a,0], \qquad h \in M^2$$

$$C(v) = C(v;h) = (x(T),Fx(T)) + \int_0^T \{(x(t),Q(t)x(t)) + (v(t),N(t)v(t))\}dt \tag{5-35}$$

where $Q \in L^\infty(0,T;\mathcal{L}(T^n))$, $N \in L^\infty(0,T;\mathcal{L}(R^m))$, $F \in \mathcal{L}(R^n)$

$$F = F^* \geq 0, \; Q(t) = Q^*(t) \geq 0, \quad N(t) = N^*(t) > 0$$

and there is a $\delta > 0$ such that

$$(v,N(t)v) \geq \delta|v|^2 \quad \text{for all} \quad t \in [0,T]$$

and with class of admissible controls

$$\mathcal{U} = \{v; \int_0^T (v(t)|^2 dt < \infty\} = L^2(0, T; R^m) \qquad (5\text{-}36)$$

The corresponding j^{th} order approximate control problem is

$$\text{minimize} \quad C_j(v,h) = (\tilde{y}_j(T), \mathcal{F}\tilde{y}_j(T))_{M^2} + \int_0^T \{(\tilde{y}_j(t), \mathcal{Q}(t)\tilde{y}_j(t))_{M^2} + (v(t), N(t)v(t))\}dt$$

$$\text{i.e. minimize} \quad (C_j(v,h) = (y_j(T), \Omega_j^*(0)F\Omega_j(0)y_j(T)) + \int_0^T \{(y_j(t), \Omega_j^*(0)Q(t)\Omega_j(0)y_j(t)) + (v(t), N(t)v(t))\}dt \qquad (5\text{-}37)$$

with admissible class of controls $\mathcal{U}$

and where

$$\begin{aligned} \dot{y}_j(t) &= \mathcal{L}_j y_j(t) + \Psi_j(0)B(t)v(t) \\ y_j(0) &= [\Psi_j, h] \end{aligned} \qquad (5\text{-}38)$$

The optimal control to (5-37), (5-38) is

$$u_j(t) = N^{-1}(t)B^*(t)\Psi_j^*(0)p_j(t) \qquad (5\text{-}39)$$

and the optimal equations are

$$\dot{y}_j(t) = \mathcal{L}_j y_j(t) - \Psi_j(0)R(t)\Psi_j(0)p_j(t); y_j(0) = [\Psi_j, h] \qquad (5\text{-}40)$$

$$\dot{p}_j(t) + \mathcal{L}_j^* p_j(t) + \Omega_j^*(0)Q(t)\Omega_j(0)y_j(t) = 0;\ p_j(T) = \Omega_j^*(0)F\Omega_j(0)y_j(T) \qquad (5\text{-}41)$$

(5-40) and (5-41) can be decoupled to obtain

$$p_j(t) = P_j(t)y_j(t) \qquad (5\text{-}42)$$

where

$P_j(t)$ is a $j \times j$ matrix satisfying the matrix Riccati differential equation

$$\dot{P}_j(t) + \mathcal{L}_j^* P_j(t) + P_j(t)\mathcal{L}_j - P_j(t)\Psi_j(0)R(t)\Psi_j^*(0)P_j(t) + \Omega_j^*(0)Q(t)\Omega_j(0) = 0 \qquad (5\text{-}43)$$

$$P_j(T) = \Omega_j^*(0)F\Omega_j(0)$$

Hence the j^{th} order approximate optimal control is given by

$$u_j(t) = - N^{-1}(t)B^*(t)\Psi_j^*(0)P_j(t)y_j(t) \qquad (5\text{-}44)$$

where $y_j(t)$ satisfies

$$\dot{y}_j(t) = \{\mathcal{L}_j - \Psi_j(0)R(t)\Psi_j^*(0)P_j(t)\}y_j(t) \; ; \; y_j(0) = [\Psi_j,h] \quad (5\text{-}45)$$

We also have an expression for the optimal cost to go at the instant $t \in [0,T]$

$$([\Psi_j,h], P_j(t)[\Psi_j,h]) = \min_{v \in \mathcal{U}_t} C_j^t(v;h) \qquad (5\text{-}46)$$

We define a positive symmetric M^2 operator $\Pi_j(t)$ by

$$(h,\Pi_j(t)k)_{M^2} = ([\Psi_j,h], P_j(t)[\Psi_j,k])$$

Theorem 5B

As $j \to \infty$

(i) $C_j(u_j) \to C(u)$ (5-48)

(ii) $u_j \to u$ strongly in $\mathcal{U} = L^2(0,T;R^m)$ (5-49)

(iii) $\Pi_j(t) \to \Pi(t)$ in weak M^2 operator topology for fixed $t \in [0,T]$ (5-50)

Proof Define $h_j = E_j h = \Omega_j[\Psi_j, h]$ for $h \in M^2$. Let $x(\cdot;h,v)$ be the solution of (5-34) with initial data h and admissible control v and let the corresponding solution of (5-38) be $y_j(t;v)$.

$$|x(t;h,v) - \Omega_j(0)y_j(t;v)|^2 \leq 2|x(t;h,v) - x(t;h_j,v)|^2$$
$$+ 2|x(t;h_j,v) - \Omega_j(0)y_j(t;v)|^2$$

As $j \to \infty$, $h_j \to h$ in M^2 (assuming the completeness of the eigenfunctions in M^2) and from the continuity of the solution with respect to the initial data, it follows that

$$|x(t;h,v) - x(t;h_j,v)|^2 \to 0 \text{ uniformly for } t \in [0,T] \quad (5\text{-}51)$$

Now $|x(t;h,v) - \Omega_j(0)y_j(t;v)|^2$

$$\leq ||\tilde{x}(t;h_j,v) - \tilde{y}_j(t,h_j,v)||^2_{M^2}$$

$$= ||\tilde{z}_j(t;h_j,v)||^2_{M^2}$$

$$= ||\int_0^t \Phi^{Z_j}(t-s)\mathcal{B}(s)v(s)ds||^2_{M^2}$$

$$\leq \alpha||v||^2_{\mathcal{U}}K^2 \int_0^t e^{2\{1+\mathrm{Re}\ \lambda_j\}(t-s)} ds$$

for some constant α and from (5-33)

$$= \alpha||v||^2_{\mathcal{U}}K^2[1 - e^{-2|1+\mathrm{Re}\ \lambda_j|t}]/2|1 + \mathrm{Re}\ \lambda_j| \qquad (5\text{-}52)$$

Now $\left| \int_0^T x(t;h,v), Q(t)\, x(t;h,v)\, dt - \int_0^T (\Omega_j(0)\, y_j(t;v), Q(t)\, \Omega_j(0)\, y_j(t;v))dt \right|$

$$= \left| \int_0^T ([x(t;h,v) + \Omega_j(0)\, y_j(t;v)], Q(t)[x(t;h,v) - \Omega_j(0)\, y_j(t;v)])\, dt \right|$$

$$\leq q \left\{ \int_0^T |x(t;h,v) + \Omega_j(0)\, y_j(t;v)|^2 dt \right\}^{1/2} \left\{ \int_0^T |x(t;h,v) - \Omega_j(0)\, y_j(t;v)|^2 dt \right\}^{1/2}$$

for some $q > 0$, since $Q(t) \in L^\infty(0,T;R^m)$

Now $\int_0^T |x(t;x,v) - \Omega_j(0)y_j(t;v)|^2 dt$

$$\leq 2 \int_0^T |x(t;h,v) - x(t;h_j,v)|^2 dt$$

$$+ 2 \int_0^T |x(t;h_j,v) - \Omega_j(0)y_j(t;v)|^2 dt$$

Now $2 \int_0^T |x(t;h,v) - x(t;h_j,v)|^2 dt \to 0$ as $j \to \infty$

from (5-51).

From (5-52)

$$\int_0^T |x(t;x_j,v) - \Omega_j(0)y_j(t;v)|^2 dt$$

$$\leq \alpha \|v\|_{\mathcal{U}}^2 K^2 \left\{ T + \left[1 - e^{-2|1+\mathrm{Re}\,\lambda_j|T}\right] / 2|1+\mathrm{Re}\,\lambda_j| \right\} / 2|1 + \mathrm{Re}\,\lambda_j|$$

$\to 0$ as $j \to \infty$ since $|1 + \mathrm{Re}\ \lambda_j| \to \infty$.

Hence $\int_0^T |x(t;h,v) - \Omega_j(0)y_j(t,v)|^2 dt \to 0$ as $j \to \infty$

$$\int_0^T |x(t;x,v) + \Omega_j(0)y_j(t,v)|^2 dt < M$$

for some bound M.

Hence

$$\left| \int_0^T (x(t;h,v),\, Q(t)\, x(t;h,v))dt - \int_0^T (\Omega_j(0)\, y_j(t;v),\, Q(t)\, \Omega_j(0)\, y_j(t;v))dt \right|$$

$\rightarrow 0$ as $j \rightarrow \infty$.

Hence for fixed $v \in \mathcal{U}$, $C_j(v) \rightarrow C(v)$ (5-53)

Now $C_j(u_j) \leq C_j(u)$

Hence $\limsup\limits_{j \rightarrow \infty} C_j(u_j) \leq C(u)$.

Now $C_j(u_j) \geq \delta ||u_j||^2_{\mathcal{U}}$

and since $\mathcal{U}$ is weakly compact, we can extract a subsequence $\{u_k\}$ such that

$u_k \rightarrow u^*$ weakly in $\mathcal{U}$.

Hence $x(\cdot;h,u_k) \rightarrow x(\cdot;h,u^*)$ weakly in $AC^2(0,T;R^n)$

and $\liminf C_k(u_k) \geq C(u^*)$.

Hence $C(u) \geq \limsup C_j(u_j) \geq \limsup C_k(u_k)$

$$\geq \liminf C_k(u_k) \geq C(u^*)$$

Hence necessarily, $u = u^*$ and $C_j(u_j) \to C(u)$ (5-54)

Now $u_j \to u$ weakly (5-55)

and since $C_j(u_j) \to C(u)$, we necessarily have

$$\int_0^T (u_j(t), N(t)u_j(t))dt \to \int_0^T (u(t), N(t)u(t))dt \tag{5-56}$$

since otherwise we have a contradiction

$$\begin{aligned} \text{Now} \quad \delta\|u_j-u\|^2_{\mathcal{U}} &\leq \int_0^T ((u_j(t)-u(t)), N(t)(u_j(t)-u(t)))dt \\ &= \int_0^T \{(u_j(t), N(t)u_j(t)) - (u(t), N(t)u_j(t)) \\ &\quad - (u_j(t), N(t)u(t)) + (u(t), N(t)u(t))\}dt \\ &\to 0 \quad \text{as} \quad j \to \infty \end{aligned}$$

Hence $u_j \to u$ strongly in $\mathcal{U}$.

We have $(h,\Pi_j(t)h)_{M^2} = \inf_{v\varepsilon\mathcal{U}_t} C_j^t(v;h)$

$$\to \inf_{v\varepsilon\mathcal{U}_t} C^t(v;h)$$

$$= (h,\Pi(t)h)_{M^2} \quad \text{as} \quad j \to \infty$$

Hence

$$(h,\Pi_j(t)k)_{M^2} = \tfrac{1}{2}((h+k),\Pi_j(t)(h+k))_{M^2} - \tfrac{1}{2}((h-k),\Pi_j(t)(h-k))_{M^2}$$

$$\to \tfrac{1}{2}((h+k),\Pi(t)(h+k))_{M^2} - \tfrac{1}{2}((h-k),\Pi(t)(h-k))_{M^2}$$

$$= (h,\Pi(t)k)_{M^2} \quad \text{as} \quad j \to \infty$$

Hence $\Pi_j(t) \to \Pi(t)$ weakly.

Q.E.D.

5.3 Finite time tracking problem

Now consider the control problem (5-34), (5-35) (5-36) with non-zero forcing term $f(t)$ in the hereditary system (5-34).

The corresponding j^{th} order approximate control problem is

$$\text{minimize} \quad C_j(v;h) = (\tilde{y}_j(T), \mathcal{F}\tilde{y}_j(T))_{M^2}$$

$$+ \int_0^T \{(\tilde{y}_j(t), \mathcal{Q}(t)\tilde{y}_j(t))_{M^2} + (v(t), N(t)v(t))\}dt$$

$$= (y_j(T), \Omega_j^*(0)F\Omega_j(0)y_j(T))$$

$$+ \int_0^T \{(y_j(t), \Omega_j^*(0)Q(t)\Omega_j(0)y_j(t)) + (v(t), N(t)v(t))\}dt \tag{5-57}$$

with admissible class of controls $\mathcal{U}$ and where

$$\dot{y}_j(t) = \mathcal{L}_j y_j(t) + \Psi_j(0)B(t)v(t) + \Psi_j(0)f(t)$$
$$y_j(0) = [\Psi_j, h] \tag{5-58}$$

The optimal control to (5-57), (5-58) is

$$u_j(t) = - N^{-1}(t)B^*(t)\Psi_j^*(0)p_j(t) \tag{5-59}$$

and the optimality equations are

$$\dot{y}_j(t) = \mathcal{L}_j y_j(t) - \Psi_j(0)R(t)\Psi_j^*(0)p_j(t) + \Psi_j(0)f(t);$$
$$y_j(0) = [\Psi_j, h] \tag{5-60}$$

$$\dot{p}_j(t) + \mathcal{L}_j^* p_j(t) + \Omega_j^*(0)\, Q(t)\, \Omega_j(0)\, y_j(t) = 0$$

$$p(T) = \Omega_j^*(0)\, F\, \Omega_j(0)\, y_j(T) \qquad (5\text{-}61)$$

These equations can be decoupled to obtain

$$p_j(t) = P_j(t)y_j(t) + d_j(t) \qquad (5\text{-}62)$$

where $P_j(t)$ is a $j \times j$ matrix satisfying the matrix Riccati differential equation (5-43) and $d_j(t)$ is a j vector satisfying the differential equation

$$\dot{d}_j(t) + [\mathcal{L}_j^* - P_j(t)\psi_j(0)R(t)\psi_j^*(0)]d_j(t) + P_j(t)\psi_j(0)f(t) = 0$$

$$d_j(T) = 0 \qquad (5\text{-}63)$$

Hence the j^{th} order approximate optimal control is given by

$$u_j(t) = -\, N^{-1}(t)B^*(t)\psi_j^*(0)[P_j(t)y_j(t) + d_j(t)] \qquad (5\text{-}64)$$

where $y_j(t)$ satisfies

$$\dot{y}_j(t) = \{\mathcal{L}_j - \Psi_j(0) R(t) \Psi_j^*(0) P_j(t)\} y_j(t) + \Psi_j(0) [f(t) - R(t) \Psi_j^*(0) d_j(t)] \tag{5-65}$$

$y_j(0) = [\Psi_j, h]$

We also have an expression for the optimal cost to go at the instant $t \in [0,T]$

$$\inf_{v \in \mathcal{U}_t} C_j^t(v;h) = ([\Psi_j,h], P_j(t)[\Psi_j,h]) + 2(d_j(t),[\Psi_j,h]) + c_j(t) \tag{5-66}$$

where the scalar $c_j(t)$ satisfies the differential equation

$$\dot{c}_j(t) = d_j^*(t) \Psi_j(0) R(t) \Psi_j^*(0) d_j(t) - f^*(t) \Psi_j^*(0) P_j^*(t) P_j(t) \Psi_j(0) f(t) \tag{5-67}$$

$c_j(T) = 0$

Define $\tilde{g}_j(t) \in M^2$ by

$$(\tilde{g}_j(t),h)_{M^2} = (d_j(t),[\Psi_j,h]) \qquad (5\text{-}68)$$

Theorem 5C

As $j \to \infty$

(i) $C_j(u_j) \to C(u)$

(ii) $u_j \to u$ strongly in $\mathcal{U} = L^2(0,T;R^m)$

(iii) $\tilde{g}_j(t) \to \tilde{g}(t)$ weakly in M^2 for $t \in [0,T]$ (5-69)

(iv) $c_j(t) \to c(t)$ for $t \in [0,T]$ (5-70)

Proof (i) and (ii) are proved exactly as in theorem 5B. To prove (iii) and (iv) we make use of the fact that

$$\inf_{v \in \mathcal{U}_t} C_j^t(v;h) \to \inf_{v \in \mathcal{U}_t} C^t(v;h).$$

By considering the case $h = 0$, we get $c_j(t) \to c(t)$. Since $(h,\Pi_j(t)h)_{M^2} \to (h,\Pi(t)h)_{M^2}$ and $c_j(t) \to c(t)$, we must have

$$(\tilde{g}_j(t),h)_{M^2} \to (\tilde{g}(t),h)_{M^2}$$

And since this holds for all $h \in M^2$,

$$\tilde{g}_j(t) \to \tilde{g}(t) \quad \text{weakly.}$$

5.4 Infinite time problem

Consider the controlled hereditary system on $[0,\infty)$

$$\frac{dx}{dt} = A_{00}x(t) + \sum_{i=1}^{N} A_i x(t+\theta_i) + \int_{-a}^{0} A_{01}(\theta)x(t+\theta)d\theta + Bv(t)$$

$$x(\theta) = h(\theta) \tag{5-71}$$

with cost functional

$$C(v;h) = C(v) = \int_0^\infty \{(x(t),Qx(t)) + (v(t),Nv(t))\}dt \tag{5-72}$$

and admissible class of controls

$$\mathcal{U} = \{v;\ \int_0^\infty |v(t)|^2 dt < \infty,\ C(v) < \infty\} \tag{5-73}$$

Let j_0 be the longest j such that

$$\text{Re } \lambda_j \geq 0, \text{ Re } \lambda_{j+1} < 0 \tag{5-74}$$

Such a j_0 exists from the ordering of the eigenvalues and since only a finite number of the eigenvalues will lie in the right half of the complex plane.

We can now state and prove a theorem due to Vandevenne [77].

Theorem 5D

The controlled hereditary system (5-71) or (4-1) is stabilizable iff the finite dimensional system

$$\dot{y}_{j_0}(t) = \mathcal{L}_{j_0} y_{j_0}(t) + \Psi_{j_0} Bv(t) \tag{5-75}$$

is completely controllable.

Proof Suppose that (5-75) is completely controllable. Hence there exists a matrix $C : R^{j_0} \to R^m$ such that all the eigenvalues of the matrix

$$(\mathcal{L}_{j_0} + \Psi_{j_0}(0)BC) \tag{5-76}$$

lie strictly in the left half plane.

Define a mapping

$$\begin{aligned} G &: M^2(-a,0;R^n) \to R^m \\ Gh &= C[\Psi_{j_0},h] \end{aligned} \tag{5-77}$$

G is a bounded linear map and

$$Gh = 0 \quad \text{for} \quad h \in Z^{j_0}$$

We want to show that $\sigma(\mathcal{A} + \mathcal{B}G)$ lies strictly in the left half plane.

Denote by $\mathcal{A} + \mathcal{B}G|_{Z_{j_0}}$ the restriction of $\mathcal{A} + \mathcal{B}G$ to Z_{j_0}

Similarly for $\mathcal{A} + \mathcal{B}G|_{Y_{j_0}}$

Now $\quad \sigma((\mathcal{A} + \mathcal{B}G)|_{Z_{j_0}}) = \sigma(\mathcal{A}|_{Z_{j_0}})$

which lies strictly in the left half plane.

Also $(\mathcal{A} + \mathcal{B}G)|_{Y_{j_0}}$ is represented by the matrix

$\mathcal{L}_{j_0} + \Psi_{j_0}(0)BC$

Hence $\sigma((\mathcal{A} + \mathcal{B}G)|_{Y_{j_0}}) = \sigma(\mathcal{L}_{j_0} + \Psi_{j_0}(0)BC)$

which lies strictly in the left half plane. Result now follows, since

$$\sigma(\mathcal{A}+\mathcal{B}G) = \sigma(\mathcal{A}+\mathcal{B}G|_{Y_{j_0}}) \cup \sigma((\mathcal{A}+\mathcal{B}G)|_{Z_{j_0}})$$

Now suppose that (5-75) is not completely controllable. From Lee and Marcus [53] pp. 99, we can decompose R^{j_0} into a controllable and an uncontrollable part such that

$$\begin{aligned} \dot{y}_{j_0}^{1}(t) &= \mathcal{L}_{j_0}^{11} y_{j_0}^{1}(t) + \mathcal{L}_{j_0}^{12} y_{j_0}^{2}(t) + (\Psi_{j_0}(0)B)^1 v(t) \\ \dot{y}_{j_0}^{2}(t) &= \mathcal{L}_{j_0}^{22} y_{j_0}^{2}(t) \end{aligned} \tag{5-78}$$

which for the initial condition

$$y_{j_0}(0) = \begin{bmatrix} 0 \\ \\ \bar{y}_{j_0}^{2}(0) \end{bmatrix} \tag{5-79}$$

will have a solution $y_{j_0}(t,v)$ bounded away from zero on a set of infinite measure for every control v since the eigenvalues of $\mathcal{L}_{j_0}^{22}$ all have real parts greater or zero. Hence (5-75) will not be stabilizable.

Q.E.D.

Remarks

1. The complete controllability of (5-75) implies the stabilizability of

$$\dot{y}_j(t) = \mathcal{L}_j y_j(t) + \psi_j(0)Bv(t) \tag{5-80}$$

for any $j > j_0$, since only the eigenvalues $(\lambda_1 \ldots \lambda_{j_0})$ of $\mathcal{L}_j$ can cause (5-80) not to be stabilizable.

2. From the results of section 2 of this chapter and section 3 of chapter 4, we can obtain an approximation to the optimal control and the optimal cost of the control problem (5-71), (5-72) (5-73) by taking j and T sufficiently large in the j^{th} order approximate control problem (5-37), (5-38) with $F = 0$.

5.5 Completeness question

There are a number of papers in the literature on whether or not any solution of an autonomous R.F.D.E. can be expressed as an infinite series of eigenfunction solutions. See for example Zverkin [82] and Bellman and Cooke [5]. However, with the exception of Pitt [67], there has been very little concern as whether or not the eigenfunctions will be complete and form a basis in some appropriate function space. In this section, we shall establish the completeness of the eigenfunctions in the space $M^2(-a,0;R)$ for the scalar R.F.D.E.

$$\frac{dx}{dt} = A_1 x(t-a), \quad A_1 \neq 0 \tag{5-81}$$

and its corresponding differential operator

$$[\mathcal{A}h](\alpha) = \begin{cases} A_1 h(-a) & \alpha = 0 \\ \frac{dh}{d\alpha} & \alpha \in [-a,0) \end{cases} \tag{5-82}$$

where $h \in \mathcal{D}(\mathcal{A})$.

We first must determine the location of its eigenvalues which will be the roots of the characteristic equation

$$\Delta_0(z) = z - A_1 e^{-az} = 0 \tag{5-83}$$

We have two possibilities; case (i) $A_1 > 0$, case (ii) $A_1 < 0$

Case (i) $A_1 > 0$

For $z = x + iy$, (5-83) reduces to

$$x = A_1 e^{-ax} \cos ay \qquad (5\text{-}84)$$

$$y = - A_1 e^{-ax} \sin ay \qquad (5\text{-}85)$$

Real roots $y = 0 \quad x = A_1 e^{-ax}$ (5-86)

(5-86) has one real root $\sigma_0 > 0$ given by

$$\sigma_0 = G(A_1) \qquad (5\text{-}87)$$

where $G(x)$ is the inverse function of

$$g(x) = xe^{ax}$$

which is monotonic increasing for $x \geq 0$.

Complex roots $y \neq 0$. The purely imaginary roots will be included in this case.

It is clear that the complex roots of (5-83) will occur in conjugate pairs and we restrict our attention only to those roots that have positive imaginary part.

From (5-84) and (5-85), we have

$$\cot ay = -\frac{x}{y}, \quad y \neq 0$$

and hence

$$-ax = ay \cot ay \tag{5-88}$$

Hence from (5-85), we have

$$y = -A_1 e^{ay \cot ay} \sin ay$$

Let $$f(y) = \frac{\sin ay}{y} e^{ay \cot ay} \tag{5-89}$$

Hence we want to find the (real) roots of

$$f(y) = -1/A_1 \tag{5-90}$$

For $p \geq 1$, in any interval $(\frac{(2p-1)\pi}{a}, \frac{2p\pi}{a})$, $f(y)$ increases monotonically from $-\infty$ to 0.

Hence $f(y) = -1/A_1$ has precisely one root

τ_p in $(\frac{(2p-1)\pi}{a}, \frac{2p\pi}{a})$

$$\text{i.e.} \quad \frac{(2p-1)\pi}{a} < \tau_p < \frac{2p\pi}{a} \tag{5-91}$$

and corresponding to τ_p we have a solution to the characteristic equation (5-83) i.e. an eigenvalue

$$\lambda_p = \sigma_p + i\tau_p \tag{5-92}$$

$$\text{where} \quad \sigma_p = -\frac{1}{a} \log \{\frac{\tau_p}{A_1 |\sin a\tau_p|}\} \tag{5-93}$$

We now want to find the asymptotic location of λ_p. From (5-91), it follows that $\tau_p \sim 2p\pi/a$

$$\sigma_p = A_1 e^{-a\sigma_p} \cos a\tau_p \tag{5-94}$$

$$\tau_p = -A_1 e^{-a\sigma_p} \sin a\tau_p \tag{5-95}$$

Taking the log of (5-95), we obtain

$$-a\sigma_p = \log \tau_p/A_1 = o(1) \tag{5-96}$$

Now $\tau_p = 2p\pi/a + o(1)$ (5-97)

$$\cos a\tau_p = \sigma_p e^{a\sigma_p}/A_1 = -\frac{1}{2p\pi}\log\frac{2p\pi}{aA_1} + O(\frac{\log p}{p})^2 \qquad (5\text{-}98)$$

Hence $\tau_p = (2p-\frac{1}{2})\frac{\pi}{a} - \frac{1}{2p\pi a}\log\frac{2p\pi}{aA_1} + O(\frac{\log p}{p})^2$ (5-99)

$$\sigma_p = -\frac{1}{a}\log\frac{(2p-\frac{1}{2})\pi}{aA_1} + O(\frac{\log p}{p})^2 \qquad (5\text{-}100)$$

Define $R_p = |\lambda_p|$ (5-101)

$$R_p^2 = \left(\frac{2p\pi}{a}\right)^2 \Big\{ 1 - \frac{1}{2p} + \frac{1}{16p^2} - \frac{1}{2p^2\pi^2}\log\left(\frac{2p\pi}{aA_1}\right) + \frac{1}{16p^4\pi^4}\left(\log\left(\frac{2p\pi}{aA_1}\right)\right)^2$$
$$+ \frac{1}{4p^3\pi^2}\log\left(\frac{2p\pi}{aA_1}\right) + \frac{1}{4p^2\pi^2}\left(\log\left(\frac{(2p-\frac{1}{2})\pi}{aA_1}\right)\right)^2 \Big]$$
$$+ \quad O\left(\frac{(\log p)^2}{p^3}\right) \qquad (5\text{-}102)$$

Hence $R_p = \frac{2p\pi}{a}(1 + \alpha(p))$ (5-103)

where $\alpha(p) = -\frac{1}{4p} + o(\frac{1}{p})$ (5-104)

Case (ii) $A_1 < 0$

Real roots $y = 0 \qquad x = A_1 e^{-ax} = -|A_1| e^{-ax}$ (5-105)

For $|A_1| > (ae)^{-1}$, no real roots

$|A_1| = (ae)^{-1}$ double root $\sigma_0 = -\frac{1}{a}$ (5-106)

$|A_1| < (ae)^{-1}$ two real roots σ_0', σ_0''

Complex roots $y \neq 0$ Again the purely imaginary case will be included and the roots will occur in conjugate pairs. Again we restrict attention only to those roots with positive imaginary part.

We want to find the (real) roots of

$$f(y) = 1/|A_1| \tag{5-107}$$

Now $f(0) = ae$ and $f(y)$ decreases monotonically from ae to 0 in interval $(0, \frac{\pi}{a})$.

Hence for $|A_1| > (ae)^{-1}$, $f(y)$ has one real root τ_0 on $(0, \frac{\pi}{a})$.

$|A_1| = (ae)^{-1}$ f(y) has (double) root $\tau_0 = 0$

$|A_1| < (ae)^{-1}$ f(y) has no roots on $(0, \frac{\pi}{a})$

For $p \geq 1$, f(y) decreases monotonically from ∞ to 0 on the interval $(\frac{2p\pi}{a}, \frac{(2p+1)\pi}{a})$ and hence $f(y) = 1/|A_1|$ has precisely one root τ_p in $(\frac{2p\pi}{a}, \frac{(2p+1)\pi}{a})$

$$\text{i.e.} \quad \frac{2p\pi}{a} < \tau_p < \frac{(2p+1)\pi}{a} \tag{5-108}$$

and corresponding to τ_p, we have an eigenvalue

$$\lambda_p = \sigma_p + i\tau_p$$

$$\text{where} \quad \sigma_p = -\frac{1}{a}\log\left\{\frac{\tau_p}{|A_1| \sin a\tau_p}\right\} \tag{5-109}$$

We want to find the asymptotic location of λ_p. From (5-108), it follows that $\tau_p \sim 2p\pi/a$

$$\text{i.e.} \quad \tau_p = 2p\pi/a + o(1) \tag{5-110}$$

$$\sigma_p = -\frac{1}{a}\log\frac{2p\pi}{a|A_1|} + o(1) \tag{5-111}$$

$$\cos a\tau_p = -\sigma_p e^{a\sigma_p}/|A_1|$$

$$\cos a\tau_p = \frac{1}{2p\pi}\log\frac{2p\pi}{a|A_1|} + O(\frac{\log p}{p})^2 \tag{5-112}$$

Hence $\tau_p = (2p - \frac{1}{2})\frac{\pi}{a} + \frac{1}{2p\pi a}\log\frac{2p\pi}{a|A_1|} + O(\frac{\log p}{p})^2$ (5-113)

$$\sigma_p = -\frac{1}{a}\log\frac{(2p+\frac{1}{2})\pi}{a|A_1|} + O(\frac{\log p}{p})^2 \tag{5-114}$$

$$R_p^2 = (\frac{2p\pi}{a})^2[1 + \frac{1}{4p} + \frac{1}{16p^2} - \frac{1}{2p^2\pi^2}\log\frac{2p\pi}{2|A_1|}$$

$$- \frac{1}{4p^3\pi^2}\log\frac{2p\pi}{a|A_1|} + \frac{1}{16p^4\pi^4}(\log\frac{2p\pi}{a|A_1|})^2$$

$$+ \frac{1}{4p^2\pi^2}(\log\frac{(2p-\frac{1}{2})\pi}{aA_1})^2] + O(\frac{(\log p)^2}{p^3})$$

$$R_p = \frac{2p\pi}{a}(1 + \alpha(p)) \tag{5-115}$$

where $\alpha(p) = \frac{1}{4p} + o(\frac{1}{p})$ (5-116)

Expressions (5-99), (5-100) and (5-110), (5-111) are well known and can be found in Pinney [66] and Wright [79].

Definition

Let $n_\lambda(t)$ be the number of $\{\lambda_p\}$ with modulus less than t

Define $$N_\lambda(R) = \int_0^R \frac{n_\lambda(t)}{t}\,dt \tag{5-117}$$

Lemma 5.4

$$\limsup_{R \to \infty} \{N_\lambda(R) - \frac{aR}{\pi} + \frac{1}{4}\log R\} = \infty \tag{5-118}$$

Proof Case (i) $A_1 > 0$

For $R_p > \sigma_0$, $n_\lambda(t) = 2p + 1 \quad R_p < t < R_{p+1}$ (5-119)

$$N_\lambda(R_p) = \int_0^{R_p} \frac{n_\lambda(t)}{t}\,dt = \sum_{m=0}^{p-1} \int_{R_m}^{R_{m+1}} \frac{2m\,dt}{t} + \int_{\sigma_0}^{R_p} \frac{dt}{t}$$

$$= \sum_{m=0}^{p-1} 2m[\log R_{m+1} - \log R_m] + \log R_p - \log \sigma_0$$

$$- 2p \log R_p - 2\sum_{m=1}^{p} \log R_m + \log R_p - \log \sigma_0 \tag{5-120}$$

From hence onwards, we shall denote any (finite) constant by A.

$$N_\lambda(R_p) - \frac{aR_p}{\pi} + \frac{1}{4}\log R_p$$

$$= 2p \log R_p - 2\sum_{m=1}^{p} \log R_m - \frac{aR_p}{\pi} + \frac{5}{4}\log R_p + A$$

$$= 2\sum_{m=1}^{p} \log \frac{R_p}{R_m} - 2p - 2p\alpha(p) + \frac{5}{4}\log p + \frac{5}{4}\log(1 + \alpha(p)) + A$$

$$= 2\log \frac{e^{-p}p^{p+\frac{1}{2}}}{p!} + 2\log(1 + \alpha(p))^p - 2\sum_{m=1}^{p}\log(1 + \alpha(m))$$

$$+ \frac{5}{4}\log(1 + \alpha(p)) - 2p\alpha(p) + A \qquad (5\text{-}121)$$

Now $\lim\limits_{p \to \infty} \dfrac{e^{-p}p^{p+\frac{1}{2}}}{p!} = \dfrac{1}{\sqrt{2\pi}}$

From (5-104), $\lim\limits_{p \to \infty} p\alpha(p) = -\frac{1}{4}$, $\lim\limits_{p \to \infty} \alpha(p) = 0$

and hence $\lim\limits_{p \to \infty} (1 + \alpha(p))^p = e^{-\frac{1}{4}}$

Also $\lim\limits_{m \to \infty} \dfrac{\log(1 + \alpha(m))}{-\frac{1}{4m}} = 1$

and hence the series $-\sum_{m=1}^{p} \log 1 + \alpha(m)$ and $\sum_{m=1}^{p} \frac{1}{4m}$

converge or diverge together.

But $\lim\limits_{p \to \infty} \sum_{m=1}^{p} \frac{1}{4m} = \infty$

and hence $\lim\limits_{p \to \infty} - 2 \sum_{m=1}^{p} \log (1 + \alpha(m)) = \infty$

Hence $\lim\limits_{R_p \to \infty} \{N_\lambda(R_p) - \frac{aR_p}{\pi} + \frac{1}{4} \log R_p\} = \infty$

Case (ii) $A_1 < 0$

For $p \geq 0$ $n_\lambda(t) = 2(p + 1)$ $R_p < t < R_{p+1}$

$$N_\lambda(R_p) = \int_0^{R_p} \frac{n_\lambda(t)}{t} dt = \sum_{m=0}^{p-1} \int_{R_m}^{R_{m+1}} \frac{2(m+1)dt}{t}$$

$$= \sum_{m=0}^{p-1} 2(m + 1)[\log R_{m+1} - \log R_m]$$

$$= 2(p + 1)\log R_p - 2 \sum_{m=0}^{p} \log R_m \qquad (5\text{-}122)$$

$$N_\lambda(R_p) - \frac{aR_p}{\pi} + \frac{1}{4}\log R_p$$

$$= 2(p+1)\log R_p - 2\sum_{m=0}^{p}\log R_m - \frac{aR_p}{\pi} + \frac{1}{4}\log R_p$$

$$= 2\sum_{m=0}^{p}\log\frac{R_p}{R_m} - 2p - 2p\alpha(p) + \frac{1}{4}\log p + \frac{1}{4}\log(1+\alpha(p))$$

$$= 2\log\frac{e^{-p}p^{p+\frac{1}{2}}}{p!} + \log p - 2p\alpha(p) + \frac{1}{4}\log p + \frac{1}{4}\log(1+\alpha(p))$$

$$+ 2\log(1+\alpha(p))^p - 2\sum_{m=1}^{p}\log(1+\alpha(m))$$

From (5-117), $\lim\limits_{p\to\infty} p\alpha(p) = \frac{1}{4}$, $\lim\limits_{p\to\infty}\alpha(p) = 0$

$$\lim_{p\to\infty}(1+\alpha(p))^p = e^{\frac{1}{4}}$$

Also $\lim\limits_{m\to\infty}\dfrac{\log(1+\alpha(m))}{\frac{1}{4m}} = 1$

Hence given $0 < \varepsilon < 1$, there exists m_0 such that

$$(1-\varepsilon)\frac{1}{4m} < \log(1+\alpha(m)) < (1+\varepsilon)\frac{1}{4m} \quad \text{for all } m \geq m_0$$

Hence $$-2\sum_{m=m_0}^{p} \log(1+\alpha(m)) \geq -(1+\varepsilon)\sum_{m=m_0}^{p} \frac{1}{2m}$$

and

$$\{\tfrac{1}{2}(1+\varepsilon)\log p - 2\sum_{m=m_0}^{p} \log(1+\alpha(m))\}$$

$$\geq \tfrac{1}{2}(1+\varepsilon)\{\log p - \sum_{m=m_0}^{p} \frac{1}{m}\}$$

But $\log p - \sum_{m=m_0}^{p} \frac{1}{m}$ is bounded as $p \to \infty$

Hence $\{\frac{1}{2}(1+\varepsilon)\log p - 2\sum_{m=m_0}^{p} \log(1+\alpha(m))\}$

is bounded from below as $p \to \infty$.

Hence $$\lim_{p\to\infty} \{N_\lambda(R_p) - \frac{aR_p}{\pi} + \frac{1}{4}\log R_p\}$$

$$\geq \lim_{p\to\infty} \{(\tfrac{3}{4} - \tfrac{1}{2}\varepsilon)\log p + A\} = \infty.$$

Theorem 5E

The eigenfunctions of $\mathcal{A}$ defined in (5-82) form a basis of $M^2(-a,0;R)$.

Proof We must first show that $\{e^{\lambda_p(\cdot)}\}$ is complete in $M^2(-a,0;R)$.

For suppose not.

Then there is a $f \in M^2(-a,0;R^n)$, $f \neq 0$ such that

$$(f,e^{\lambda_p(\cdot)})_{M^2} = f(0) + \int_{-a}^{0} f(\theta)e^{\lambda_p\theta}d\theta = 0 \quad \text{for all } p$$

$$\text{Define} \quad F(z) = f(0) + \int_{-a}^{0} f(\theta)e^{z\theta}d\theta \tag{5-123}$$

$F(z)$ is an entire function of z and $F(\lambda_p) = 0$ for all p

$$|F(z)| \leq |f(0)| + \left|\int_{-a}^{0} f(\theta)e^{z\theta}d\theta\right|$$

$$\int_{-a}^{0} f(\theta)e^{z\theta}d\theta = \int_{-a}^{-a+\varepsilon} + \int_{-a+\varepsilon}^{0} f(\theta)e^{z\theta}d\theta$$

$$\left|\int_{-a}^{0} f(\theta)e^{z\theta}d\theta\right| \leq \int_{-a}^{-a+\varepsilon} + \int_{-a+\varepsilon}^{0} |f(\theta)|e^{x\theta}d\theta$$

$$\leq \left(\int_{-a}^{-a+\varepsilon} e^{2x\theta}d\theta\right)^{\frac{1}{2}}\left(\int_{-a}^{-a+\varepsilon} |f(\theta)|^2d\theta\right)^{\frac{1}{2}}$$

$$+ \left(\int_{-a+\varepsilon}^{0} e^{2x\theta}d\theta\right)^{\frac{1}{2}}\left(\int_{-a+\varepsilon}^{0} |f(\theta)|^2d\theta\right)^{\frac{1}{2}}$$

Case (a) $x < 0$ $x = -|x|$

$$\int_{-a}^{-a+\varepsilon} e^{2x\theta} d\theta = \frac{1}{2|x|}[e^{-2|x|\theta}]_{-a+\varepsilon}^{-a} \leq \frac{1}{2|x|} e^{2|x|a}$$

$$\int_{-a+\varepsilon}^{0} e^{2x\theta} d\theta = \frac{1}{2|x|}[e^{-2|x|\theta}]_{0}^{-a+\varepsilon} \leq \frac{1}{2|x|} e^{2|x|(a-\varepsilon)}$$

Let $\delta(\varepsilon) = (\int_{-a}^{-a+\varepsilon} |f(\theta)|^2 d\theta)^{1/2}; \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$

Hence $|F(z)| \leq K_1(\frac{1}{2|x|})^{\frac{1}{2}} e^{|x|a}(e^{-\varepsilon|x|} + \delta(\varepsilon))$

and $\frac{1}{2\pi}\{\int_{-\pi}^{-\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi} \log |F(Re^{i\theta})| d\theta\}$

$$\leq -\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} \frac{1}{2} \log R \, d\theta - \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} \frac{1}{2} \log |\cos \theta| d\theta$$

$$+ \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} aR|\cos \theta| d\theta + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} \log (e^{-\varepsilon R|\cos \theta|} + \delta) d\theta + A$$

$$= -\frac{1}{4} \log R + \frac{aR}{\pi} + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} \log (e^{-\varepsilon R|\cos \theta|} + \delta) d\theta + A \tag{5-124}$$

Case (b) $x > 0 \quad x = |x|$

$$\int_{-a}^{-a+\varepsilon} e^{2x\theta}d\theta = \frac{1}{2|x|}\left[e^{2|x|\theta}\right]_{-a}^{-a+\varepsilon} \leq \frac{1}{2|x|}$$

$$\int_{-a+\varepsilon}^{0} e^{2x\theta}d\theta = \frac{1}{2|x|}\left[e^{2|x|\theta}\right]_{-a+\varepsilon}^{0} \leq \frac{1}{2|x|}$$

Hence $|F(z)| \leq f(0) + \dfrac{K_2}{2|x|^{\frac{1}{2}}}$ K_2 some constant

$$|F(Re^{i\theta})| \leq \begin{cases} 2|f(0)|^2 & |\sec\theta| \leq K_4|f(0)|^2R \\ 2K_3R^{-\frac{1}{2}}|\sec|^{\frac{1}{2}} & |\sec\theta| \geq K_4|f(0)|^2R \end{cases}$$

where K_3 and K_4 are constants.

Hence $\dfrac{1}{2\pi}\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log|F(Re^{i\theta})|d\theta \leq K_5$, for sufficiently large R and constant K_5

(5-125)

Now Jensen's theorem states that

$$N_F(R) = \int_0^R \frac{n_F(t)}{t}\, dt = \frac{1}{2\pi} \int_0^{2\pi} \log |F(Re^{i\theta})| d\theta - \log |F(0)| \tag{5-126}$$

where $n_F(t)$ is the number of zeros of F with modulus less than t.

Without any loss in generality, we can assume that $F(0) \neq 0$, since if $F(0) = 0$ with multiplicity r.
We can write

$$F(z) = z^r F_1(z)$$

where $F_1(z)$ is an entire function of z, $F_1(0) \neq 0$ and apply Jensen's theorem to F_1
Since $F(\lambda_p) = 0$ for all p, we have

$$N_\lambda(R) \leq N_F(R) \tag{5-127}$$

Hence from (5-124), (5-125) and (5-126) we have

$$N_\lambda(R) - \frac{aR}{\pi} + \frac{1}{4} \log R \leq \frac{1}{\pi} \int_{\frac{\pi}{2}}^{\pi} \log (e^{-\varepsilon R|\cos \theta|} + \delta) d\theta + A \tag{5-128}$$

If $\varepsilon > 0$ is small enough, and $R > 0$ is large enough, we can make

$$\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} \log\,(e^{-\varepsilon R|\cos\,\theta|} + \delta)d\theta$$

less than any arbitrary large negative number.

Hence $$\lim_{R \to \infty}\sup\,(N_\lambda(R) - \frac{aR}{\pi} + \frac{1}{4}\log R) = -\infty \qquad (5\text{-}129)$$

But this contradicts (5-118) lemma 5.4.

Hence either $f = 0$, in which case we have proved completeness or $F = 0$.

So suppose $F = 0$

Hence $$f(0) + \int_{-a}^{0} f(\theta)e^{z\theta}d\theta = 0 \qquad (5\text{-}130)$$

We differentiate to get

$$\int_{-a}^{0} \theta f(\theta)e^{z\theta}d\theta = 0$$

Putting $z = iy$ $$\int_{-a}^{0} \theta f(\theta)e^{iy\theta}d\theta = 0 \qquad (5\text{-}131)$$

Hence by the Fourier transform theorem,

$$\theta f(\theta) = 0 \quad \text{a.e. on } [-a,0].$$

Hence $f(\theta) = 0$ a.e. on $[-a,0]$

From (5-130), it follows that

$$f(0) = 0$$

Hence $f = 0$ as an element of $M^2(-a,0;R)$ and we have proved completeness.

To complete the proof, we have to show that $\{e^{\lambda_p(\cdot)}\}$ is strongly linear independent, i.e. that no member can be approximated by a linear combination of the others. Put another way, if

$$h_n(\theta) = \sum_{p=1}^{n} a_p^{(n)} e^{\lambda_p \theta} \quad \theta \in [-a,0]$$

and $h_n \to 0$ in M^2, then we have to show that

$$a_q^{(n)} \to 0 \text{ for fixed } q.$$

From corollary to lemma 5.1

$$\mathcal{H}(e^{-\lambda_q(\cdot)},h_n) = \mathcal{H}(e^{-\lambda_q(\cdot)},a_q^{(n)}e^{\lambda_q(\cdot)})$$

$$= a_q^{(n)}\{1 + A_1 \int_{-a}^{0} e^{-\lambda_q(\xi+a)} e^{\lambda_q\xi} d\xi\}$$

$$= a_q^{(n)}\{1 + aA_1e^{-\lambda_q a}\} = a_q^{(n)}\{1 + a\lambda_q\}.$$

Hence $a_q^{(n)} = \mathcal{H}(e^{-\lambda_q(\cdot)},h_n)/\{1 + a\lambda_q\}$

$$= \{h_n(0) + A_1 \int_{-a}^{0} e^{-\lambda_q(\xi+a)} h_n(\xi)\}/\{1 + a\lambda_q\}$$

$$\to 0 \quad \text{as} \quad h_n \to 0 \quad \text{in} \quad M^2$$

Hence proof.

Remark

1. The proof of the completeness of the eigenfunctions follows the work of Levinson [56], Boas [6] and Levin [54]. The concise proof that $F = 0 \Rightarrow f = 0$ is due to Levinson [57].

2. From the completeness of the eigenfunctions of (5-81) in $M^2(-a,0;R)$ and the continuity of solutions to a R.F.D.E. with respect to the initial data, it follows that any solution of (5-81) can be arbitrarily approximated by the eigenfunction solutions. Thus we have arrived at the series expansion of a solution to (5-81) as, for example, has been discussed in Bellman and Cooke [5] pp. 102-110.

Corollary 1

The eigenfunctions corresponding to the scalar R.F.D.E.

$$\frac{dx}{dt} = A_{00}x(t) + A_1 x(t-a), \quad A_1 \neq 0 \tag{5-132}$$

form a basis of $M^2(-a,0;R)$

Proof The characteristic equation yielding the eigenvalues is

$$\begin{aligned} \Delta(z) &= z - A_{00} - A_1 e^{-az} \\ &= (z - A_{00}) - (A_1 e^{-aA_{00}})e^{-a(z-A_{00})} = 0 \end{aligned} \tag{5-133}$$

and comparing with (5-83)

$$\Delta_0(\bar{z}) = \bar{z} - (A_1 e^{-aA_{00}})e^{-a\bar{z}} = 0 \qquad (5\text{-}134)$$

where $\bar{z} = z - A_{00}$

i.e. the eigenvalues are

$$\lambda_p' = \lambda_p + A_{00}$$

where λ_p are the roots of (5-83) with A_1 replaced by $A_1 e^{-aA_{00}}$.

Hence the proof is as in the proof of the theorem and its preceding development.

Corollary 2

The eigenfunctions corresponding to the R.F.D.E.

$$\frac{dx}{dt} = A_1 x(t-a) \qquad (5\text{-}135)$$

where $A_1 \in \mathcal{L}(R^n)$ and has real distinct non-zero eigenvalues, forms a basis of $M^2(-a,0;R^n)$

Proof Follows from decomposing (5-135) into n scalar R.F.D.E.'s and apply the previous theorem.

Counterexample 5.5

The eigenfunctions corresponding to a R.F.D.E. will not always be complete in M^2. Consider for example

$$\frac{dx}{dt} = A_1 x(t-1) \qquad (5\text{-}136)$$

where $A_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$

The characteristic equation is

$$\Delta(z) = \det \{zI - A_1 e^{-z}\} = z(z + e^{-z}) = 0$$

The eigenvalues are $\lambda = 0$ and λ_p

where λ_p satisfies $\lambda + e^{-\lambda} = 0$

The eigenfunctions are

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda_p \theta} , \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \qquad \theta \in [-1,0]$$

which are clearly not complete in $M^2(-a,0;R^2)$.

Conjecture 5.6

Consider the autonomous R.F.D.E.

$$\frac{dx}{dt} = A_{00}x(t) + \sum_{i=1}^{N} A_i x(t + \theta_i) + \int_{-a}^{0} A_{01}(\theta)x(t + \theta)d\theta \tag{5-137}$$

A_{00}, A_i $(i = 1\ldots N) \in \mathcal{L}(R^n)$, $A_{01}(\cdot) \in L^2(-a,0;\mathcal{L}(R^n))$

and suppose that either

(i) det $A_N \neq 0$

or (ii) det $A_{01}(\theta) \neq 0$ a.e. on some set $[-a,-a+\varepsilon]$ for some $\varepsilon > 0$

Then the eigenfunctions corresponding to (5-137) will be complete and form a basis in $M^2(-a,0;R^n)$.

(5.6) Example

We will now work out an example to illustrate our method of generating approximate optimal controls.

We consider the one dimensional controlled hereditary system

$$\frac{dx}{dt} = A_1 x(t-1) + Bv(t)$$

$$x(\theta) = h(\theta), \qquad \theta \in [-1,0] \tag{5-138}$$

with quadratic cost

$$C(v) = \int_0^T \{Q|x(t)|^2 + N|v(t)|^2\}dt \tag{5-139}$$

and class of admissible controls

$$\mathcal{U} = \{v; \int_0^T |v(t)|^2 dt < \infty\} \tag{5-140}$$

We take $A_1 > 0$. From the results of the last section, the eigenfunctions of the differential operator

$$[\mathcal{A}h](\alpha) = \begin{cases} A_1 h(-1) & \alpha = 0 \\ \dfrac{dh}{d\alpha} & \alpha \in [-1,0) \end{cases} \tag{5-141}$$

will be a basis for $M^2(-1,0;R^n)$

We take a finite set of the eigenvalues of

$$\Lambda_n = \{\sigma_0, \lambda_1, \lambda_{-1}, \dots \lambda_n, \lambda_{-n}\} \qquad (5\text{-}142)$$

$$\lambda_{\pm n} = \sigma_n \pm i\tau_n$$

and where the ordering is as in the previous section. σ_j and τ_j satisfy for $j \geq 1$

$$\sigma_j = A_1 e^{-\sigma_j} \cos \tau_j ; \quad \tau_j = - A_1 e^{-\sigma_j} \sin \tau_j \qquad (5\text{-}143)$$

$$\sigma_0 \text{ satisfies } \sigma_0 = A_1 e^{-\sigma_0} \qquad (5\text{-}144)$$

$$\Omega_n = \text{column } \{\phi_0, \phi_1, \phi_{-1}, \dots \phi_n, \phi_{-n}\} \qquad (5\text{-}145)$$

where for $-1 \leq \theta \leq 0$,

$$\begin{aligned}
\phi_0(\theta) &= e^{\sigma_0\theta} \\
\phi_1(\theta) &= e^{\sigma_1\theta}\sin \tau_1\theta \\
\phi_{-1}(\theta) &= e^{\sigma_1\theta}\cos \tau_1\theta \\
&\cdots \\
&\cdots \\
\phi_n(\theta) &= e^{\sigma_n\theta}\sin \tau_n\theta \\
\phi_{-n}(\theta) &= e^{\sigma_n\theta}\cos \tau_n\theta
\end{aligned} \tag{5-146}$$

$$\Psi_n' = \text{row}\,\{\psi_0', \psi_1', \psi_{-1}' \ldots \psi_n', \psi_{-n}'\} \tag{5-147}$$

where for $0 \leq \alpha \leq 1$

$$\psi'_0(\alpha) = e^{-\sigma_0\alpha}$$

$$\psi'_1(\alpha) = e^{-\sigma_1\alpha}\sin \tau_1\alpha$$

$$\psi'_{-1}(\alpha) = e^{-\sigma_1\alpha}\cos \tau_1\alpha \qquad (5\text{-}148)$$

$$\cdots$$

$$\cdots$$

$$\psi'_n(\alpha) = e^{-\sigma_n\alpha}\sin \tau_n\alpha$$

$$\psi'_{-n}(\alpha) = e^{-\sigma_n\alpha}\cos \tau_n\alpha$$

$$\mathcal{H}(\psi,\phi) = \psi(0)\phi(0) + A_1 \int_{-1}^{0} \psi(\xi+1)\phi(\xi)d\xi \qquad (5\text{-}149)$$

For $j \geq 1, \mathcal{H}(\psi'_j,\phi_j) = A_1 \int_{-1}^{0} e^{-\sigma_j(\xi+1)} \sin \tau_j(\xi+1)e^{\sigma_j\xi} \sin (\tau_j\xi)d\tau$

$$= \frac{1}{2}(1 + \sigma_j) \qquad (5\text{-}150)$$

$$\mathcal{H}(\psi_j',\phi_j) = A_1 \int_{-1}^{0} e^{-\sigma_j(\xi+1)} \sin \tau_j(\xi+1) e^{\sigma_j \xi} \cos (\tau_j \xi) d\xi$$

$$= -\frac{1}{2}\tau_j \tag{5-151}$$

$$\mathcal{H}(\psi_{-j}',\phi_j) = A_1 \int_{-1}^{0} e^{-\sigma_j(\xi+1)} \cos \tau_j(\xi+1) e^{\sigma_j \xi} \sin (\tau_j \xi) d\xi$$

$$= \frac{1}{2}\tau_j \tag{5-152}$$

$$\mathcal{H}(\psi_{-j}',\phi_j) = 1 + A_1 \int_{-1}^{0} e^{-\sigma_j(\xi+1)} \cos \tau_j(\xi+1) e^{\sigma_j \xi} \cos (\tau_j \xi) d\xi$$

$$= \frac{1}{2}(1 + \sigma_j) \tag{5-153}$$

From corollary to lemma 5.1 it follows that

$$\mathcal{H}(\psi_j',\phi_k) = 0 \quad \text{for} \quad |j| \neq |k|$$

Also $\mathcal{H}(\psi_0',\phi_0) = 1 + \sigma_0$ (5-154)

Hence

$$(\Psi_n',\Omega_n) = \begin{bmatrix} 1+\sigma_0 & 0 & 0 & & & \\ 0 & \frac{1}{2}(1+\sigma_1) & -\frac{1}{2}\tau_1 & & & \\ 0 & \frac{1}{2}\tau_1 & \frac{1}{2}(1+\sigma_1) & & & \\ & & & & & \\ & & & & 0 & 0 \\ & & & 0 & \frac{1}{2}(1+\sigma_n) & -\frac{1}{2}\tau_n \\ & & & 0 & \frac{1}{2}\tau_n & \frac{1}{2}(1+\sigma_n) \end{bmatrix} \tag{5-155}$$

Define $\mu_j = 4/\{(1+\sigma_j)^2 + \tau_j^2\} \quad j \geq 1$

$$\mu_0 = 1/(1+\sigma_0) \tag{5-156}$$

$\mu_0 \neq 0$, $\mu_j \neq 0$ for any $j \geq 1$

$$(\Psi_n', \Omega_n)^{-1} = \begin{bmatrix} \mu_0 & 0 & 0 & & & \\ 0 & \frac{1}{2}\mu_1(1+\sigma_1) & \frac{1}{2}\mu_1\tau_1 & & & \\ 0 & -\frac{1}{2}\mu_1\tau_1 & \frac{1}{2}\mu_1(1+\sigma_1) & & & \\ & & & & & \\ & & & & 0 & 0 \\ & & & 0 & \frac{1}{2}\mu_n(1+\sigma_n) & \frac{1}{2}\mu_n\tau_n \\ & & & 0 & -\frac{1}{2}\mu_n\tau_n & \frac{1}{2}\mu_n(1+\sigma_n) \end{bmatrix} \tag{5-157}$$

Define $\Psi_n = (\Psi_n', \Omega_n)^{-1}\Psi_n'$ (5-158)

For $0 \leq \alpha \leq 1$

$$\Psi_n(\alpha) = \begin{bmatrix} \mu_0 e^{-\sigma_0\alpha} \\ \frac{1}{2}\mu_1 e^{-\sigma_1\alpha}[(1+\sigma_1)\sin\tau_1\alpha + \tau_1\cos\tau_1\alpha] \\ \frac{1}{2}\mu_1 e^{-\sigma_1\alpha}[-\tau_1\sin\tau_1\alpha + (1+\sigma_1)\cos\tau_1\alpha] \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \frac{1}{2}\mu_n e^{-\sigma_n\alpha}[(1+\sigma_n)\sin\tau_n\alpha + \tau_n\cos\tau_n\alpha] \\ \frac{1}{2}\mu_n e^{-\sigma_n\alpha}[-\tau_n\sin\tau_n\alpha + (1+\sigma_n)\cos\tau_n\alpha] \end{bmatrix} \tag{5-159}$$

$$\Psi_n(0) = \begin{bmatrix} \mu_0 \\ \frac{1}{2}\mu_1\tau_1 \\ \frac{1}{2}\mu_1(1+\sigma_1) \\ \cdots \\ \frac{1}{2}\mu_n\tau_n \\ \frac{1}{2}\mu_n(1+\sigma_n) \end{bmatrix} \qquad (5\text{-}160)$$

For $-1 \leq \theta \leq 0$

$$\Omega_n(\theta) = \left[e^{\sigma_0\theta}, e^{\sigma_1\theta}\sin\tau_1\theta, e^{\sigma_1\theta}\cos\tau_1\theta, \ldots\ldots e^{\sigma_n\theta}\sin\tau_n\theta, e^{\sigma_n\theta}\cos\tau_n\theta\right]$$

Hence $\Omega_n(0) = [1, 0, 1, \ldots 0, 1]$ (5-162)

Since $\mathcal{A}\Omega_n = \Omega_n\mathcal{L}_n$, we obtain

$$\mathcal{L}_n = \begin{bmatrix} \sigma_0 & 0 & 0 & & & & \\ 0 & \sigma_1 & -\tau_1 & & & & \\ 0 & \tau_1 & \sigma_1 & & & & \\ 0 & 0 & & & & & \\ & & & & & & \\ & & & & & 0 & 0 \\ & & & & 0 & \sigma_n & -\tau_n \\ & & & & 0 & \tau_n & \sigma_n \end{bmatrix} \tag{5-163}$$

The $(2n+1)^{th}$ order approximate control problem is minimize

$$C_n(v;h) = \int_0^T \{(y_n(t), \Omega_n^*(0)Q\Omega_n(0)y_n(t)) + N|v(t)|^2\}dt \tag{5-164}$$

where $y_n(t)$ is a $(2n+1)$ vector satisfying

$$\begin{aligned} \dot{y}_n(t) &= \mathcal{L}_n y_n(t) + \Psi_n(0)Bv(t) \\ y_n(0) &= [\Psi_n, h] \end{aligned} \tag{5-165}$$

where for instance, the first component of $y_n(t)$ is

$$\mathscr{H}(\mu_0 e^{-\sigma(\cdot)}, h) = \mu_0 h(0) + \mu_0 A_1 \int_{-1}^{0} e^{-\sigma_0(\xi+1)} h(\xi) d\xi$$

The other components can be calculated in a similar fashion.

If for instance $h(0) \neq 0 \quad h^1 = 0$

$$y_n(0) = \begin{bmatrix} \mu_0 h(0) \\ \frac{1}{2}\mu_1 \tau_1 h(0) \\ \frac{1}{2}\mu_1 (1+\sigma_1) h(0) \\ \cdots \\ \frac{1}{2}\mu_n \tau_n h(0) \\ \frac{1}{2}\mu_n (1+\sigma_n) h(0) \end{bmatrix}$$

The $(2n+1)^{th}$ approximate optimal control is

$$u_n(t) = - N^{-1} B \Psi_n^*(0) P_n(t) y_n(t) \tag{5-166}$$

where the (2n+1) × (2n+1) matrix $P_n(t)$ satisfies the matrix Riccati differential equation

$$\begin{aligned}\dot{P}_n(t) + \mathcal{L}_n^* P_n(t) + P_n(t)\mathcal{L}_n - P_n(t)\Psi_n(0)B^2N^{-1}\Psi_n^*(0)P_n(t) \\ + \Omega_n^*(0)\, Q\, \Omega_n(0) = 0\end{aligned} \qquad (5\text{-}167)$$

$$P_n(T) = 0$$

and the (2n+1) vector $y_n(t)$ satisfies the differential equation

$$\begin{aligned}\dot{y}_n(t) &= \{\mathcal{L}_n - \Psi_n(0)B^2N^{-1}\Psi_n^*(0)P_n(t)\}y_n(t) \\ y_n(0) &= [\Psi_n, h]\end{aligned} \qquad (5\text{-}168)$$

There are standard numerical methods of solving (5-167) to obtain (approximately) $P_n(t)$ with knowledge of $P_n(t)$, the $(2n+1)^{th}$ order approximate control problem is completely solved.

Remark

Note the resemblance between the $(2n+1)^{th}$ order approximate control for the scalar R.F.D.E. and that for a system governed by a scalar P.D.E. of hyperbolic or parabolic type.

Chapter 6

Applications, Suggestions for Further Research, Conclusions

(6.1) The business cycle, or a control theorist looks at economics.

Mayr ([58] pp. 128) has an interesting discussion on the possible influence of control mechanisms on economic thought. So for instance the Baroque preoccupation with an inflexible predetermined feedforward control, as evidenced by the countless inventions of automatons, led to the Mercantilistic economics of a rigidly planned centrally directed economy. The increase in the use of feedback devices at the start of the Industrial Revolution led to Adam Smith's free enterprise economic philosophy that the economy would automatically swing into equilibrium at optimal conditions without governmental interference.

However, by the nineteen thirties, Adam Smith's laissez faire economics was no longer viable and the business cycle, with its alternate successions of severe depressions and runaway inflations, was a terrible scourge on the capitalistic economies. Marxist economists claimed that the business cycle was an inherent trait of a capitalistic economy. Nonetheless, a cure was forthcoming. Keynes [43] was largely responsible for the recommendation that the government should intervene and regulate the economy by means of taxation and public spending. It is interesting to note that since the second world war there has not been a severe depression nor indeed does anybody seriously anticipate its reoccurence.

What, though is the optimal government policy? We shall try to answer this question by considering a R.F.D.E. model of the business cycle due to Kalechi [40] along with a quadratic objective functional. Kalechi's model takes into account the fact that there will be a time lag between the decision to invest in a capital good and the completion of the finished product. From this we can obtain an R.F.D.E. for the rate of investment.

There are three stages of an investment: the order for the capital good, the production and the delivery. We denote by

$I(t)$ the rate of investment orders at time t

$J(t)$ the rate of production of capital goods at time t

$L(t)$ the rate of delivery of capital goods at time t

Let a be the time lag between investment decision and completion of capital good i.e. the gestation period. The relation between L and I is simple

$$L(t) = I(t-a) \tag{6-1}$$

Let $W(t)$ be the total volume of unfilled investment orders at time t. We have

$$W(t) = \int_{t-a}^{t} I(\tau)d\tau \tag{6-2}$$

since no order during the period $[t-a, t]$ is yet finished while all the orders before that period have been completed.

The rate of production must be

$$J(t) = \frac{1}{a} W(t) = \frac{1}{a} \int_{t-a}^{t} I(\tau)d\tau \tag{6-3}$$

Let $K(t)$ be the stock of capital goods at time t.

The derivative will be given by

$$\dot{K}(t) = L(t) - D + v(t) \tag{6-4}$$

where D is the depreciation of the capital stock and v(t) is the rate of governmental investment, positive for public spending and negative for taxation. v(t) is therefore the controlling input of the government independent of considerations of profitability.

Let B(t) be the gross profit and C(t) the consumption at time t. C(t) is assumed to consist of a constant part C_1 and a part proportional to B(t) i.e. $\lambda B(t)$

i.e. $$C(t) = C_1 + \lambda B(t) \tag{6-5}$$

Also $$B(t) = C(t) + J(t) \tag{6-6}$$

From (6-5) and (6-6), $$B(t) = (C_1 + J(t))/(1 - \lambda) \tag{6-7}$$

Kalechi assumes that the relative investment rate is a linear function of the relative profit rate and obtains

$$I(t) = m(C_1 + J(t)) - nK(t) \tag{6-8}$$

where m and n are constants.

Let I_0 be the desired rate of investment and $x(t) = I(t) - I_0$ the deviation from the desired rate of investment.

Differentiating (6-8), (6-3) and combining with (6-4) we obtain Kalechi's equation

$$\frac{dx}{dt} = A_{00}x(t) + A_1x(t - a) + Bv(t) + f \tag{6-9}$$

where $A_{00} = \frac{m}{a}$, $A_1 = -(\frac{m}{a} + n)$, $B = -n$, $f = -n(I_0 - D)$ (6-10)

We consider (6-9) over some time interval $[0,T]$ and with initial condition

$$x(\theta) = h(\theta), \quad \theta \in [-a,0], \quad h \in M^2 \tag{6-11}$$

Not surprisingly, we complete the control problem by considering the objective function

$$\text{minimize} \quad C(v;\, h) = \int_0^T \{q|x(t)|^2 + |v(t)|^2\}dt \qquad (6\text{-}12)$$

with admissible class of controls

$$= \{v;\, \int_0^T (v(t)|^2 dt < \infty\} = L^2(0,\, T;\, R) \qquad (6\text{-}13)$$

where $q \geq 0$ is a weight indicative of the trade off between deviation and the magnitude of the control.

Remarks

1. Central to Kalechi's model is the role played by $K(t)$. Investment activity is directly related to profitability and prices do not enter into the picture.

2. Very few economic variables appear in the model.

3. A mathematical analysis of Kalechi's equation (6-9), a study of its spectrum and eigensolutions and of its stability can be found in Frisch and Holme [31].

4. With no governmental intervention $v = 0$ and with $f = 0$, Kalechi's model yields the following dilemma

of the capitalistic system: growth and instability or stability and stagnation. For further details see Lange [51] chapter 5.

5. From an examination of U.S.A. economic data for the years 1909-1918, Kalechi obtains $m = 0.95$ and $n = 0.121$. He takes $a = 0.6$ years from a consideration of the lag between orders and deliveries in the industrial trades.

Now from the results of chapter 2, namely corollary to theorem 2B, we can write an exact closed form solution to Kalechi's equation (6-9) with initial condition (6-11). It is

$$x(t) = \Phi^0(t)h(0) + \int_{-a}^{\min(0,t-a)} \Phi^0(t-\alpha-a)h(\alpha)d\alpha + \int_0^t \Phi^0(t-s)\{Bv(s) + f\}ds \tag{6-14}$$

$$\text{where} \quad \Phi^0(t) = \begin{cases} e^{A_{00}t} \sum\limits_{j=0}^{p} \dfrac{(e^{-aA_{00}}A_1)^j (t-ja)^j}{j!} & t \in [p\alpha, (p+1)\alpha] \\ & p \in \mathbb{Z}^+ \\ 0 & t < 0 \end{cases} \tag{6-15}$$

From the results of chapter 3, we have a unique optimal control to (6-10), (6-11) (6-12) (6-13) given in feedback form by

$$u(t) = -B\{\Pi_{00}(t)x(t) + \int_{-a}^{0} \Pi_{01}(t,\alpha)x(t+\alpha)d\alpha + g_0(t)\} \qquad (6\text{-}16)$$

where

$$\frac{d\Pi_{00}(t)}{dt} + 2A_{00}\Pi_{00}(t) - B^2\Pi_{00}^2(t) + 2\Pi_{01}(t,0) + q = 0$$

$$\Pi_{00}(T) = 0 \qquad (6\text{-}17)$$

$$\left[\frac{\partial}{\partial t} - \frac{\partial}{\partial \alpha}\right]\Pi_{01}(t,\alpha) + A_{00}\Pi_{01}(t,\alpha) - B^2\Pi_{00}(t)\Pi_{01}(t,\alpha) + \Pi_{11}(t,0,\alpha) = 0$$

$$(6\text{-}18)$$

$$\Pi_{01}(T,\alpha) = \theta \quad \text{a.e.} \quad \alpha \in [-a,0]; \quad \Pi_{01}(t,-a) = \Pi_{00}(t)A_1$$

$$\left[\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \alpha}\right]\Pi_{11}(t,\theta,\alpha) = B^2\Pi_{01}(t,\theta)\Pi_{01}(t,\alpha)$$

$$\Pi_{11}(T,\theta,\alpha) = 0 \quad \text{a.e.} \quad (\theta,\alpha) \in [-a,0] \times [-a,0] \qquad (6\text{-}19)$$

$$\Pi_{11}(t,-a,\alpha) = A_1\Pi_{01}(t,\alpha); \quad \Pi_{11}(t,\theta,-a) = A_1\Pi_{01}(t,\theta)$$

$$\frac{dg_0(t)}{dt} + A_{00}g_0(t) - B^2\Pi_{00}(t)g_0(t) + \Pi_{00}(t)f + g_1(t,0) = 0$$

(6-20)

$$g(T) = 0$$

$$[\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta}]g_1(t,\theta) - B^2\Pi_{01}(t,\theta)g_1(t,\theta) + \Pi_{01}(t,\theta)f = 0$$

(6-21)

$$g_1(T,\theta) = 0 \quad \text{a.e.} \quad \theta \in [-a,0], \; g(t,-a) = A_1g_0(t)$$

Of economic significance is that the optimal policy will be in feedback form. There is no known solution of equations (6-16) through (6-21) and so to obtain an answer in concrete form we have to apply the approximation method of chapter 5. From corollary 1 to theorem 5E the eigenfunctions corresponding to Kalechi's equation (6-9) will form a basis in $M^2(-a,0;R)$ and so we can apply the results of section 3 to obtain a concrete approximate answer.

The eigenfunctions of Kalechi's equation (6-9) have already been used to study the stability of the solutions of the equation, Frisch and Holme [31]. Here we have extended their use to another purpose - that of finding an approximate optimal control to the control problem (6-9), (6-11), (6-12), (6-13).

6.2 Suggestions for further research; Conclusions

Mention has already been made of the several advantages of setting the R.F.D.E. problem in the functional space M^2. The main conclusion of this thesis is that M^2 is the appropriate functional space in tackling the quadratic criterion problem for hereditary systems. The solution to the problem follows naturally from the structure of the M^2 framework, use of the Lions' direct method provides us with a vastly superior approach in terms of elegance, aesthetics and generality, and concrete results are obtained rigorously without having to make ad-hoc assumptions.

Success in one area does not necessarily guarantee success in another, but it provides a strong incentive to try. In this light, the following topics seem worthy of further attention:

1) The formulation and the solution of the control problem for systems governed by neutral functional differential equation with quadratic cost and within some suitable analogue of the M^2 function space.

2) The formulation and solution of the stochastic control problem for hereditary systems with quadratic

cost functional within the M^2 framework (or its analogue for the neutral functional differential).

3) It would be interesting and hopefully fruitful to pose problems of interest in the control theory of hereditary systems within the M^2 framework. In particular, one such study could be the realization theory for hereditary systems.

Finally, there was one question raised within the M^2 framework and which was not answered in full generality. That is the conjecture 5.6 on the completeness of the eigensolutions of $\mathcal{A}$. That problem is of mathematical interest in its own right and it ought to be possible to supply an answer and a proof.

Appendix 1 - Proof of theorem 3C

From eqn (3-92), we have

$$(h, \Pi(t)k)_{M^2} = (\Phi(T,t)h, \mathcal{F}\,\Phi(T,t)k)_{M^2}$$

$$+\int_t^T ds \{ (\Phi(s,t)h, Q(s)\,\Phi(s,t)k)_{M^2} + (\Pi(s)\,\Phi(s,t)h, R(s)\Pi(s)\,\Phi(s,t)k)_{M^2} \}$$

Now
$$(\Phi(s,t)h)(\theta) = \begin{cases} \Phi^{\circ}(s+\theta,t)\,h(0) + \int_{-a}^{0} \Phi'(s+\theta,t,\alpha)\,h(\alpha) & s+\theta \geq t \\ h(s-t+\theta) & s+\theta < t \end{cases}$$
(A.1-1)

where $\Phi^{\circ}(\cdot,\cdot)$ is defined in eqn. (3-86)
and $\Phi'(\cdot,\cdot,\cdot)$ is defined in eqn. (3-87)

Recall that $\Phi^{\circ}(s,t) = 0$ for $s < t$
For convenience in notation and in computation
we shall define for both $s+\theta < t$ and $s+\theta \geq t$

$$(\Phi(s,t)h)(\theta) = \Phi^{\circ}(s+\theta,t)\,h(0) + \int_{-a}^{0} \Phi'(s+\theta,t,\alpha)\,h(\alpha)\,d\alpha \qquad (A.1\text{-}2)$$

where the meaning of (A.1-2) is to be taken as that
given by (A.1-2).

$$(R(s)\,\Pi(s)\,\Phi(s,t)\,k)(0) = R(s)\,\Pi_{00}(s)\,\Phi(s,t)\,k(0) + \int_{-a}^{0} R(s)\Pi_{00}(s)\,\Phi'(s,t,\alpha)\,k(\alpha)\,d\alpha$$

$$+\int_{-a}^{0} R(s)\,\Pi_{01}(s,\beta)\,\Phi^{\circ}(s+\beta,t)\,k(0)\,d\beta$$

$$+\int_{-a}^{0} d\alpha \int_{-a}^{0} d\beta\; R(s)\,\Pi_{01}(s,\beta)\,\Phi'(s+\beta,t,\alpha)\,k(\alpha)$$

Notation
$$\int_a^{\{b\}} f(\theta)\,d\theta = \begin{cases} 0 & a \geq b \\ \int_a^b f(\theta)\,d\theta & a < b \end{cases}$$

Writing out (3-92) in full we have

$$(h(0), \Pi_{00}(t)\, k(0)) + \int_{-a}^{0} d\alpha\, (h(0), \Pi_{01}(t,\alpha)\, k(\alpha))$$

$$+ \int_{-a}^{0} d\theta\, (h(\theta), \Pi_{10}(t,\theta)\, k(0)) + \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha\, (h(\theta), \Pi_{11}(t,\theta,\alpha)\, k(\alpha))$$

$$= (\Phi^{0}(T,t)\, h(0), F\, \Phi^{0}(T,t)\, k(0))$$

$$+ \int_{-a}^{0} d\alpha\, (\Phi^{0}(T,t)\, h(0), F\, \Phi'(T,t,\alpha)\, k(\alpha))$$

$$+ \int_{-a}^{0} d\theta\, (\Phi'(T,t,\theta)\, h(\theta), F\, \Phi^{0}(T,t)\, k(0))$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha\, (\Phi'(T,t,\theta)\, h(\theta), F\, \Phi'(T,t,\alpha)\, k(\alpha))$$

$$+ \int_{t}^{T} ds\, (\Phi^{0}(s,t)\, h(0), [Q(s) + \Pi_{00}(s)\, R(s)\, \Pi_{00}(s)]\, \Phi^{0}(s,t)\, k(0))$$

$$+ \int_{t}^{T} ds \int_{-a}^{0} d\alpha\, (\Phi^{0}(s,t)\, h(0), Q(s)\, \Phi'(s,t,\alpha)\, k(\alpha))$$

$$+ \int_{t}^{T} ds \int_{-a}^{0} d\theta\, (\Phi'(s,t,\theta)\, h(\theta), Q(s)\, \Phi^{0}(s,t)\, k(0))$$

$$+ \int_{t}^{T} ds \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha\, (\Phi'(s,t,\theta)\, h(\theta), Q(s)\, \Phi'(s,t,\alpha)\, k(\alpha))$$

$$+ \int_{t}^{T} ds \int_{-a}^{0} d\eta\, (\Pi_{01}(s,\eta)\, \Phi^{0}(s+\eta,t)\, h(0), R(s)\, \Pi_{00}(s)\, \Phi^{0}(s,t)\, k(0))$$

$$+ \int_{t}^{T} ds \int_{-a}^{0} d\beta\, (\Pi_{00}(s)\, \Phi^{0}(s,t)\, h(0), R(s)\, \Pi_{01}(s,\beta)\, \Phi^{0}(s+\beta,t)\, k(0))$$

$$+ \int_t^T ds \int_{-a}^0 d\eta \int_{-a}^0 d\beta \; \Pi_{01}(s,\eta) \, \Phi^0(s+\eta,t) \, h(0), \; R(s) \, \Pi_{01}(s,\beta) \, \Phi^0(s+\beta,t) \, k(0)$$

$$+ \int_t^T ds \int_{-a}^0 d\alpha \, (\Pi_{00}(s) \, \Phi^0(s,t) \, h(0), \; R(s) \, \Pi_{00}(s) \, \Phi^1(s,t,\alpha) \, k(\alpha))$$

$$+ \int_t^T ds \int_{-a}^0 d\alpha \int_{-a}^0 d\beta \, (\Pi_{00}(s) \, \Phi^0(s,t) \, h(0), \; R(s) \, \Pi_{01}(s,\beta) \, \Phi^1(s+\beta,t,\alpha) \, k(\alpha))$$

$$+ \int_t^T ds \int_{-a}^0 d\eta \int_{-a}^0 d\alpha \, (\Pi_{01}(s,\eta) \, \Phi^0(s+\eta,t) \, h(0), \; R(s) \, \Pi_{00}(s) \; \Phi^0(s,t,\alpha) \, k(\alpha))$$

$$+ \int_t^T ds \int_{-a}^0 d\eta \int_{-a}^0 d\beta \int_{-a}^0 d\alpha \, (\Pi_{01}(s,\eta) \, \Phi^0(s+\eta,t) \, h(0), \; R(s) \, \Pi_{01}(s,\beta) \, \Phi^0(s+\beta,t,\alpha) \, k(\alpha))$$

$$+ \int_t^T ds \int_{-a}^0 d\theta \, (\Pi_{00}(s) \, \Phi^1(s,t,\theta) \, h(\theta), \; R(s) \, \Pi_{00}(s) \, \Phi^0(s,t) \, k(0))$$

$$+ \int_t^T ds \int_{-a}^0 d\beta \int_{-a}^0 d\theta \, (\Pi_{00}(s) \, \Phi^1(s,t,\theta) \, h(\theta), \; R(s) \, \Pi_{01}(s,\beta) \, \Phi^0(s+\beta,t) \, k(0))$$

$$+ \int_t^T ds \int_{-a}^0 d\eta \int_{-a}^0 d\theta \, (\Pi_{01}(s,\eta) \, \Phi^1(s+\eta,t,\theta) \, h(\theta), \; R(s) \, \Pi_{00}(s) \, \Phi^0(s,t) \, k(0))$$

$$+ \int_t^T ds \int_{-a}^0 d\beta \int_{-a}^0 d\eta \int_{-a}^0 d\theta \, (\Pi_{01}(s,\eta) \, \Phi^1(s+\eta,t,\theta) \, h(\theta), \; R(s) \, \Pi_{01}(s,\beta) \, \Phi^0(s+\beta,t) \, k(0))$$

$$+ \int_t^T ds \int_{-a}^0 d\theta \int_{-a}^0 d\alpha \, (\Pi_{00}(s) \, \Phi^1(s,t,\theta) \, h(\theta), \; R(s) \, \Pi_{00}(s) \, \Phi^1(s,t,\alpha) \, k(\alpha))$$

$$+ \int_t^T ds \int_{-a}^0 d\beta \int_{-a}^0 d\theta \int_{-a}^0 d\alpha \, (\Pi_{00}(s) \, \Phi^1(s,t,\theta) \, h(\theta), \; R(s) \, \Pi_{01}(s,\beta) \, \Phi^1(s+\beta,t,\alpha) \, k(\alpha))$$

$$+ \int_t^T ds \int_{-a}^0 d\eta \int_{-a}^0 d\theta \int_{-a}^0 d\alpha \, (\Pi_{01}(s,\eta) \, \Phi^1(s+\eta,t,\theta) \, h(\theta), \; R(s) \, \Pi_{00}(s) \, \Phi^1(s,t,\alpha) \, k(\alpha))$$

$$+ \int_t^T ds \int_{-a}^0 d\eta \int_{-a}^0 d\beta \int_{-a}^0 d\theta \int_{-a}^0 d\alpha \, (\Pi_{01}(s,\eta) \, \Phi^1(s+\eta,t,\theta) \, h(\theta), \; R(s) \, \Pi_{01}(s,\beta) \, \Phi^1(s+\beta,t,\alpha) \, k(\alpha))$$

(A.1-3).

Putting $h' = k' = 0$ in (A.1-3) we obtain

$$
\begin{aligned}
(h(0), \Pi_{00}(t)\, k(0)) &= (\Phi^{0}(T,t)\, h(0), F\, \Phi^{0}(T,t)\, k(0)) \\
&+ \int_t^T ds\, (\Phi^{0}(s,t)\, h(0), [Q(s) + \Pi_{00}(s) R(s) \Pi_{00}(s)]\, \Phi^{0}(s,t)\, k(0)) \\
&+ \int_t^T ds \int_{\max(-a,\, t-s)}^{0} d\eta\, (\Pi_{01}(s,\eta)\, \Phi^{0}(s+\eta, t)\, h(0), R(s)\, \Pi_{00}(s)\, \Phi^{0}(s,t)\, k(0)) \\
&+ \int_t^T ds \int_{\max(-a,\, t-s)}^{0} d\beta\, (\Pi_{00}(s)\, \Phi^{0}(s,t)\, h(0), R(s)\, \Pi_{01}(s,\beta)\, \Phi^{0}(s+\beta, t)\, k(0)) \\
&+ \int_t^T ds \int_{\max(-a,\, t-s)}^{0} d\eta \int_{\max(-a,\, t-s)}^{0} d\beta\, (\Pi_{01}(s,\eta)\, \Phi^{0}(s+\eta, t)\, h(0), R(s)\, \Pi_{01}(s,\beta)\, \Phi^{0}(s+\beta, t)\, k(0)) \\
&= (h(0), \Phi^{0*}(T,t)\, F\, \Phi^{0}(T,t)\, k(0)) \\
&+ \int_t^T ds\, (h(0), \Phi^{0*}(s,t)\, [Q(s) + \Pi_{00}(s) R(s) \Pi_{00}(s)]\, \Phi^{0}(s,t)\, k(0)) \\
&+ \int_{-a}^{0} d\eta \int_{t-\eta}^{\{T\}} ds\, (h(0), \Phi^{0*}(s+\eta, t)\, \Pi_{01}^{*}(s,\eta)\, R(s)\, \Pi_{00}(s)\, \Phi^{0}(s,t)\, k(0)) \\
&+ \int_{-a}^{0} d\beta \int_{t-\beta}^{\{T\}} ds\, (h(0), \Phi^{0*}(s,t)\, \Pi_{00}(s)\, R(s)\, \Pi_{01}(s,\beta)\, \Phi^{0}(s+\beta, t)\, k(0)) \\
&+ \int_{-a}^{0} d\eta \int_{-a}^{0} d\beta \int_{\max(t-\eta,\, t-\beta)}^{\{T\}} ds\, (h(0), \Phi^{0*}(s+\eta, t)\, \Pi_{01}^{*}(s,\eta)\, R(s)\, \Pi_{01}(s,\beta)\, \Phi^{0}(s+\beta, t)\, k(0))
\end{aligned}
$$

interchanging order of integration by Fubini (A.1-4)

Since (A.1-4) holds for all $h(0), k(0) \in R^n$, we have that

$$
\begin{aligned}
\Pi_{00}(t) = \quad & \Phi^{0*}(T,t)\, F\, \Phi^{0}(T,t) \\
& + \int_t^T ds \quad \Phi^{0*}(s,t)\left[Q(s) + \Pi_{00}(s) R(s) \Pi_{00}(s)\right] \Phi^{0}(s,t) \\
& + \int_{-a}^{0} d\eta \int_{t-\eta}^{\{T\}} ds \quad \Phi^{0*}(s+\eta,t)\, \Pi_{01}^{*}(s,\eta)\, R(s)\, \Pi_{00}(s)\, \Phi^{0}(s,t) \\
& + \int_{-a}^{0} d\beta \int_{t-\beta}^{\{T\}} ds \quad \Phi^{0*}(s,t)\, \Pi_{00}(s)\, R(s)\, \Pi_{01}(s,\beta)\, \Phi^{0}(s+\beta,t) \\
& + \int_{-a}^{0} d\eta \int_{-a}^{0} d\beta \int_{\max(t-\beta,t-\eta)}^{\{T\}} ds \quad \Phi^{0*}(s+\eta,t)\, \Pi_{01}^{*}(s,\eta)\, R(s)\, \Pi_{01}(s,\beta)\, \Phi^{0}(s+\beta,t).
\end{aligned}
$$

$\Pi_{00}(T) = F$ and the map
$t \mapsto \Pi_{00}(t) \quad [t_0, T] \to \mathcal{L}(R^n)$
is clearly absolutely continuous.

Putting $h' = 0$, $k(0) = 0$ in (A.1-3), we obtain

$$
\begin{aligned}
\int_{-a}^{0} d\alpha\, (h(0), \Pi_{01}(t,\alpha)\, k(\alpha)) = \quad & \int_{-a}^{0} d\alpha\, (\Phi^{0}(T,t)\, h(0),\, F\, \Phi^{1}(T,t,\alpha)\, k(\alpha)) \\
+ \int_t^T ds \int_{-a}^{0} d\alpha\, & (\Phi^{0}(s,t)\, h(0),\, [Q(s) + \Pi_{00}(s) R(s) \Pi_{00}(s)]\, \Phi^{1}(s,t,\alpha)\, k(\alpha)) \\
+ \int_t^T ds \int_{-a}^{0} d\beta \int_{-a}^{0} d\alpha\, & (\Pi_{00}(s)\, \Phi^{0}(s,t)\, h(0),\, R(s)\, \Pi_{01}(s,\beta)\, \Phi^{1}(s+\beta,t,\alpha)\, k(\alpha)) \\
+ \int_t^T ds \int_{-a}^{0} d\eta \int_{-a}^{0} d\beta \int_{-a}^{0} d\alpha\, & (\Pi_{01}(s,\eta)\, \Phi^{0}(s+\eta,t)\, h(0),\, R(s)\, \Pi_{01}(s,\beta)\, \Phi^{1}(s+\beta,t,\alpha)\, k(\alpha)) \\
+ \int_t^T ds \int_{-a}^{0} d\eta \int_{-a}^{0} d\alpha\, & (\Pi_{01}(s,\eta)\, \Phi^{0}(s+\eta,t)\, h(0),\, R(s)\, \Pi_{00}(s)\, \Phi^{1}(s,t,\alpha)\, k(\alpha))
\end{aligned}
$$

$$= \sum_{i=1}^{N} \int_{\theta_i}^{\min(0,\,\theta_i+T-t)} d\alpha \, \left(\Phi^0(T,t)\,h(0),\, F\,\Phi^0(T,\, t+\alpha-\theta_i)\,A_i(t+\alpha-\theta_i)\,k(\alpha) \right)$$

$$+ \int_{-a}^{0} d\alpha \int_{\max(-a,\,\alpha-T+t)}^{\alpha} dy \, \left(\Phi^0(T,t)\,h(0),\, F\,\Phi^0(T,\, t+\alpha-y)\,D_{01}(t+\alpha-y,y)\,k(\alpha) \right)$$

$$+ \sum_{i=1}^{N} \int_{t}^{T} ds \int_{\theta_i}^{\min(0,\,\theta_i+s-t)} d\alpha \, \left(\Phi^0(s,t)\,h(0),\, [Q(s)+\Pi_{00}(s)R(s)\Pi_{00}(s)]\,\Phi^0(s,\,t+\alpha-\theta_i)\,A_i(t+\alpha-\theta_i)\,k(\alpha) \right)$$

$$+ \int_{t}^{T} ds \int_{-a}^{0} d\alpha \int_{\max(-a,\,\alpha-s+t)}^{\alpha} dy \, \left(\Phi^0(s,t)\,h(0),\, [Q(s)+\Pi_{00}(s)R(s)\Pi_{00}(s)]\,\Phi^0(s,\,t+\alpha-y)\,D_{01}(t+\alpha-y,y)\,k(\alpha) \right)$$

$$+ \sum_{i=1}^{N} \int_{t}^{T} ds \int_{\max(-a,\,t-s)}^{0} d\beta \int_{\theta_i}^{\min(0,\,\theta_i+s+\beta-t)} d\alpha \, \left(\Pi_{00}(s)\,\Phi^0(s,t)\,h(0),\, R(s)\,\Pi_{01}(s,\beta)\,\Phi^0(s+\beta,\, t+\alpha-\theta_i)\,A_i(t+\alpha-\theta_i)\,k(\alpha) \right)$$

$$+ \int_{t}^{T} ds \int_{\max(-a,\,t-s)}^{0} d\beta \int_{-a}^{0} d\alpha \int_{\max(-a,\,\alpha-s-\beta+t)}^{\alpha} dy \, \left(\Pi_{00}(s)\,\Phi^0(s,t)\,h(0),\, R(s)\,\Pi_{01}(s,\beta)\,\Phi^0(s+\beta,\, t+\alpha-y)\,D_{01}(t+\alpha-y,y)\,k(\alpha) \right)$$

$$+ \int_{t}^{T} ds \int_{-a}^{\max(-a,\,-t)} d\beta \, \left(\Pi_{00}(s)\,\Phi^0(s,t)\,h(0),\, R(s)\,\Pi_{01}(s,\beta)\,k(s+\beta-t) \right)$$

$$+ \sum_{i=1}^{N} \int_{t}^{T} ds \int_{\max(-a,\,t-s)}^{0} d\eta \int_{\theta_i}^{\min(0,\,\theta_i+s-t)} d\alpha \, \left(\Pi_{01}(s,\eta)\,\Phi^0(s+\eta,t)\,h(0),\, R(s)\,\Pi_{00}(s)\,\Phi^0(s,\, t+\alpha-\theta_i)\,A_i(t+\alpha-\theta_i)\,k(\alpha) \right)$$

$$+ \int_{t}^{T} ds \int_{\max(-a,\,t-s)}^{0} d\eta \int_{-a}^{0} d\alpha \int_{\max(-a,\,\alpha-s+t)}^{\alpha} dy \, \left(\Pi_{01}(s,\eta)\,\Phi^0(s+\eta,t)\,h(0),\, R(s)\,\Pi_{00}(s)\,\Phi^0(s,\, t+\alpha-y)\,D_{01}(t+\alpha-y,y)\,k(\alpha) \right)$$

$$+ \sum_{i=1}^{N} \int_{t}^{T} ds \int_{\max(-a,\,t-s)}^{0} d\eta \int_{\max(-a,\,t-s)}^{0} d\beta \int_{\theta_i}^{\min(0,\,\theta_i+s+\beta-t)} d\alpha \, \left(\Pi_{01}(s,\eta)\,\Phi^0(s+\eta,t)\,h(0),\, R(s)\,\Pi_{01}(s,\beta)\,\Phi^0(s+\beta,\, t+\alpha-\theta_i)\,A_i(t+\alpha-\theta_i)\,k(\alpha) \right)$$

$$+ \int_{t}^{T} ds \int_{\max(-a,\,t-s)}^{0} d\eta \int_{\max(-a,\,t-s)}^{0} d\beta \int_{-a}^{0} d\alpha \int_{\max(-a,\,\alpha-s-\beta+t)}^{\alpha} dy \, \left(\Pi_{01}(s,\eta)\,\Phi^0(s+\eta,t)\,h(0),\, R(s)\,\Pi_{01}(s,\beta)\,\Phi^0(s+\beta,\, t+\alpha-y)\,D_{01}(t+\alpha-y,y)\,k(\alpha) \right)$$

$$+ \int_{t}^{T} ds \int_{-a}^{\max(-a,\,t-s)} d\eta \int_{\max(-a,\,t-s)}^{0} d\beta \; \Pi_{01}(s,\eta)\,\Phi^0(s+\eta,t)\,h(0),\, R(s)\,\Pi_{01}(s,\beta)\,k(s+\beta-t)$$

$$= \sum_{i=1}^{N} \int_{\theta_i}^{\min(0,\,\theta_i+T-t)} d\alpha \, \left(h(0),\, \Phi^{0*}(T,t)\,F\,\Phi^0(T,\, t+\alpha-\theta_i)\,A_i(t+\alpha-\theta_i)\,k(\alpha) \right)$$

$$+ \int_{-a}^{0} d\alpha \int_{\max(-a,\,\alpha-T+t)}^{\alpha} dy \, \left(h(0),\, \Phi^{0*}(T,t)\,F\,\Phi^0(T,\, t+\alpha-y)\,D_{01}(t+\alpha-y,\, y)\,k(\alpha) \right)$$

$$+ \sum_{i=1}^{N} \int_{\theta_i}^{0} d\alpha \int_{\alpha-\theta_i+t}^{\{T\}} ds \, \big(h(0), \Phi^{0*}(s,t) \left[Q(s) + \Pi_{00}(s) R(s) \Pi_{00}(s)\right] \Phi^{0}(s, t+\alpha-\theta_i) A_i(t+\alpha-\theta_i) k(\alpha)\big)$$

$$+ \sum_{i=1}^{N} \int_{\theta_i}^{0} d\alpha \int_{-a}^{0} d\beta \int_{\alpha-\theta_i+t-\beta}^{\{T\}} ds \, \big(h(0), \Phi^{0*}(s,t) \Pi_{00}(s) R(s) \Pi_{01}(s,\beta) \Phi^{0}(s+\beta, t+\alpha-\theta_i) A_i(t+\alpha-\theta_i) k(\alpha)\big)$$

$$+ \sum_{i=1}^{N} \int_{\theta_i}^{0} d\alpha \int_{-a}^{0} d\eta \int_{\max(t-\eta,\, \alpha-\theta_i+t)}^{\{T\}} ds \, \big(h(0), \Phi^{0*}(s+\eta,t) \Pi_{01}^{*}(s,\eta) R(s) \Pi_{00}(s) \Phi^{0}(s, t+\alpha-\theta_i) A_i(t+\alpha-\theta_i) k(\alpha)\big)$$

$$+ \sum_{i=1}^{N} \int_{\theta_i}^{0} d\alpha \int_{-a}^{0} d\eta \int_{-a}^{0} d\beta \int_{\max(t-\eta,\, \alpha-\theta_i-\beta+t)}^{\{T\}} ds \, \big(h(0), \Phi^{0*}(s+\eta,t) \Pi_{01}^{*}(s,\eta) R(s) \Pi_{a}(s,\beta) \Phi^{0}(s+\beta, t+\alpha-\theta_i) A_i(t+\alpha-\theta_i) k(\alpha)\big)$$

$$+ \int_{-a}^{0} d\alpha \int_{-a}^{\alpha} dY \int_{\alpha-Y+t}^{\{T\}} ds \, \big(h(0), \Phi^{0*}(s,t) \left[Q(s) + \Pi_{00}(s) R(s) \Pi_{00}(s)\right] \Phi^{0}(s, t+\alpha-Y) D_{01}(t+\alpha-Y, Y) k(\alpha)\big)$$

$$+ \int_{-a}^{0} d\alpha \int_{-a}^{\alpha} dY \int_{-a}^{0} d\beta \int_{\alpha-Y-\beta+t}^{\{T\}} ds \, \big(h(0), \Phi^{0*}(s,t) \Pi_{00}(s) R(s) \Pi_{01}(s,\beta) \Phi^{0}(s+\beta, t+\alpha-Y) D_{01}(t+\alpha-Y, Y) k(\alpha)\big)$$

$$+ \int_{-a}^{0} d\alpha \int_{-a}^{\alpha} dY \int_{-a}^{0} d\eta \int_{\max(t-\eta,\, \alpha-Y+t)}^{\{T\}} ds \, \big(h(0), \Phi^{0*}(s+\eta,t) \Pi_{01}^{*}(s,\eta) R(s) \Pi_{00}(s) \Phi^{0}(s, t+\alpha-Y) D_{01}(t+\alpha-Y, Y) k(\alpha)\big)$$

$$+ \int_{-a}^{0} d\alpha \int_{-a}^{\alpha} dY \int_{-a}^{0} d\eta \int_{-a}^{0} d\beta \int_{\max(t-\eta,\, \alpha-Y-\beta+t)}^{\{T\}} ds \, \big(h(0), \Phi^{0*}(s+\eta,t) \Pi_{01}^{*}(s,\eta) R(s) \Pi_{01}(s,\beta) \Phi^{0}(s+\beta, t+\alpha-Y) D_{01}(t+\alpha-Y, Y) k(\alpha)\big)$$

$$+ \int_{-a}^{0} d\alpha \int_{t}^{\{\min(T,\, t+\alpha+a)\}} ds \, \big(h(0), \Phi^{0*}(s,t) \Pi_{00}^{*}(s) R(s) \Pi_{01}(s, \alpha+t-s) k(\alpha)\big)$$

$$+ \int_{-a}^{0} d\alpha \int_{t}^{\{\min(T,\, t+\alpha+a)\}} ds \int_{t-s}^{0} d\eta \, \big(h(0), \Phi^{0*}(s+\eta,t) \Pi_{01}^{*}(s,\eta) R(s) \Pi_{01}(s, \alpha+t-s) k(\alpha)\big) \tag{A.1-5}$$

Now define $\Pi_{01}^{i}(t,\alpha)$ for $i = 1 \ldots N-1$ by

$$\Pi_{01}^{i}(t,\alpha) = 0 \qquad \alpha < \theta_i$$

and for $\alpha \geq \theta_i$

$$\Pi_{01}^{i}(t,\alpha) = \int_{\alpha-\theta_i+t}^{\{T\}} ds \; \Phi^{0*}(s,t) \left[Q(s) + \Pi_{00}(s) R(s) \Pi_{00}(s)\right] \Phi^{0}(s, t+\alpha-\theta_i) A_i(t+\alpha-\theta_i)$$

$$+ \int_{-a}^{0} d\beta \int_{\alpha-\theta_i+t-\beta}^{\{T\}} ds \; \Phi^{0*}(s,t) \Pi_{00}(s) R(s) \Pi_{01}(s,\beta) \Phi^{0}(s+\beta, t+\alpha-\theta_i) A_i(t+\alpha-\theta_i)$$

$$+ \int_{-a}^{0} d\eta \int_{\max(t-\eta,\, \alpha-\theta_i+t)}^{\{T\}} ds \; \Phi^{0*}(s+\eta,t) \Pi_{01}^{*}(s,\eta) R(s) \Pi_{00}(s) \Phi^{0}(s, t+\alpha-\theta_i) A_i(t+\alpha-\theta_i)$$

$$+ \int_{-a}^{0} d\eta \int_{-a}^{0} d\beta \int_{\max(t-\eta,\, \alpha-\theta_i+t-\beta)}^{\{T\}} ds \;\; \Phi^{o*}(s+\eta,t)\, \Pi_{01}^{*}(s,\eta)\, R(s)\, \Pi_{01}(s,\beta)\, \Phi^{o}(s+\beta, t+\alpha-\theta_i)\, A_i(t+\alpha-\theta_i)$$

$$+ \begin{cases} \Phi^{o*}(T,t)\, F\, \Phi^{o}(T, t+\alpha-\theta_i)\, A_i(t+\alpha-\theta_i) & \theta_i \leq \alpha < \min(0, \theta_i+T-t) \\ 0 & \text{otherwise} \end{cases}$$

Since (A.1-5) holds for all $h(0) \in R^n$, $k \in L^2(-a,0;R^n)$, we have that

$$\Pi_{01}(t,\alpha) = \sum_{i=1}^{N} \Pi_{01}^{i}(t,\alpha)$$

$$+ \int_{\max(-a,\alpha-T+t)}^{\alpha} dY \;\; \Phi^{o*}(T,t)\, F\, \Phi^{o}(T, t+\alpha-Y)\, D_{01}(t+\alpha-Y, Y)$$

$$+ \int_{-a}^{\alpha} dY \int_{\alpha-Y+t}^{\{T\}} ds \;\; \Phi^{o*}(s,t)\, [Q(s) + \Pi_{00}(s)\, R(s)\, \Pi_{00}(s)]\, \Phi^{o}(s, t+\alpha-Y)\, D_{01}(t+\alpha-Y, Y)$$

$$+ \int_{-a}^{\alpha} dY \int_{-a}^{0} d\beta \int_{\alpha-Y-\beta+t}^{\{T\}} ds \;\; \Phi^{o*}(s,t)\, \Pi_{00}(s)\, R(s)\, \Pi_{01}(s,\beta)\, \Phi^{o}(s+\beta, t+\alpha-Y)\, D_{01}(t+\alpha-Y, Y)$$

$$+ \int_{-a}^{\alpha} dY \int_{-a}^{0} d\eta \int_{\max(t-\eta,\, \alpha-Y+t)}^{\{T\}} ds \;\; \Phi^{o*}(s+\eta,t)\, \Pi_{01}^{*}(s,\eta)\, R(s)\, \Pi_{00}(s)\, \Phi^{o}(s, t+\alpha-Y)\, D_{01}(t+\alpha-Y, Y)$$

$$+ \int_{-a}^{\alpha} dY \int_{-a}^{0} d\eta \int_{-a}^{0} d\beta \int_{\max(t-\eta,\, \alpha-Y-\beta+t)}^{\{T\}} ds \;\; \Phi^{o*}(s+\eta,t)\, \Pi_{01}^{*}(s,\eta)\, R(s)\, \Pi_{01}(s,\beta)\, \Phi^{o}(s+\beta, t+\alpha-Y)\, D_{01}(t+\alpha-Y, Y)$$

$$+ \int_{t}^{\{\min(T,\, t+\alpha+a)\}} ds \;\; \Phi^{o*}(s,t)\, \Pi_{00}(s)\, R(s)\, \Pi_{01}(s, \alpha+t-s)$$

$$+ \int_{-a}^{0} d\eta \int_{t-\eta}^{\{\min(T,\, t+\alpha+a)\}} ds \;\; \Phi^{o*}(s+\eta,t)\, \Pi_{01}^{*}(s,\eta)\, R(s)\, \Pi_{01}(s, \alpha+t-s) \qquad \text{(A.1-6)}$$

$\Pi_{01}(t,\alpha)$ has jumps at $\alpha = \theta_i$ of magnitude

$$\Pi_{01}^{i}(t,\theta_i) = \Pi_{00}(t)\, A_i(t)$$

and for i such that $\theta_i + T - t < 0$ at $\alpha = \theta_i + T - t$ of magnitude $\Phi^{o*}(T,t)\, F\, A_i(T) = 0$

It is clear from (A.1-6) that the map

$$\alpha \to \Pi_{01}(t,\alpha) \; : \; [-a,0] \to \mathcal{L}(R^n)$$

is piecewise absolutely continuous and is absolutely continuous in the intervals (θ_{i+1}, θ_i) $i = 0, .. N-1$

Also it follows from (A.1-6) that for $\alpha = \theta_i$ $i = 1 . . . N$ that the map

$$t \to \Pi_{01}(t,\alpha) \; : \; [t_0, T] \to \mathcal{L}(R^n)$$

is **absolutely** continuous

$\Pi_{01}(T,\alpha) = 0$ a.e $\alpha \in [-a,0]$ follows from the definition of $\Pi(t)$ or from (A.1-6)

Putting $h(0) = k(0) = 0$ in (A.1-3) we obtain

$$\int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \, (h(\theta), \Pi_{11}(t,\theta,\alpha)\, k(\alpha)) = \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \, (\Phi'(T,t,\theta)\, h(\theta), F\, \Phi'(T,t,\alpha)\, h(\alpha))$$

$$+ \int_{t}^{T} ds \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \, (\Phi'(s,t,\theta)\, h(\theta), [Q(s) + \Pi_{00}(s) R(s) \Pi_{00}(s)]\, \Phi'(s,t,\alpha)\, k(\alpha))$$

$$+ \int_{t}^{T} ds \int_{-a}^{0} d\beta \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \, (\Pi_{00}(s)\, \Phi'(s,t,\theta)\, h(\theta), R(s)\, \Pi_{01}(s,\beta)\, \Phi'(s+\beta, t, \alpha)\, k(\alpha))$$

$$+ \int_{t}^{T} ds \int_{-a}^{0} d\eta \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \, (\Pi_{01}(s,\eta)\, \Phi'(s+\eta, t, \theta)\, h(\theta), R(s)\, \Pi_{00}(s)\, \Phi'(s,t,\alpha)\, k(\alpha))$$

$$+ \int_{t}^{T} ds \int_{-a}^{0} d\eta \int_{-a}^{0} d\beta \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \, (\Pi_{01}(s,\eta)\, \Phi'(s+\eta, t, \theta)\, h(\theta), R(s)\, \Pi_{01}(s,\beta)\, \Phi'(s+\beta, t, \alpha)\, k(\alpha))$$

$$
\begin{aligned}
= {} & \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{\theta_i}^{\min(0,\,\theta_i+T-t)} d\theta \int_{\theta_j}^{\min(0,\,\theta_j+T-t)} d\alpha \, \left(\overset{\circ}{\Phi}(T, t+\theta-\theta_i) A_i(t+\theta-\theta_i) h(\theta), \; F \overset{\circ}{\Phi}(T, t+\alpha-\theta_j) A_j(t+\alpha-\theta_j) k(\alpha) \right) \\
& + \sum_{i=1}^{N} \int_{\theta_i}^{\min(0,\,\theta_i+T-t)} d\theta \int_{-a}^{0} d\alpha \int_{\max(-a,\,\alpha-T+t)}^{\alpha} d\gamma \, \left(\overset{\circ}{\Phi}(T, t+\theta-\theta_i) A_i(t+\theta-\theta_i) h(\theta), \; F \overset{\circ}{\Phi}(T, t+\alpha-\gamma) D_{01}(t+\alpha-\gamma, \gamma) k(\alpha) \right) \\
& + \sum_{j=1}^{N} \int_{-a}^{0} d\theta \int_{\max(a,\,\theta-T+t)}^{\theta} d\delta \int_{\theta_j}^{\min(0,\,\theta_j+T-t)} d\alpha \, \left(\overset{\circ}{\Phi}(T, t+\theta-\delta) D_{01}(t+\theta-\delta, \delta) h(\theta), \; F \overset{\circ}{\Phi}(T, t+\alpha-\theta_j) A_j(t+\alpha-\theta_j) k(\alpha) \right) \\
& + \int_{-a}^{0} d\theta \int_{\max(-a,\,\theta-T+t)}^{\theta} d\delta \int_{-a}^{0} d\alpha \int_{\max(a,\,\alpha-T+T)}^{\alpha} d\gamma \;\; \overset{\circ}{\Phi}(T, t+\theta-\delta) D_{01}(t+\theta-\delta, \delta) h(\theta), \; F \overset{\circ}{\Phi}(T, t+\alpha-\gamma) D_{01}(t+\alpha-\gamma, \gamma) k(\alpha)
\end{aligned}
$$

$$
\begin{aligned}
& + \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{t}^{T} ds \int_{\theta_i}^{\min(0,\,\theta_i+s-t)} d\theta \int_{\theta_j}^{\min(0,\,\theta_j+s-t)} d\alpha \, \Big(\overset{\circ}{\Phi}(s, t+\theta-\theta_i) A_i(t+\theta-\theta_i) h(\theta), \\
& \qquad\qquad [Q(s) + \Pi_{00}(s) R(s) \Pi_{00}(s)] \overset{\circ}{\Phi}(s, t+\alpha-\theta_j) A_j(t+\alpha-\theta_j) k(\alpha) \Big) \\
& + \sum_{i=1}^{N} \int_{t}^{T} ds \int_{\theta_i}^{\min(0,\,\theta_i+s-t)} d\theta \int_{-a}^{0} d\alpha \int_{\max(-a,\,\alpha-s+t)}^{\alpha} d\gamma \, \Big(\overset{\circ}{\Phi}(s, t+\theta-\theta_i) A_i(t+\theta-\theta_i) h(\theta), \\
& \qquad\qquad [Q(s) + \Pi_{00}(s) R(s) \Pi_{00}(s)] \overset{\circ}{\Phi}(s, t+\alpha-\gamma) D_{01}(t+\alpha-\gamma, \gamma) k(\alpha) \Big) \\
& + \sum_{j=1}^{N} \int_{t}^{T} ds \int_{-a}^{0} d\theta \int_{\max(-a,\,\theta-s+t)}^{\theta} d\delta \int_{\theta_j}^{\min(0,\,\theta_j+s-t)} d\alpha \, \Big(\overset{\circ}{\Phi}(s, t+\theta-\delta) D_{01}(t+\theta-\delta, \delta) h(\theta), \\
& \qquad\qquad [Q(s) + \Pi_{00}(s) R(s) \Pi_{00}(s)] \overset{\circ}{\Phi}(s, t+\alpha-\theta_j) A_j(t+\alpha-\theta_j) k(\alpha) \Big) \\
& + \int_{t}^{T} ds \int_{-a}^{0} d\theta \int_{\max(-a,\,\theta-s+t)}^{\theta} d\delta \int_{a}^{0} d\alpha \int_{\max(-a,\,\alpha-s+t)}^{\alpha} d\gamma \, \Big(\overset{\circ}{\Phi}(s, t+\theta-\delta) D_{01}(t+\theta-\delta, \delta) h(\theta), \\
& \qquad\qquad [Q(s) + \Pi_{00}(s) R(s) \Pi_{00}(s)] \overset{\circ}{\Phi}(s, t+\alpha-\gamma) D_{01}(t+\alpha-\gamma, \gamma) k(\alpha) \Big) \\
& + \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{t}^{T} ds \int_{\theta_i}^{\min(0,\,\theta_i+t-s)} d\theta \int_{\max(-a,\,t-s)}^{0} d\beta \int_{\theta_j}^{\min(0,\,\theta_j+s+\beta-t)} d\alpha \, \Big(\Pi_{00}(s) \overset{\circ}{\Phi}(s, t+\theta-\theta_i) A_i(t+\theta-\theta_i) h(\theta), \\
& \qquad\qquad R(s) \Pi_{01}(s, \beta) \overset{\circ}{\Phi}(s+\beta, t+\alpha-\theta_j) A_j(t+\alpha-\theta_j) k(\alpha) \Big) \\
& + \sum_{i=1}^{N} \int_{t}^{T} ds \int_{\theta_i}^{\min(0,\,\theta_i+t-s)} d\theta \int_{\max(-a,\,t-s)}^{0} d\beta \int_{-a}^{0} d\alpha \int_{\max(-a,\,\alpha-s-\beta+t)}^{\alpha} d\gamma \, \Big(\Pi_{00}(s) \overset{\circ}{\Phi}(s, t+\theta-\theta_i) A_i(t+\theta-\theta_i) h(\theta), \\
& \qquad\qquad R(s) \Pi_{01}(s, \beta) \overset{\circ}{\Phi}(s+\beta, t+\alpha-\gamma) D_{01}(t+\alpha-\gamma, \gamma) k(\alpha) \Big) \\
& + \sum_{j=1}^{N} \int_{t}^{T} ds \int_{-a}^{0} d\theta \int_{\max(-a,\,\theta-s+t)}^{\theta} d\delta \int_{\max(-a,\,t-s)}^{0} d\beta \int_{\theta_j}^{\min(0,\,\theta_j+s+\beta-t)} d\alpha \, \Big(\Pi_{00}(s) \overset{\circ}{\Phi}(s, t+\theta-\delta) D_{01}(t+\theta-\delta, \delta) h(\theta), \\
& \qquad\qquad R(s) \Pi_{01}(s, \beta) \overset{\circ}{\Phi}(s+\beta, t+\alpha-\theta_j) A_j(t+\alpha-\theta_j) k(\alpha) \Big)
\end{aligned}
$$

$$+ \int_t^T ds \int_{-a}^{0} d\theta \int_{\max(-a,\theta-s+t)}^{\theta} d\delta \int_{\max(-a,t-s)}^{0} d\beta \int_{a}^{0} d\alpha \int_{\max(-a,\alpha-s-\beta+t)}^{\alpha} d\gamma \left(\begin{aligned} &\Pi_{00}(s)\, \Phi^{0}(s,t+\theta-\delta)\, D_{01}(t+\theta-\delta,\delta)\, h(\theta), \\ &R(s)\, \Pi_{01}(s,\beta)\, \Phi^{0}(s+\beta,t+\alpha-\gamma)\, D_{01}(t+\alpha-\gamma,\gamma)\, k(\alpha) \end{aligned} \right)$$

$$+ \sum_{i=1}^{N} \int_t^T ds \int_{\theta_i}^{\min(0,\theta_i+t-s)} d\theta \int_{-a}^{\max(-a,t-s)} d\beta \left(\Pi_{00}(s)\, \Phi^{0}(s,t+\theta-\theta_i)\, A_i(t+\theta-\theta_i)\, h(\theta),\ R(s)\, \Pi_{01}(s,\beta)\, k(s+\beta-t) \right)$$

$$+ \int_t^T ds \int_{-a}^{0} d\theta \int_{\max(-a,\theta-s+t)}^{\theta} d\delta \int_{-a}^{\max(-a,t-s)} d\beta\ \Pi_{00}(s)\, \Phi^{0}(s,t+\theta-\delta)\, D_{01}(t+\theta-\delta,\delta)\, h(\theta),\ R(s)\, \Pi_{01}(s,\beta)\, k(s+\beta-t)$$

$$+ \sum_{i=1}^{N} \sum_{j=1}^{N} \int_t^T ds \int_{\max(-a,t-s)}^{0} d\eta \int_{\theta_i}^{\min(0,\theta_i+s+\eta-t)} d\theta \int_{\theta_j}^{\min(0,\theta_j+s-t)} d\alpha \left(\begin{aligned} &\Pi_{01}(s,\eta)\, \Phi^{0}(s+\eta,t+\theta-\theta_i)\, A_i(t+\theta-\theta_i)\, h(\theta), \\ &R(s)\, \Pi_{00}(s)\, \Phi^{0}(s,t+\alpha-\theta_j)\, A_j(t+\alpha-\theta_j)\, k(\alpha) \end{aligned} \right)$$

$$+ \sum_{i=1}^{N} \int_t^T ds \int_{\max(-a,t-s)}^{0} d\eta \int_{\theta_i}^{\min(0,\theta_i+s+\eta-t)} d\theta \int_{-a}^{0} d\alpha \int_{\max(-a,\alpha-s+t)}^{\alpha} d\gamma \left(\begin{aligned} &\Pi_{01}(s,\eta)\, \Phi^{0}(s+\eta,t+\theta-\theta_i)\, A_i(t+\theta-\theta_i)\, h(\theta), \\ &R(s)\, \Pi_{00}(s)\, \Phi^{0}(s,t+\alpha-\gamma)\, D_{01}(t+\alpha-\gamma,\gamma)\, k(\alpha) \end{aligned} \right)$$

$$+ \sum_{j=1}^{N} \int_t^T ds \int_{\max(-a,t-s)}^{0} d\eta \int_{-a}^{0} d\theta \int_{\max(-a,\theta-s-\eta+t)}^{\theta} d\delta \int_{\theta_j}^{\min(0,\theta_j+s-t)} \left(\begin{aligned} &\Pi_{01}(s,\eta)\, \Phi^{0}(s+\eta,t+\theta-\delta)\, D_{01}(t+\theta-\delta,\delta)\, h(\theta), \\ &R(s)\, \Pi_{00}(s)\, \Phi^{0}(s,t+\alpha-\theta_j)\, A_j(t+\alpha-\theta_j)\, k(\alpha) \end{aligned} \right)$$

$$+ \int_t^T ds \int_{\max(-a,t-s)}^{0} d\eta \int_{-a}^{0} d\theta \int_{\max(0,\theta-s-\eta+t)}^{\theta} d\delta \int_{-a}^{0} d\alpha \int_{\max(-a,\alpha-s+t)}^{\alpha} d\gamma \left(\begin{aligned} &\Pi_{01}(s,\eta)\, \Phi^{0}(s+\eta,t+\theta-\delta)\, D_{01}(t+\theta-\delta,\delta)\, h(\theta), \\ &R(s)\, \Pi_{00}(s)\, \Phi^{0}(s,t+\alpha-\gamma)\, D_{01}(t+\alpha-\gamma,\gamma)\, k(\alpha) \end{aligned} \right)$$

$$+ \sum_{j=1}^{N} \int_t^T ds \int_{-a}^{\max(-a,t-s)} d\eta \int_{\theta_j}^{\min(0,\theta_j+t-s)} d\alpha \left(\Pi_{01}(s,\eta)\, h(s+\eta-t),\ R(s)\, \Pi_{00}(s)\, \Phi^{0}(s,t+\alpha-\theta_j)\, A_j(t+\alpha-\theta_j)\, k(\alpha) \right)$$

$$+ \int_t^T ds \int_{-a}^{\max(-a,t-s)} d\eta \int_{-a}^{0} d\alpha \int_{\max(-a,\alpha-s+t)}^{\alpha} d\gamma \left(\Pi_{01}(s,\eta)\, h(s+\eta-t),\ R(s)\, \Pi_{00}\, \Phi^{0}(s,t+\alpha-\gamma)\, D_{01}(t+\alpha-\gamma)\, k(\alpha) \right)$$

$$+ \sum_{i=1}^{N} \sum_{j=1}^{N} \int_t^T ds \int_{\max(-a,t-s)}^{0} d\eta \int_{\theta_i}^{\min(0,\theta_i+s+\eta-t)} d\theta \int_{\max(-a,t-s)}^{0} d\beta \int_{\theta_j}^{\min(0,\theta_j+s+\beta-t)} d\alpha \left(\begin{aligned} &\Pi_{01}(s,\eta)\, \Phi^{0}(s+\eta,t+\theta-\theta_i)\, A_i(t+\theta-\theta_i)\, h(\theta), \\ &R(s)\, \Pi_{01}(s,\beta)\, \Phi^{0}(s+\beta,t+\alpha-\theta_j)\, A_j(t+\alpha-\theta_j)\, k(\alpha) \end{aligned} \right)$$

$$+ \sum_{i=1}^{N} \int_t^T ds \int_{\max(-a,t-s)}^{0} d\eta \int_{\theta_i}^{\min(0,\theta_i+s+\eta-t)} d\theta \int_{\max(-a,t-s)}^{0} d\beta \int_{a}^{0} d\alpha \int_{\max(-a,\alpha-s-\beta+t)}^{\alpha} d\gamma \left(\begin{aligned} &\Pi_{01}(s,\eta)\, \Phi^{0}(s+\eta,t+\theta-\theta_i)\, A_i(t+\theta-\theta_i)\, h(\theta), \\ &R(s)\, \Pi_{01}(s,\beta)\, \Phi^{0}(s+\beta,t+\alpha-\gamma)\, D_{01}(t+\alpha-\gamma,\gamma)\, k(\alpha) \end{aligned} \right)$$

$$+\sum_{j=1}^{N}\int_t^T ds\int_{\max(-a,t-s)}^{0} d\eta\int_{-a}^{0} d\theta\int_{\max(-a,\theta-s-\eta+t)}^{\theta} d\delta\int_{\theta_j}^{\min(0,\theta_j+s+\beta-t)} d\theta \begin{pmatrix}\Pi_{01}(s,\eta)\,\Phi^0(s+\eta,t+\theta-\delta)D_{01}(t+\theta-\delta,\delta)h(\theta),\\ R(s)\,\Pi_{01}(s,\beta)\,\Phi^0(s+\beta,t+\alpha-\theta_j)A_j(t+\alpha-\theta_j)k(\alpha)\end{pmatrix}$$

$$+\int_t^T ds\int_{\max(-a,t-s)}^{0} d\eta\int_{-a}^{0} d\theta\int_{\max(-a,\theta-s-\eta+t)}^{\theta} d\delta\int_{\max(-a,t-s)}^{0} d\beta\int_{-a}^{0} d\alpha\int_{\max(-a,\alpha-s-\beta+t)}^{\alpha} d\gamma \begin{pmatrix}\Pi_{01}(s,\eta)\Phi^0(s+\eta,t+\theta-\delta)D_{01}(t+\theta-\delta,\delta)h(\theta),\\ R(s)\Pi_{01}(s,\beta)\Phi^0(s+\beta,t+\alpha-\gamma)D_{01}(t+\alpha-\gamma,\gamma)k(\alpha)\end{pmatrix}$$

$$+\sum_{i=1}^{N}\int_t^T ds\int_{\max(-a,t-s)}^{0} d\eta\int_{\theta_i}^{\min(0,\theta_i+s+\eta-t)} d\theta\int_{-a}^{\max(-a,t-s)} d\beta\,\big(\Pi_{01}(s,\eta)\Phi^0(s+\eta,t+\theta-\theta_i)A_i(t+\theta-\theta_i)h(\theta),R(s)\Pi_{01}(s,\beta)k(s+\beta-t)\big)$$

$$+\int_t^T ds\int_{\max(-a,t-s)}^{0} d\eta\int_{-a}^{0} d\theta\int_{\max(-a,\theta-s-\eta+t)}^{\theta} d\delta\int_{-a}^{\max(-a,t-s)} d\beta\,\big(\Pi_{01}(s,\eta)\Phi^0(s+\eta,t+\theta-\delta)D_{01}(t+\theta-\delta,\delta)h(\theta),\ R(s)\Pi_{01}(s,\beta)k(s+\beta-t)\big)$$

$$+\sum_{j=1}^{N}\int_t^T ds\int_{-a}^{\max(-a,t-s)} d\eta\int_{\max(-a,t-s)}^{0} d\beta\int_{\theta_j}^{\min(0,\theta_j+s+\beta-t)} d\alpha\,\big(\Pi_{01}(s,\eta)\,h(s+\eta-t),R(s)\Pi_{01}(s,\beta)\Phi^0(s+\beta,t+\alpha-\theta_j)A_j(t+\alpha-\theta_j)k(\alpha)\big)$$

$$+\int_t^T ds\int_{-a}^{\max(-a,t-s)} d\eta\int_{\max(-a,t-s)}^{0} d\beta\int_{-a}^{0} d\alpha\int_{\max(-a,\alpha-s-\beta+t)}^{\alpha} d\gamma\,\big(\Pi_{01}(s,\eta)h(s+\eta-t),R(s)\Pi_{01}(s,\beta)\Phi^0(s+\beta,t+\alpha-\gamma)D_{01}(t+\alpha-\gamma,\gamma)k(\alpha)\big)$$

$$+\int_t^T ds\int_{-a}^{\max(-a,t-s)} d\eta\int_{-a}^{\max(-a,t-s)} d\beta\,\big(\Pi_{01}(s,\eta)h(s+\eta-t),\ R(s)\Pi_{01}(s,\beta)\,k(s+\beta-t)\big)$$

$$=\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\theta_i}^{\min(0,\theta_i+T-t)} d\theta\int_{\theta_j}^{\min(0,\theta_j+T-t)} d\alpha\,\big(\Phi^0(T,t+\theta-\theta_i)A_i(t+\theta-\theta_i)h(\theta),\ F\,\Phi^0(T,t+\alpha-\theta_j)A_j(t+\alpha-\theta_j)k(\alpha)\big)$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\theta_i}^{0} d\theta\int_{\theta_j}^{0} d\alpha\int_{\max(\theta-\theta_i+t,\alpha-\theta_j+t)}^{\{T\}} ds \begin{pmatrix}\Phi^0(s,t+\theta-\theta_i)A_i(t+\theta-\theta_i)h(\theta),\\ [Q(s)+\Pi_{00}(s)R(s)\Pi_{00}(s)]\,\Phi^0(s,t+\alpha-\theta_j)A_j(t+\alpha-\theta_j)k(\alpha)\end{pmatrix}$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\theta_i}^{0} d\theta\int_{\theta_j}^{0} d\alpha\int_{-a}^{0} d\beta\int_{\max(\theta-\theta_i+t,\alpha-\theta_j-\beta+t)}^{\{T\}} ds \begin{pmatrix}\Pi_{00}(s)\,\Phi^0(s,t+\theta-\theta_i)A_i(t+\theta-\theta_i)h(\theta),\\ R(s)\Pi_{01}(s,\beta)\,\Phi^0(s+\beta,t+\alpha-\theta_j)A_j(t+\alpha-\theta_j)k(\alpha)\end{pmatrix}$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\theta_i}^{0} d\theta\int_{\theta_j}^{0} d\alpha\int_{-a}^{0} d\eta\int_{\max(\theta-\theta_i-\eta+t,\alpha-\theta_j+t}^{\{T\}} ds \begin{pmatrix}\Pi_{01}(s,\eta)\,\Phi^0(s+\eta,t+\theta-\theta_i)\,A_i(t+\theta-\theta_i)h(\theta),\\ R(s)\,\Pi_{00}(s)\,\Phi^0(s,t+\alpha-\theta_j)A_j(t+\alpha-\theta_j)\,k(\alpha)\end{pmatrix}$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\theta_i}^{0}d\theta\int_{\theta_j}^{0}d\alpha\int_{-a}^{0}d\eta\int_{-a}^{0}d\beta\int_{\max(\theta-\theta_i-\eta+t,\,\alpha-\theta_j-\beta+t)}^{\{T\}}ds\Big(\Pi_{01}(s,\eta)\,\Phi^0(s+\eta,t+\theta-\theta_i)\,A_i(t+\theta-\theta_i)\,h(\theta),\ R(s)\,\Pi_{01}(s,\beta)\,\Phi^0(s+\beta,t+\alpha-\theta_j)\,A_j(t+\alpha-\theta_j)\,k(\alpha)\Big)$$

$$+\sum_{i=1}^{N}\int_{\theta_i}^{\min(0,\,\theta_i+T-t)}d\theta\int_{-a}^{0}d\alpha\int_{\max(-a,\,\alpha-T+t)}^{\alpha}d\gamma\ \Phi^0(T,t+\theta-\theta_i)\,A_i(t+\theta-\theta_i),\ F\,\Phi^0(T,t+\alpha-\gamma)\,D_{01}(t+\alpha-\gamma,\gamma)\,k(\alpha)$$

$$+\sum_{i=1}^{N}\int_{\theta_i}^{0}d\theta\int_{-a}^{0}d\alpha\int_{-a}^{\alpha}d\gamma\int_{\max(\theta-\theta_i+t,\,\alpha-\gamma+t)}^{\{T\}}ds\Big(\Phi^0(s,t+\theta-\theta_i)\,A_i(t+\theta-\theta_i)\,h(\theta),\ [Q(s)+\Pi_{00}(s)R(s)\Pi_{00}(s)]\,\Phi^0(s,t+\alpha-\gamma)\,D_{01}(t+\alpha-\gamma,\gamma)\,k(\alpha)\Big)$$

$$+\sum_{i=1}^{N}\int_{\theta_i}^{0}d\theta\int_{-a}^{0}d\alpha\int_{-a}^{\alpha}d\gamma\int_{-a}^{0}d\beta\int_{\max(\theta-\theta_i+t,\,\alpha-\gamma-\beta+t)}^{\{T\}}ds\Big(\Pi_{00}(s)\,\Phi^0(s,t+\theta-\theta_i)\,A_i(t+\theta-\theta_i)\,h(\theta),\ R(s)\,\Pi_{01}(s,\beta)\,\Phi^0(s+\beta,t+\alpha-\gamma)\,D_{01}(t+\alpha-\gamma,\gamma)\,k(\alpha)\Big)$$

$$+\sum_{i=1}^{N}\int_{\theta_i}^{0}d\theta\int_{-a}^{0}d\alpha\int_{-a}^{\alpha}d\gamma\int_{-a}^{0}d\eta\int_{\max(\theta-\theta_i-\eta+t,\,\alpha-\gamma-t)}^{\{T\}}ds\Big(\Pi_{01}(s,\eta)\,\Phi^0(s+\eta,t+\theta-\theta_i)\,A_i(t+\theta-\theta_i)\,h(\theta),\ R(s)\,\Pi_{00}(s)\,\Phi^0(s,t+\alpha-\gamma)\,D_{01}(t+\alpha-\gamma,\gamma)\,k(\alpha)\Big)$$

$$+\sum_{i=1}^{N}\int_{\theta_i}^{0}d\theta\int_{-a}^{0}d\alpha\int_{-a}^{\alpha}d\gamma\int_{-a}^{0}d\eta\int_{-a}^{0}d\beta\int_{\max(\theta-\theta_i-\eta+t,\,\alpha-\gamma-\beta+t)}^{\{T\}}ds\Big(\Pi_{01}(s,\eta)\,\Phi^0(s+\eta,t+\theta-\theta_i)\,A_i(t+\theta-\theta_i)\,h(\theta),\ R(s)\,\Pi_{01}(s,\beta)\,\Phi^0(s+\beta,t+\alpha-\gamma)\,D_{01}(t+\alpha-\gamma,\gamma)\,k(\alpha)\Big)$$

$$+\sum_{i=1}^{N}\int_{\theta_i}^{0}d\theta\int_{-a}^{0}d\alpha\int_{\theta-\theta_i+t}^{\{\min(T,\,t+\alpha+a)\}}ds\ \big(\Pi_{00}(s)\,\Phi^0(s,t+\theta-\theta_i)\,A_i(t+\theta-\theta_i)\,h(\theta),\ R(s)\,\Pi_{01}(s,\alpha+t-s)\,k(\alpha)\big)$$

$$+\sum_{i=1}^{N}\int_{\theta_i}^{0}d\theta\int_{-a}^{0}d\alpha\int_{-a}^{0}d\eta\int_{\theta-\theta_i-\eta+t}^{\{\min(T,\,t+\alpha+a)\}}ds\ \big(\Pi_{01}(s,\eta)\,\Phi^0(s+\eta,t+\theta-\theta_i)\,A_i(t+\theta-\theta_i)\,h(\theta),\ R(s)\,\Pi_{01}(s,\alpha+t-s)\,k(\alpha)\big)$$

$$+\sum_{j=1}^{N}\int_{-a}^{0}d\theta\int_{\theta_j}^{0}d\alpha\int_{-a}^{\theta}d\delta\int_{\max(\theta-\delta+t,\,\alpha-\theta_j+t)}^{\{T\}}ds\Big(\Phi^0(s,t+\theta-\delta)\,D_{01}(t+\theta-\delta,\delta)\,h(\theta),\ [Q(s)+\Pi_{00}(s)R(s)\Pi_{00}(s)]\,\Phi^0(s,t+\alpha-\theta_j)\,A_j(t+\alpha-\theta_j)\,k(\alpha)\Big)$$

$$+\sum_{j=1}^{N}\int_{-a}^{0}d\theta\int_{\theta_j}^{\min(0,\,\theta_j+T-t)}d\alpha\int_{\max(-a,\,\theta-T+t)}^{\theta}d\delta\ \big(\Phi^0(T,t+\theta-\delta)\,D_{01}(t+\theta-\delta,\delta)\,h(\theta),\ F\,\Phi^0(T,t+\alpha-\theta_j)\,A_j(t+\alpha-\theta_j)\,k(\alpha)\big)$$

$$+\sum_{j=1}^{N}\int_{-a}^{0}d\theta\int_{\theta_j}^{0}d\alpha\int_{-a}^{\theta}d\delta\int_{-a}^{0}d\beta\int_{\max(\theta-\delta+t,\,\alpha-\theta_j-\beta+t)}^{\{T\}}ds\Big(\Pi_{00}(s)\,\Phi^0(s,t+\theta-\delta)\,D_{01}(t+\theta-\delta,\delta)\,h(\theta),\ R(s)\,\Pi_{01}(s,\beta)\,\Phi^0(s+\beta,t+\alpha-\theta_j)\,A_j(t+\alpha-\theta_j)\,k(\alpha)\Big)$$

$$\begin{aligned}&+\sum_{j=1}^{N}\int_{-a}^{0}d\theta\int_{\theta_j}^{0}d\alpha\int_{-a}^{\theta}d\delta\int_{-a}^{0}d\eta\int_{\max(\theta-\delta-\eta+t,\,\alpha-\theta_j-\beta+t)}^{\{T\}}ds\,\Big(\Pi_{01}(s,\eta)\,\Phi^{0}(s+\eta,t+\theta-\delta)D_{01}(t+\theta-\delta,\delta)\,h(\theta),\\&\qquad R(s)\Pi_{00}(s)\,\Phi^{0}(s,t+\alpha-\theta_j)\,A_j(t+\alpha-\theta_j)\,k(\alpha)\Big)\end{aligned}$$

$$\begin{aligned}&+\sum_{j=1}^{N}\int_{-a}^{0}d\theta\int_{\theta_j}^{0}d\alpha\int_{-a}^{\theta}d\alpha\int_{-a}^{0}d\eta\int_{-a}^{0}d\beta\int_{\max(\theta-\delta-\eta+t,\,\alpha-\theta_j-\beta+t)}^{\{T\}}ds\,\Big(\Pi_{01}(s,\eta)\,\Phi^{0}(s+\eta,t+\theta-\delta)\,D_{01}(t+\theta-\delta,\delta)\,h(\theta),\\&\qquad R(s)\Pi_{01}(s,\beta)\Phi^{0}(s+\beta,t+\alpha-\theta_j)\,A_j(t+\alpha-\theta_j)\,k(\alpha)\Big)\end{aligned}$$

$$+\sum_{j=1}^{N}\int_{-a}^{0}d\theta\int_{\theta_j}^{0}d\alpha\int_{\alpha-\theta_j+t}^{\{\min(T,\,t+\theta+a)\}}ds\,\big(\Pi_{01}(s,\theta+t-s)\,h(\theta),\;R(s)\,\Pi_{00}\,\Phi^{0}(s,t+\alpha-\theta_j)\,A_j(t+\alpha-\theta_j)\,k(\alpha)\big)$$

$$+\sum_{j=1}^{N}\int_{-a}^{0}d\theta\int_{\theta_j}^{0}d\alpha\int_{-a}^{0}d\beta\int_{\alpha-\theta_j-\beta+t}^{\{\min(T,\,t+\theta+a)\}}ds\,\big(\Pi_{01}(s,\theta+t-s)\,h(\theta),\;R(s)\Pi_{01}(s,\beta)\,\Phi^{0}(s+\beta,t+\alpha-\theta_j)\,A_j(t+\alpha-\theta_j)\,k(\alpha)\big)$$

$$\begin{aligned}&+\int_{-a}^{0}d\theta\int_{-a}^{0}d\alpha\int_{-a}^{\theta}d\delta\int_{-a}^{\alpha}d\gamma\int_{\max(\theta-\delta+t,\,\alpha-\gamma+t)}^{\{T\}}ds\,\Big(\Phi^{0}(s,t+\theta-\delta)D_{01}(t+\theta-\delta,\delta)\,h(\theta),\\&\qquad [Q(s)+\Pi_{00}(s)R(s)\Pi_{00}(s)]\,\Phi^{0}(s,t+\alpha-\gamma)\,D_{01}(t+\alpha-\gamma,\gamma)\,k(\alpha)\Big)\end{aligned}$$

$$\begin{aligned}&+\int_{-a}^{0}d\theta\int_{-a}^{0}d\alpha\int_{-a}^{\theta}d\delta\int_{-a}^{\alpha}d\gamma\int_{-a}^{0}d\beta\int_{\max(\theta-\delta+t,\,\alpha-\gamma-\beta+t)}^{\{T\}}ds\,\Big(\Pi_{00}(s)\Phi^{0}(s,t+\theta-\delta)D_{01}(t+\theta-\delta,\delta)\,h(\theta),\\&\qquad R(s)\,\Pi_{01}(s,\beta)\,\Phi^{0}(s+\beta,t+\alpha-\gamma)D_{01}(t+\alpha-\gamma,\gamma)\,k(\alpha)\Big)\end{aligned}$$

$$\begin{aligned}&+\int_{-a}^{0}d\theta\int_{-a}^{0}d\alpha\int_{-a}^{\theta}d\delta\int_{-a}^{\alpha}d\gamma\int_{-a}^{0}d\eta\int_{\max(\theta-\delta-\eta+t,\,\alpha-\gamma+t)}^{\{T\}}ds\,\Big(\Pi_{01}(s,\eta)\,\Phi^{0}(s+\eta,t+\theta-\delta)\,D_{01}(t+\theta-\delta,\delta)\,h(\theta),\\&\qquad R(s)\,\Pi_{00}(s)\,\Phi^{0}(s,t+\alpha-\gamma)D_{01}(t+\alpha-\gamma,\gamma)\,k(\alpha)\Big)\end{aligned}$$

$$\begin{aligned}&+\int_{-a}^{0}d\theta\int_{-a}^{0}d\alpha\int_{-a}^{\theta}d\delta\int_{-a}^{\alpha}d\gamma\int_{-a}^{0}d\beta\int_{-a}^{0}d\eta\int_{\max(\theta-\delta-\eta+t,\,\alpha-\gamma-\beta+t)}^{\{T\}}ds\,\Big(\Pi_{01}(s,\eta)\,\Phi^{0}(s+\eta,t+\theta-\delta)\,D_{01}(t+\theta-\delta,\delta)\,h(\theta),\\&\qquad R(s)\Pi_{01}(s,\beta)\Phi^{0}(s+\beta,t+\alpha-\gamma)\,D_{01}(t+\alpha-\gamma,\gamma)\,k(\alpha)\Big)\end{aligned}$$

$$+\int_{-a}^{0}d\theta\int_{-a}^{0}d\alpha\int_{-a}^{\theta}d\delta\int_{\theta-\delta+t}^{\{\min(T,\,t+\alpha+a)\}}ds\,\big(\Pi_{00}(s)\,\Phi^{0}(s,t+\theta-\delta)\,D_{01}(t+\theta-\delta,\delta)\,h(\theta),\;R(s)\Pi_{01}(s,\alpha+t-s)\,k(\alpha)\big)$$

$$+\int_{-a}^{0}d\theta\int_{-a}^{0}d\alpha\int_{-a}^{\alpha}d\gamma\int_{\alpha-\gamma+t}^{\{\min(T,\,t+\theta+a)\}}ds\,\big(\Pi_{01}(s,\theta+t-s)\,h(\theta),\;R(s)\Pi_{00}(s)\,\Phi^{0}(s,t+\alpha-\gamma)D_{01}(t+\alpha-\gamma,\gamma)\,k(\alpha)\big)$$

$$+\int_{-a}^{0}d\theta\int_{-a}^{0}d\alpha\int_{t}^{\{\min(T+\theta+a,\,T+\alpha+a)\}}ds\,\big(\Pi_{01}(s,\theta+t-s)\,h(\theta),\;R(s)\,\Pi_{01}(s,\alpha+t-s)\,k(\alpha)\big)$$

(A.1-7)

We define $\Pi_{11}^{i,j}(t,\theta,\alpha)$ for $i=1 \ldots N-1$, $j=1 \ldots N-1$
by $\Pi_{11}^{i,j}(t,\theta,\alpha) = 0$ for $\theta_i < \theta_i$, $\alpha < \theta_j$
and for $\theta \geq \theta_i$, $\alpha \geq \alpha_j$

$$\Pi_{11}^{i,j}(t,\theta,\alpha) = \int_{\max(\theta-\theta_i+t,\,\alpha-\theta_j+t)}^{\{T\}} ds \; A_i^*(t+\theta-\theta_i)\,\Phi^{0*}(s,t+\theta-\theta_i)\big[Q(s)+\Pi_{00}(s)R(s)\Pi_{00}(s)\big]\Phi^0(s,t+\alpha-\theta_j)A_j(t+\alpha-\theta_j)$$

$$+\int_{-a}^{0} d\beta \int_{\max(\theta-\theta_i+t,\,\alpha-\theta_j-\beta+t)}^{\{T\}} ds \; A_i^*(t+\theta-\theta_i)\,\Phi^{0*}(s,t+\theta-\theta_i)\Pi_{00}(s)R(s)\Pi_{01}(s,\beta)\Phi^0(s+\beta,t+\alpha-\theta_j)A_j(t+\alpha-\theta_j)$$

$$+\int_{-a}^{0} d\eta \int_{\max(\theta-\theta_i-\eta+t,\,\alpha-\theta_j+t)}^{\{T\}} ds \; A_i^*(t+\theta-\theta_i)\,\Phi^{0*}(s+\eta,t+\theta-\theta_i)\Pi_{01}^*(s,\eta)R(s)\Pi_{00}(s)\Phi^0(s,t+\alpha-\theta_j)A_j(t+\alpha-\theta_j)$$

$$+\int_{-a}^{0} d\beta \int_{-a}^{0} d\eta \int_{\max(\theta-\theta_i-\eta+t,\,\alpha-\theta_j-\beta+t)}^{\{T\}} ds \; A_i^*(t+\theta-\theta_i)\,\Phi^{0*}(s+\eta,t+\theta-\theta_i)\Pi_{01}^*(s,\eta)R(s)\Pi_{01}(s,\beta)\Phi^0(s+\beta,t+\alpha-\theta_j)A_j(t+\alpha-\theta_j)$$

$$+\begin{cases} A_i^*(t+\theta-\theta_i)\,\Phi^{0*}(T,t+\theta-\theta_i)\,F\,\Phi^0(T,t+\alpha-\theta_j)A_j(t+\alpha-\theta_j) & \begin{array}{l}\theta_i \leq \theta < \min(0,\theta_i+T-t)\\ \theta_j \leq \alpha \leq \min(0,\theta_j+T-t)\end{array} \\ 0 & \text{otherwise} \end{cases}$$

Define $\Pi_{11}^{i,0}(t,\theta,\alpha)$ for $i=1 \ldots N-1$ by
$\Pi_{11}^{i,0}(t,\theta,\alpha) = 0 \qquad \theta < \theta_i$
and for $\theta \geq \theta_i$

$$\Pi_{11}^{i,0}(t,\theta,\alpha) = \int_{-a}^{\alpha} dY \int_{\max(\theta-\theta_i+t,\,\alpha-Y+t)}^{\{T\}} ds \; A_i^*(t+\theta-\theta_i)\,\Phi^{0*}(s,t+\theta-\theta_i)\big[Q(s)+\Pi_{00}(s)R(s)\Pi_{00}(s)\big]\Phi^0(s,t+\alpha-Y)D_{01}(t+\alpha-Y,Y)$$

$$+\int_{-a}^{\alpha} dY \int_{-a}^{0} d\beta \int_{\max(\theta-\theta_i+t,\,\alpha-Y-\beta+t)}^{\{T\}} ds \; A_i^*(t+\theta-\theta_i)\,\Phi^{0*}(s,t+\theta-\theta_i)\Pi_{00}(s)R(s)\Pi_{01}(s,\beta)\Phi^0(s+\beta,t+\alpha-Y)D_{01}(t+\alpha-Y,Y)$$

$$+\int_{-a}^{\alpha} dY \int_{-a}^{0} d\eta \int_{\max(\theta-\theta_i-\eta+t,\,\alpha-Y+t)}^{\{T\}} ds \; A_i^*(t+\theta-\theta_i)\,\Phi^{0*}(s+\eta,t+\theta-\theta_i)\Pi_{01}^*(s,\eta)R(s)\Pi_{00}(s)\Phi^0(s,t+\alpha-Y)D_{01}(t+\alpha-Y,Y)$$

$$+\int_{-a}^{\alpha} dY \int_{-a}^{0} d\beta \int_{-a}^{0} d\beta \int_{\max(\theta-\theta_i-\eta+t,\,\alpha-Y-\beta+t)}^{\{T\}} ds \; A_i^*(t+\theta-\theta_i)\,\Phi^{0*}(s+\eta,t+\theta-\theta_i)\Pi_{01}^*(s,\eta)R(s)\Pi_{01}(s,\beta)\Phi^0(s+\beta,t+\alpha-Y)D_{01}(t+\alpha-Y,Y)$$

$$+ \begin{cases} \int_{-a}^{\alpha} d\gamma \; A_i^*(t+\theta-\theta_i)\, \Phi^{0*}(T, t+\theta-\theta_i)\, F\, \Phi^0(T, t+\alpha-\gamma)\, D_{01}(t+\alpha-\gamma, \gamma) & \theta_i \le \theta \le \min(0, \theta_i+T-t) \\ 0 & \text{otherwise} \end{cases}$$

$\Pi_{11}^{0,j}(t,\theta,\alpha)$ is defined similarly for $j = 1 \dots N-1$

Hence from (A.1-7), we have

$$\Pi_{11}(t,\theta,\alpha) = \sum_{i=1}^{N} \sum_{j=1}^{N} \Pi_{11}^{i,j}(t,\theta,\alpha) + \sum_{i=1}^{N} \Pi_{11}^{i,0}(t,\theta,\alpha) + \sum_{j=1}^{N} \Pi_{11}^{0,j}(t,\theta,\alpha)$$

$$+ \int_{-a}^{\theta} d\delta \int_{-a}^{\alpha} d\gamma \int_{\max(\theta-\delta+t,\, \alpha-\gamma+t)}^{\{T\}} ds\; D_{01}^*(t+\theta-\delta, \delta)\, \Phi^{0*}(s, t+\theta-\delta) [Q(s) + \Pi_{00}(s) R(s) \Pi_{00}(s)]\, \Phi^0(s, t+\alpha-\gamma)\, D_{01}(t+\alpha-\gamma, \gamma)$$

$$+ \int_{-a}^{\theta} d\delta \int_{-a}^{\alpha} d\gamma \int_{-a}^{0} d\beta \int_{\max(\theta-\delta+t,\, \alpha-\gamma-\beta+t)}^{\{T\}} ds\; D_{01}^*(t+\theta-\delta, \delta)\, \Phi^{0*}(s, t+\theta-\delta)\, \Pi_{00}(s) R(s) \Pi_{01}(s,\beta)\, \Phi^0(s+\beta, t+\alpha-\gamma) D_{01}(t+\alpha-\gamma, \gamma)$$

$$+ \int_{-a}^{\theta} d\delta \int_{-a}^{\alpha} d\gamma \int_{-a}^{0} d\eta \int_{\max(\theta-\delta-\eta+t,\, \alpha-\gamma+t)}^{\{T\}} ds\; D_{01}^*(t+\theta-\delta, \delta)\, \Phi^{0*}(s, t+\theta-\delta)\, \Pi_{01}^*(s,\eta)\, R(s) \Pi_{00}(s)\, \Phi^0(s, t+\alpha-\gamma)\, D_{01}(t+\alpha-\gamma, \gamma)$$

$$+ \int_{-a}^{\theta} d\delta \int_{-a}^{\alpha} d\gamma \int_{-a}^{0} d\beta \int_{-a}^{0} d\eta \int_{\max(\theta-\delta-\eta+t,\, \alpha-\gamma-\beta+t)}^{\{T\}} ds\; D_{01}^*(t+\theta-\delta, \delta)\, \Phi^{0*}(s, t+\theta-\delta)\, \Pi_{01}^*(s,\eta)\, R(s) \Pi_{01}(s,\beta)\, \Phi^0(s+\beta, t+\alpha-\gamma)\, D_{01}(t+\alpha-\gamma, \gamma)$$

$$+ \int_{-a}^{\theta} d\delta \int_{\theta-\delta+t}^{\{\min(T,\, t+\alpha+a)\}} ds\; D_{01}^*(t+\theta-\delta, \delta)\, \Phi^{0*}(s, t+\theta-\delta)\, \Pi_{00}(s) R(s)\, \Pi_{01}(s, \alpha+t-s)$$

$$+ \int_{-a}^{\alpha} d\gamma \int_{\alpha-\gamma+t}^{\{\min(T,\, t+\theta+a)\}} ds\; \Pi_{01}^*(s, \theta+t-s)\, R(s) \Pi_{00}(s)\, \Phi^0(s, t+\alpha-\gamma)\, D_{01}(t+\alpha-\gamma, \gamma)$$

$$+ \int_{t}^{\{\min(T+\theta+a,\, T+\alpha+a)\}} ds\; \Pi_{01}^*(s, \theta+t-s)\, R(s)\, \Pi_{01}(s, \alpha+t-s) \tag{A.1-8}$$

The stated properties of $\Pi_{11}(t,\theta,\alpha)$ follow from (A.1-8)

Appendix 2 - Proof of theorem 3 E

$$(\Pi(t)k)(\theta) = \begin{cases} \Pi_{00}(t)\,k(0) + \int_{-a}^{0}\Pi_{01}(t,\alpha)\,k(\alpha)\,d\alpha & \theta = 0 \\ \Pi_{10}(t,\theta)\,k(0) + \int_{-a}^{0}\Pi_{11}(t,\theta,\alpha)\,k(\alpha)\,d\alpha & -a \le \theta < 0 \end{cases}$$

$$(\Pi(t)R(t)\Pi(t)k)(\theta) = \begin{cases} \Pi_{00}(t)R(t)\Pi_{00}(t)k(0) + \int_{-a}\Pi_{00}(t)R(t)\Pi_{01}(t,\alpha)\,k(\alpha)\,d\alpha & \theta = 0 \\ \Pi_{10}(t,\theta)\,R(t)\Pi_{00}(t)k(0) + \int_{-a}^{0}\Pi_{10}(t,\theta)R(t)\Pi_{01}(t,\alpha)\,k(\alpha)\,d\alpha & -a \le \theta < 0 \end{cases}$$

$$(\Pi(t)\mathcal{A}(t)k)(\theta) = \begin{cases} \left.\begin{array}{l} \Pi_{00}(t)\left[A_{00}(t)\,k(0) + \int_{-a}^{0}A_{01}(t,\alpha)\,k(\alpha)\,d\alpha + \sum_{i=1}^{N}A_i(t)\,k(\theta_i)\right] \\ \quad + \int_{-a}^{0}\Pi_{01}(t,\alpha)\frac{dk}{d\alpha}\,d\alpha \end{array}\right\} & \theta = 0 \\ \left.\begin{array}{l} \Pi_{10}(t,\theta)\left[A_{00}(t)\,k(0) + \int_{-a}^{0}A_{01}(t,\alpha)\,k(\alpha)\,d\alpha + \sum_{i=1}^{N}A_i(t)\,k(\theta_i)\right] \\ \quad + \int_{-a}^{0}\Pi_{11}(t,\theta,\alpha)\frac{dk}{d\alpha}\,d\alpha \end{array}\right\} & -a \le \theta < 0 \end{cases}$$

Writing out equation (3-101) in full we have

$$\left(h(0), \frac{d\Pi_{00}(t)}{dt}k(0)\right) + \int_{-a}^{0} d\alpha\left(h(0), \frac{\partial \Pi_{01}(t,\alpha)}{\partial t}k(\alpha)\right) + \int_{-a}^{0} d\theta\left(h(\theta), \frac{\partial \Pi_{01}^{*}(t,\theta)}{\partial t}k(0)\right)$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha\left(h(\theta), \frac{\partial \Pi_{11}(t,\theta,\alpha)}{\partial t}k(\alpha)\right) + \left(A_{00}(t)h(0), \Pi_{00}(t)k(0)\right) + \int_{-a}^{0} d\alpha\left(A_{00}(t)h(0), \Pi_{01}(t,\alpha)k(\alpha)\right)$$

$$+ \int_{-a}^{0} d\theta\left(A_{01}(t,\theta)h(\theta), \Pi_{00}(t)k(0)\right) + \sum_{i=1}^{N}\left(A_i(t)h(\theta_i), \Pi_{00}(t)k(0)\right)$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha\left(A_{01}(t,\theta)h(\theta), \Pi_{01}(t,\alpha)k(\alpha)\right) + \sum_{i=1}^{N}\int_{-a}^{0} d\alpha\left(A_i(t)h(\theta_i), \Pi_{01}(t,\alpha)k(\alpha)\right)$$

$$+ \int_{-a}^{0} d\theta\left(\frac{dh}{d\theta}, \Pi_{01}^{*}(t,\theta)k(0)\right) + \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha\left(\frac{dh}{d\theta}, \Pi_{11}(t,\theta,\alpha)k(\alpha)\right) + \left(h(0), \Pi_{00}(t)A_{00}(t)k(0)\right)$$

$$+ \int_{-a}^{0} d\alpha\left(h(0), \Pi_{00}(t)A_{01}(t,\alpha)k(\alpha)\right) + \sum_{i=1}^{N}\left(h(0), \Pi_{00}(t)A_i(t)k(\theta_i)\right) + \int_{-a}^{0} d\alpha\left(h(0), \Pi_{01}(t,\alpha)\frac{dk}{d\alpha}\right)$$

$$+ \int_{-a}^{0} d\theta \, (h(\theta), \Pi_{01}^{*}(t,\theta) A_{00}(t) k(0)) + \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \, (h(\theta), \Pi_{01}^{*}(t,\theta) A_{01}(t,\alpha) k(\alpha))$$

$$+ \sum_{i=1}^{N} \int_{-a}^{0} d\theta \, (h(\theta), \Pi_{01}^{*}(t,\theta) A_i(t) k(\theta_i)) + \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \, (h(\theta), \Pi_{11}(t,\theta,\alpha) \frac{dk}{d\alpha})$$

$$- (h(0), \Pi_{00}(t) R(t) \Pi_{00}(t) k(0)) - \int_{-a}^{0} d\alpha \, (h(0), \Pi_{00}(t) R(t) \Pi_{01}(t,\alpha) k(\alpha))$$

$$- \int_{-a}^{0} d\theta \, (h(\theta), \Pi_{10}^{*}(t,\theta) R(t) \Pi_{00}(t) k(0)) - \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \, (h(\theta), \Pi_{01}^{*}(t,\theta) R(t) \Pi_{01}(t,\alpha) k(\alpha))$$

$$+ (h(0), Q(t) k(0))$$

$$= 0 \qquad \text{for } h, k \in \mathcal{D}(A(t)) = AC^2(-a, 0; R^n) \qquad (A.2-1)$$

For $m, n \geq n_0 = a/|\theta_1|$ define

$$h_n(\theta) = \begin{cases} h(0)\left(1 + \frac{n\theta}{a}\right) & -a/n \leq \theta \leq 0 \\ 0 & -a \leq \theta < -a/n \end{cases}$$

$$k_m(\alpha) = \begin{cases} k(0)\left(1 + \frac{m\alpha}{a}\right) & -a/m \leq \theta \leq 0 \\ 0 & -a \leq \theta < -a/m \end{cases}$$

Substituting into (A.2-1) we obtain

$$(h(0), \frac{d\Pi_{00}(t)}{dt} k(0)) + \int_{-a/m}^{0} d\alpha \, (h(0), \frac{\partial \Pi_{01}(t,\alpha)}{\partial t} k_m(\alpha)) + \int_{-a/n}^{0} d\theta \, (h_n(\theta), \frac{\partial \Pi_{01}^{*}(t,\theta)}{\partial t} k(0))$$

$$+ \int_{-a/n}^{0} d\theta \int_{-a/m}^{0} d\alpha \, (h_n(\theta), \frac{\partial \Pi_{11}(t,\theta,\alpha)}{\partial t} k_m(\alpha)) + (A_{00}(t) h(0), \Pi_{00}(t) k(0))$$

$$+ \int_{-a/m}^{0} d\alpha \, (A_{00}(t) h(0), \Pi_{01}(t,\alpha) k_m(\alpha)) + \int_{-a/n}^{0} d\theta \, (A_{01}(t,\theta) h_n(\theta), \Pi_{00}(t) k(0))$$

$$+ \int_{-a/n}^{0} d\theta \int_{-a/m}^{0} d\alpha \, (A_{01}(t,\theta) h_n(\theta), \Pi_{01}(t,\alpha) k_m(\alpha)) + \frac{n}{a} \int_{-a/n}^{0} d\theta \, (h(0), \Pi_{01}^{*}(t,\theta) k(0))$$

$$+ \frac{n}{a} \int_{-a/n}^{0} d\theta \int_{-a/m}^{0} d\alpha \, (h(0), \Pi_{11}(t,\theta,\alpha) k_m(\alpha)) + (h(0), \Pi_{00}(t) A_{00}(t) k(0)) + \int_{a/m}^{0} d\alpha \, (h(0), \Pi_{00}(t) A_{01}(t,\alpha) k_m(\alpha))$$

$$+ \frac{m}{a}\int_{-a/m}^{0} d\alpha \, (h(0), \Pi_{01}(t,\alpha)\, k(0)) + \int_{-a/n}^{0} d\theta \, (h_n(\theta), \Pi_{01}^{*}(t,\theta)\, A_{00}(t) k(0))$$

$$+ \int_{-a/n}^{0} d\theta \int_{-a/m}^{0} d\alpha \, (h_n(\theta), \Pi_{01}^{*}(t,\theta)\, A_{01}(t,\alpha)\, k_m(\alpha)) + \int_{-a/n}^{0} d\theta \, \frac{m}{a}\int_{-a/m}^{0} (h_n(\theta), \Pi_{11}(t,\theta,\alpha)\, k(0))$$

$$- (h(0), \Pi_{00}(t) R(t) \Pi_{00}(t) k(0)) - \int_{-a/m}^{0} d\alpha \, (h(0), \Pi_{00}(t) R(t) \Pi_{01}(t,\alpha) k_m(\alpha))$$

$$- \int_{-a/n}^{0} d\theta \, (h_n(\theta), \Pi_{01}^{*}(t,\theta) R(t) \Pi_{00}(t) k(0)) - \int_{-a/n}^{0} d\theta \int_{-a/m}^{0} d\alpha \, (h_n(\theta), \Pi_{01}^{*}(t,\theta) R(t) \Pi_{01}(t,\alpha) k_m(\alpha))$$

$$+ (h(0), Q(t) k(0)). = 0 \qquad (A.2-2)$$

Now for any $f \in L^2(-a, 0; R^n)$,

$$\left| \int_{-a/n}^{0} (h_n(\theta), f(\theta))\, d\theta \right| = \left| \int_{-a}^{0} (h_n(\theta), f(\theta))\, d\theta \right|$$

$$\leq \|f\|_{L^2} \|h_n\|_{L^2} = \|f\|_{L^2} |h(0)| \left(\frac{a}{3n}\right)^{1/2}$$

$$\longrightarrow 0 \quad \text{as} \quad n \to \infty$$

If f is continuous at 0, then

$$\frac{n}{a}\int_{-a/n}^{0} f(\theta)\, d\theta \longrightarrow f(0) \quad \text{as} \quad n \to \infty$$

Taking the limit $m, n \to \infty$ in (A.2-2) we obtain

$$\left(h(0), \frac{d\Pi_{00}(t)}{dt} k(0)\right) + (h(0), A_{00}^{*}(t)\Pi_{00}(t)\, k(0)) + (h(0), \Pi_{01}^{*}(t,0)\, k(0))$$

$$+ (h(0), \Pi_{01}(t,0)\, k(0)) - (h(0), \Pi_{00}(t) R(t) \Pi_{00}(t) k(0)) + (h(0), Q(t) k(0))$$

$$+ (h(0), \Pi_{00}(t) A_{00}(t) k(0))$$

$$= 0 \qquad (A.2-3)$$

Equation (3-105) since (A.2-3) holds for all $h(0), k(0) \in R^n$

Combining (A.2-3) and (A.2-1) we have

$$\int_{-a}^{0} d\alpha \left(h(0), \frac{\partial \Pi_{01}(t,\alpha)}{\partial t} k(\alpha)\right) + \int_{-a}^{0} d\theta \left(h(\theta), \frac{\partial \Pi_{01}(t,\theta)}{\partial t} k(0)\right)$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \left(h(\theta), \frac{\partial \Pi_{11}(t,\theta,\alpha)}{\partial t} k(\alpha)\right) + \int_{-a}^{0} d\alpha \left(A_{00}(t) h(0), \Pi_{01}(t,\alpha) k(\alpha)\right)$$

$$+ \int_{-a}^{0} d\theta \left(A_{01}(t,\theta) h(\theta), \Pi_{00}(t) k(0)\right) + \sum_{i=1}^{N} \left(A_i(t) h(\theta_i), \Pi_{00}(t) k(0)\right)$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \left(A_{01}(t,\theta) h(\theta), \Pi_{01}(t,\alpha) k(\alpha)\right) + \sum_{i=1}^{N} \int_{-a}^{0} d\alpha \left(A_i(t) h(\theta_i), \Pi_{01}(t,\alpha) k(\alpha)\right)$$

$$+ \int_{-a}^{0} d\theta \left(\frac{dh}{d\theta}, \Pi_{01}^{*}(t,\theta) k(0)\right) - \left(h(0), \Pi_{01}(t,0) k(0)\right) + \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \left(\frac{dh}{d\theta}, \Pi_{11}(t,\theta,\alpha) k(\alpha)\right)$$

$$+ \int_{-a}^{0} d\alpha \left(h(0), \Pi_{00}(t) A_{01}(t,\alpha) k(\alpha)\right) + \sum_{i=1}^{N} \left(h(0), \Pi_{00}(t) A_i(t) k(\theta_i)\right) + \int_{-a}^{0} d\alpha \left(h(0), \Pi_{01}(t,\alpha) \frac{dk}{d\alpha}\right)$$

$$- \left(h(0), \Pi_{01}(t,0) k(0)\right) + \int_{-a}^{0} d\theta \left(h(\theta), \Pi_{01}^{*}(t,\theta) A_{00}(t) k(0)\right)$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \left(h(\theta), \Pi_{01}^{*}(t,\theta) A_{01}(t,\alpha) k(\alpha)\right) + \sum_{i=1}^{N} \int_{-a}^{0} d\theta \left(h(\theta), \Pi_{01}^{*}(t,\theta) A_i(t) k(\theta_i)\right)$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \left(h(\theta), \Pi_{11}(t,\theta,\alpha) \frac{dk}{d\alpha}\right) - \int_{-a}^{0} d\alpha \left(h(0), \Pi_{00}(t) R(t) \Pi_{01}(t,\alpha) k(\alpha)\right)$$

$$- \int_{-a}^{0} d\theta \left(h(\theta), \Pi_{01}^{*}(t,\theta) R(t) \Pi_{00}(t) k(0)\right) - \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \left(h(\theta), \Pi_{01}^{*}(t,\theta) R(t) \Pi_{01}(t,\alpha) k(\alpha)\right)$$

$$= 0 \qquad \text{(A.2-4)}$$

As before define $h_n(\theta) = \begin{cases} h(0)\left(1 + \frac{n\theta}{a}\right) & -a/n \le \theta \le 0 \\ 0 & -a \le \theta < -a/n \end{cases}$

and take k absolutely continuous and with support in (θ_{i+1}, θ_i)

Substituting into (A.2-4) and integrating by parts in the α variable we have

$$\int_{\theta_{i+1}}^{\theta_i} d\alpha \left(h(0), \frac{\partial \Pi_{01}(t,\alpha)}{\partial t} k(\alpha)\right) + \int_{-a/n}^{0} d\theta \int_{\theta_{i+1}}^{\theta_i} d\alpha \left(h_n(\theta), \frac{\partial \Pi_{11}(t,\theta,\alpha)}{\partial t} k(\alpha)\right)$$
$$+ \int_{\theta_{i+1}}^{\theta_i} d\alpha \left(A_{00}(t)h(0), \Pi_{01}(t,\alpha)k(\alpha)\right) + \int_{-a/n}^{0} d\theta \int_{\theta_{i+1}}^{\theta_i} d\alpha \left(A_{01}(t,\theta)h_n(\theta), \Pi_{01}(t,\alpha)k(\alpha)\right)$$
$$+ \frac{n}{a}\int_{-a/n}^{0} d\theta \int_{\theta_{i+1}}^{\theta_i} d\alpha \left(h(0), \Pi_{11}(t,\theta,\alpha)k(\alpha)\right) + \int_{\theta_{i+1}}^{\theta_i} d\alpha \left(h(0), \Pi_{00}(t)A_{01}(t,\alpha)k(\alpha)\right)$$
$$- \int_{\theta_{i+1}}^{\theta_i} d\alpha \left(h(0), \frac{\partial \Pi_{01}(t,\alpha)}{\partial \alpha} k(\alpha)\right) + \int_{-a/n}^{0} d\theta \int_{\theta_{i+1}}^{\theta_i} d\alpha \left(h_n(\theta), \Pi_{01}^{*}(t,\theta)A_{01}(t,\alpha)k(\alpha)\right)$$
$$- \int_{-a/n}^{0} d\theta \int_{\theta_{i+1}}^{\theta_i} d\alpha \left(h_n(\theta), \frac{\partial \Pi_{11}(t,\theta,\alpha)}{\partial \alpha} k(\alpha)\right) - \int_{\theta_{i+1}}^{\theta_i} d\alpha \left(h(0), \Pi_{00}(t)R(t)\Pi_{01}(t,\alpha)k(\alpha)\right)$$
$$- \int_{-a/n}^{0} d\theta \int_{\theta_{i+1}}^{\theta_i} d\alpha \left(h_n(\theta), \Pi_{01}^{*}(t,\theta)R(t)\Pi_{01}(t,\alpha)k(\alpha)\right)$$
$$= 0 \tag{A.2-5}$$

Taking limit $n \to \infty$ in (A.2-5) we have

$$\int_{\theta_{i+1}}^{\theta_i} d\alpha \left(h(0), \frac{\partial \Pi_{01}(t,\alpha)}{\partial t} k(\alpha)\right) + \int_{\theta_{i+1}}^{\theta_i} d\alpha \left(A_{00}(t)h(0), \Pi_{01}(t,\alpha)k(\alpha)\right)$$
$$+ \int_{\theta_{i+1}}^{\theta_i} d\alpha \left(h(0), \Pi_{11}(t,0,\alpha)k(\alpha)\right) + \int_{\theta_{i+1}}^{\theta_i} d\alpha \left(h(0), \Pi_{00}(t)A_{01}(t,\alpha)k(\alpha)\right)$$
$$- \int_{\theta_{i+1}}^{\theta_i} d\alpha \left(h(0), \frac{\partial \Pi_{01}(t,\alpha)}{\partial t} k(\alpha)\right) - \int_{\theta_{i+1}}^{\theta_i} d\alpha \left(h(0), \Pi_{00}(t)R(t)\Pi_{01}(t,\alpha)k(\alpha)\right)$$
$$= 0 \tag{A.2-6}$$

Equation (3-106) follows from (A.2-6) since the absolutely continuous functions with support in (θ_{i+1}, θ_i) is dense in $L^2(\theta_{i+1}, \theta_i\,;\, R^n)$.

Combining (A.2-6) and (A.2-4) we have

$$
\begin{aligned}
&\int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \left(h(\theta), \frac{\partial \Pi_{11}(t,\theta,\alpha)}{\partial t} k(\alpha)\right) + \sum_{i=1}^{N} \left(A_i(t) h(\theta_i), \Pi_{00}(t) k(0)\right) \\
&+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \left(A_{01}(t,\theta) h(\theta), \Pi_{01}(t,\alpha) k(\alpha)\right) + \sum_{i=1}^{N} \int_{-a}^{0} d\alpha \left(A_i(t) h(\theta_i), \Pi_{01}(t,\alpha) k(\alpha)\right) \\
&+ \int_{-a}^{0} d\theta \left(\frac{dh}{d\theta}, \Pi_{01}^{*}(t,\theta) k(0)\right) - \left(h(0), \Pi_{01}^{*}(t,0) k(0)\right) + \int_{-a}^{0} d\theta \left(h(\theta), \frac{\partial \Pi_{01}^{*}(t,\theta)}{\partial t} k(0)\right) \\
&+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \left(\frac{dh}{d\theta}, \Pi_{11}(t,\theta,\alpha) k(\alpha)\right) - \int_{-a}^{0} d\alpha \left(h(0), \Pi_{11}(t,0,\alpha) k(\alpha)\right) \\
&+ \sum_{i=1}^{N} \left(h(0), \Pi_{00}(t) A_i(t) k(\theta_i)\right) + \int_{-a}^{0} d\alpha \left(h(0), \Pi_{01}(t,\alpha) \frac{dk}{d\alpha}\right) - \left(h(0), \Pi_{01}(t,0) k(0)\right) \\
&+ \int_{-a}^{0} d\alpha \left(h(0), \frac{\partial \Pi_{01}(t,\alpha)}{\partial \alpha} k(\alpha)\right) + \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \left(h(\theta), \Pi_{01}^{*}(t,\theta) A_{01}(t,\alpha) k(\alpha)\right) \\
&+ \sum_{i=1}^{N} \int_{-a}^{0} d\theta \left(h(\theta), \Pi_{01}^{*}(t,\theta) A_i(t) k(\theta_i)\right) + \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \left(h(\theta), \Pi_{11}(t,\theta,\alpha) \frac{dk}{d\alpha}\right) \\
&- \int_{-a}^{0} d\theta \left(h(\theta), \Pi_{11}(t,\theta,0) k(0)\right) - \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \left(h(\theta), \Pi_{01}^{*}(t,\theta) R(t) \Pi_{01}(t,\alpha) k(\alpha)\right)
\end{aligned}
\tag{A.2-7}
$$

$$= 0$$

Now take h, k absolutely continuous,
supp $h \subset (\theta_{i+1}, \theta_i)$, supp $k \subset (\theta_{j+1}, \theta_j)$.

Substituting into (A.2-7) and integrating by parts, we obtain

$$
\begin{aligned}
&\int_{\theta_{i+1}}^{\theta_i} d\theta \int_{\theta_{j+1}}^{\theta_j} d\alpha \left(h(\theta), \frac{\partial \Pi_{11}(t,\theta,\alpha)}{\partial t} k(\alpha)\right) + \int_{\theta_{i+1}}^{\theta_i} d\theta \int_{\theta_{j+1}}^{\theta_j} d\alpha \left(A_{01}(t,\theta) h(\theta), \Pi_{01}(t,\alpha) k(\alpha)\right) \\
&- \int_{\theta_{i+1}}^{\theta_i} d\theta \int_{\theta_{j+1}}^{\theta_j} d\alpha \left(h(\theta), \frac{\partial \Pi_{11}(t,\theta,\alpha)}{\partial \theta} k(\alpha)\right) + \int_{\theta_{i+1}}^{\theta_i} d\theta \int_{\theta_{j+1}}^{\theta_j} d\alpha \left(h(\theta), \Pi_{01}^{*}(t,\theta) A_{01}(t,\alpha) k(\alpha)\right)
\end{aligned}
$$

$$- \int_{\theta_{i+1}}^{\theta_i} d\theta \int_{\theta_{j+1}}^{\theta_j} d\alpha \left(h(\theta), \frac{\partial \Pi_{11}(t,\theta,\alpha)}{\partial \alpha} k(\alpha)\right) - \int_{\theta_{i+1}}^{\theta_i} d\theta \int_{\theta_{j+1}}^{\theta_j} d\alpha \left(h(\theta), \Pi_{01}^*(t,\theta) R(t) \Pi_{01}(t,\alpha) k(\alpha)\right)$$

$$= 0 \qquad \text{(A.2-8)}$$

Hence equation (3-107) follows from (A.2-8) since the absolutely continuous functions with support in (θ_{i+1}, θ_i) is dense in $L^2(\theta_{i+1}, \theta_i; R^n)$

To determine the boundary conditions, we consider

$$h_n(\theta) = \begin{cases} h(-a)\left(-\frac{n\theta}{a} + 1 - n\right) & -a \le \theta \le -a(1-\frac{1}{n}) \\ 0 & -a(1-\frac{1}{n}) < \theta \le 0 \end{cases}$$

defined for $n \geq n_0 > a/|\theta_N - \theta_{N-1}|$

and $$k_m(\alpha) = \begin{cases} k(0)\left(1 + \frac{m\alpha}{a}\right) & -a/m \le \alpha \le 0 \\ 0 & -a \le \alpha < -a/m \end{cases}$$

defined for $m \geq m_0 > a/|\theta_1|$

Substituting into (A.2-1) we obtain

$$\int_{-a}^{-a(1-\frac{1}{n})} d\theta \left(h_n(\theta), \frac{\partial \Pi_{01}^*(t,\theta)}{\partial t} k(0)\right) + \int_{-a}^{-a(1-\frac{1}{n})} d\theta \int_{-a/m}^{0} d\alpha \left(h_n(\theta), \frac{\partial \Pi_{11}(t,\theta,\alpha)}{\partial t} k_m(\alpha)\right)$$

$$+ \int_{-a}^{-a(1-\frac{1}{n})} d\theta \left(A_{01}(t,\theta) h_n(\theta), \Pi_{00}(t) k(0)\right) + \left(A_N(t) h(-a), \Pi_{00}(t) k(0)\right)$$

$$+ \int_{-a}^{-a(1-\frac{1}{n})} d\theta \int_{-a/m}^{0} d\alpha \left(A_{01}(t,\theta) h_n(\theta), \Pi_{01}(t,\alpha) k_m(\alpha)\right) + \int_{-a/m}^{0} d\alpha \left(A_N(t) h(-a), \Pi_{01}(t,\alpha) k_m(\alpha)\right)$$

$$- \frac{n}{a} \int_{-a}^{-a(1-\frac{1}{n})} d\theta \left(h(-a), \Pi_{01}^*(t,\theta) k(0)\right) + \int_{-a}^{-a(1-\frac{1}{n})} d\theta \left(h_n(\theta), \Pi_{01}^*(t,\theta) A_{00}(t) k(0)\right)$$

$$+ \int_{-a}^{-a(1-\frac{1}{n})} d\theta \int_{-a/m}^{0} d\alpha \left(h_n(\theta), \Pi_{01}^*(t,\theta) A_{01}(t,\alpha) k_m(\alpha)\right) + \int_{-a}^{-a(1-\frac{1}{n})} d\theta \frac{m}{a} \int_{-a/m}^{0} d\alpha \left(h_n(\theta), \Pi_{11}(t,\theta,\alpha) k(0)\right)$$

$$- \int_{-a}^{-a(1-\frac{1}{n})} d\theta \, (h_n(\theta), \Pi_{01}^*(t,\theta) R(t) \Pi_{00}(t) k(0)) - \int_{-a}^{-a(1-\frac{1}{n})} d\theta \int_{-a/m}^{0} d\alpha (h_n(\theta), \Pi_{01}^*(t,\theta) R(t) \Pi_{01}(t,\alpha) k_m(\alpha)$$

$$= 0 \qquad (A.2-9)$$

Taking limit $m, n \to \infty$ we have

$$(A_N(t) h(-a), \Pi_{00}(t) k(0)) - (h(-a), \Pi_{01}^*(t,-a) k(0)) = 0$$

and since this holds for all $h(-a), k(0) \in R^n$, we have

$$\Pi_{01}(t,-a) = \Pi_{00}(t) A_N(t).$$

Taking h_n as before and k absolutely continuous with support in (θ_{i+1}, θ_i) we have

$$\int_{-a}^{-a(1-\frac{1}{n})} d\theta \int_{\theta_{i+1}}^{\theta_i} d\alpha (h_n(\theta), \frac{\partial \Pi_{11}(t,\theta,\alpha)}{\partial t} k(\alpha)) + \int_{-a}^{-a(1-\frac{1}{n})} d\theta \int_{\theta_{i+1}}^{\theta_i} d\alpha \, (A_{01}(t,\theta) h_n(\theta), \Pi_{01}(t,\alpha) k(\alpha))$$

$$+ \int_{\theta_{i+1}}^{\theta_i} d\alpha \, (A_N(t) h(-a), \Pi_{01}(t,\alpha) k(\alpha)) - \frac{n}{a} \int_{-a}^{-a(1-\frac{1}{n})} d\theta \int_{\theta_{i+1}}^{\theta_i} d\alpha \, (h(-a), \Pi_{11}(t,\theta,\alpha) k(\alpha))$$

$$- \int_{-a}^{-a(1-\frac{1}{n})} d\theta \int_{\theta_{i+1}}^{\theta_i} d\alpha \, (h_n(\theta), \frac{\partial \Pi_{11}(t,\theta,\alpha)}{\partial \alpha} k(\alpha)) - \int_{-a}^{-a(1-\frac{1}{n})} d\theta \int_{\theta_{i+1}}^{\theta_i} d\alpha \, (h_n(\theta), \Pi_{01}^*(t,\theta) R(t) \Pi_{01}(t,\alpha) k(\alpha))$$

$$= 0$$

Taking limit $n \to \infty$, we have

$$\int_{\theta_{i+1}}^{\theta_i} d\alpha \, (A_N(t) h(-a), \Pi_{01}(t,\alpha) k(\alpha)) - \int_{\theta_{i+1}}^{\theta_i} d\alpha \, (h(-a), \Pi_{11}(t,-a,\alpha) k(\alpha)) = 0 \qquad (A.2-10)$$

Hence $\Pi_{11}(t,-a,\alpha) = A_N^*(t) \Pi_{01}(t,\alpha)$ a.e.
from the density of the absolutely continuous functions with support in (θ_{i+1}, θ_i) in $L^2(\theta_{i+1}, \theta_i; R^n)$.

Appendix 3 – Proof of theorem 3 G

$$(\tilde{g}(t), h)_{M^2} = (g_0(t), h(0)) + \int_{-a}^{0} (g_1(t,\theta), h(\theta))\, d\theta$$

$$= \int_t^T ds\, (\Phi(T,t)h, \mathcal{F}\,\Phi(T,s)\,\tilde{f}_0(s))_{M^2}$$

$$+ \int_t^T ds\, (\Pi(s)\,\Phi(s,t)h, R(s)\tilde{g}(s))_{M^2}$$

$$+ \int_t^T ds \int_t^s ds_1\, (\Pi(s)\,\Phi(s,t)h, R(s)\Pi(s)\,\Phi(s,s_1)\,\tilde{f}_0(s_1))_{M^2}$$

$$+ \int_t^T ds \int_t^s ds_1\, (\Phi(s,t)\,h, Q(s)\,\Phi(s,s_1)\,\tilde{f}_0(s_1))_{M^2}$$

$$= \int_t^T ds\, (\Phi^0(T,t)\,h(0), F\,\Phi^0(T,s)\,f_0(s))$$

$$+ \int_t^T ds \int_{-a}^{0} d\theta\, (\Phi^1(T,t,\theta)\,h(\theta), F\,\Phi^0(T,s)\,f_0(s))$$

$$+ \int_t^T ds\, (\Pi_{00}(s)\,\Phi^0(s,t)\,h(0), R(s)\,g_0(s))$$

$$+ \int_t^T ds \int_{-a}^{0} d\theta\, (\Pi_{00}(s)\,\Phi^1(s,t,\theta)\,h(\theta), R(s)\,g_0(s))$$

$$+ \int_t^T ds \int_{-a}^{0} d\eta\, (\Pi_{01}(s,\eta)\,\Phi^0(s+\eta,t)\,h(0), R(s)\,g_0(s))$$

$$+ \int_t^T ds \int_{-a}^{0} d\theta \int_{-a}^{0} d\eta\, (\Pi_{01}(s,\eta)\,\Phi^1(s+\eta,t,\theta)\,h(\theta), R(s)\,g_0(s))$$

$$+ \int_t^T ds \int_t^s ds_1\, (\Pi_{00}(s)\,\Phi^0(s,t)\,h(0), R(s)\Pi_{00}(s)\,\Phi^0(s,s_1)\,f_0(s_1))$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^{0} d\alpha\, (\Pi_{00}(s)\,\Phi^0(s,t)\,h(0), R(s)\,\Pi_{01}(s,\alpha)\,\Phi^0(s+\alpha,s_1)\,f_0(s_1))$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^0 d\theta \; \Pi_{00}(s) \Phi^1(s, t, \theta) h(\theta), R(s) \Pi_{00}(s) \Phi(s, s_1) f_0(s_1)$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^0 d\theta \int_{-a}^0 d\alpha \; \Pi_{00}(s) \Phi^1(s, t, \theta) h(\theta), R(s) \Pi_{01}(s, \alpha) \Phi^0(s+\alpha, s_1) f_0(s_1)$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^0 d\eta \; \Pi_{01}(s, \eta) \Phi^0(s+\eta, t) h(0), R(s) \Pi_{00}(s) \Phi^0(s, s_1) f_0(s_1)$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^0 d\eta \int_{-a}^0 d\alpha \left(\Pi_{01}(s, \eta) \Phi^0(s+\eta, t) h(0), R(s) \Pi_{01}(s, \alpha) \Phi^0(s+\alpha, s_1) f_0(s_1)\right)$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^0 d\theta \int_{-a}^0 d\eta \left(\Pi_{01}(s, \eta) \Phi^1(s+\eta, t, \theta) h(\theta), R(s) \Pi_{00}(s) \Phi^0(s, s_1) f_0(s_1)\right)$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^0 d\theta \int_{-a}^0 d\eta \int_{-a}^0 d\alpha \left(\Pi_{01}(s, \eta) \Phi^1(s+\eta, t, \theta) h(\theta), R(s) \Pi_{01}(s, \alpha) \Phi^0(s+\alpha, s_1) f_0(s_1)\right)$$

$$+ \int_t^T ds \int_t^s ds_1 \left(\Phi^0(s, t) h(0), Q(s) \Phi^0(s, s_1) f_0(s_1)\right)$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^0 d\theta \left(\Phi^1(s, t, \theta) h(\theta), Q(s) \Phi^0(s, s_1) f_0(s_1)\right) \qquad (A.3-1)$$

Putting $h^1 = 0$, we have

$$(g_0(t), h(0)) = \int_t^T ds \left(\Phi^0(T, t) h(0), F \Phi^0(T, s) f_0(s)\right)$$

$$+ \int_t^T ds \left(\Pi_{00}(s) \Phi^0(s, t) h(0), R(s) g_0(s)\right)$$

$$+ \int_t^T ds \int_{-a}^0 d\eta \left(\Pi_{01}(s, \eta) \Phi^0(s+\eta, t) h(0), R(s) g_0(s)\right)$$

$$+ \int_t^T ds \int_t^s ds_1 \left(\Pi_{00}(s) \Phi^0(s, t) h(0), R(s) \Pi_{00}(s) \Phi^0(s, s_1) f_0(s_1)\right)$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^0 d\alpha \left(\Pi_{00}(s) \Phi^0(s, t) h(0), R(s) \Pi_{01}(s, \alpha) \Phi(s+\alpha, s_1) f_0(s_1)\right)$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^0 d\eta \left(\Pi_{01}(s, \eta) \Phi^0(s+\eta, t) h(0), R(s) \Pi_{00}(s) \Phi^0(s, s_1) f_0(s_1)\right)$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^0 d\eta \int_{-a}^0 d\alpha \left(\Pi_{01}(s,\eta)\, \Phi^0(s+\eta,t)\, h(0),\ R(s)\, \Pi_{01}(s,\alpha)\, \Phi^0(s+\alpha, s_1)\, f_0(s_1) \right)$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^0 d\theta \left(\Phi^0(s,t)\, h(0),\ Q(s)\, \Phi^0(s,s_1)\, f_0(s_1) \right) \tag{A.3-2}$$

Since (A.3-2) holds for all $h(0) \in R^n$, by interchanging the order of integration we obtain:

$$\begin{aligned} g_0(t) = {} & \int_t^T ds\ \Phi^{0*}(T,t)\, F\, \Phi^0(T,s)\, f_0(s) \\ & + \int_t^T ds\ \Phi^{0*}(s,t)\, \Pi_{00}^*(s)\, R(s)\, g_0(s) \\ & + \int_{-a}^0 d\eta \int_{-\eta}^{\{T\}} ds\ \Phi^{0*}(s+\eta,t)\, \Pi_{01}^*(s,\eta)\, R(s)\, g_0(s) \\ & + \int_t^T ds_1 \int_{s_1}^T ds\ \Phi^{0*}(s,t)\, \Pi_{00}^*(s)\, R(s)\, \Pi_{00}(s)\, \Phi^0(s,s_1)\, f_0(s_1) \\ & + \int_{-a}^0 d\alpha \int_t^T ds_1 \int_{s_1-\alpha}^{\{T\}} ds\ \Phi^{0*}(s,t)\, \Pi_{00}^*(s)\, R(s)\, \Pi_{01}(s,\alpha)\, \Phi^0(s+\alpha,s_1)\, f_0(s_1) \\ & + \int_{-a}^0 d\eta \int_t^T ds_1 \int_{\max(s_1,t-\eta)}^0 ds\ \Phi^{0*}(s+\eta,t)\, \Pi_{01}^*(s,\eta)\, R(s)\, \Pi_{00}(s)\, \Phi^0(s,s_1)\, f_0(s_1) \\ & + \int_{-a}^0 d\eta \int_{-a}^0 d\alpha \int_t^T ds_1 \int_{\max(s_1-\alpha,t-\eta)}^{\{T\}} ds\ \Phi^{0*}(s+\eta,t)\, \Pi_{01}^*(s,\eta)\, R(s)\, \Pi_{01}(s,\alpha)\, \Phi^0(s+\alpha,s_1)\, f_0(s_1) \\ & + \int_t^T ds_1 \int_{s_1}^T ds\ \Phi^{0*}(s,t)\, Q(s)\, \Phi^0(s,s_1)\, f_0(s_1) \end{aligned}$$

Hence clearly the map $t \mapsto g_0(t)$ is absolutely continuous and $g(T) = 0$

Putting $h(0)=0$ in (A.3-1) we obtain:

$$\int_{-a}^{0} (g_1(t,\theta), h(\theta))\, d\theta$$

$$= \int_t^T ds \int_{-a}^{0} d\theta \, (\Phi^1(T,t,\theta) h(\theta),\ F\, \Phi^0(T,s) f_0(s))$$

$$+ \int_t^T ds \int_{-a}^{0} d\theta \, (\Pi_{00}(s)\, \Phi^1(s,t,\theta) h(\theta),\ R(s)\, g_0(s))$$

$$+ \int_t^T ds \int_{-a}^{0} d\theta \int_{-a}^{0} d\eta \, (\Pi_{01}(s,\eta)\, \Phi^1(s+\eta,t,\theta) h(\theta),\ R(s)\, g_0(s))$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^{0} d\theta \, (\Pi_{00}(s)\, \Phi^1(s,t,\theta) h(\theta),\ R(s)\, \Pi_{00}(s)\, \Phi^0(s,s_1) f_0(s_1))$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \, (\Pi_{00}(s)\, \Phi^1(s,t,\theta) h(\theta),\ R(s)\, \Pi_{01}(s,\alpha)\, \Phi^0(s+\alpha,s_1) f_0(s_1))$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^{0} d\theta \int_{-a}^{0} d\eta \, (\Pi_{01}(s,\eta)\, \Phi^1(s+\eta,t,\theta) h(\theta),\ R(s)\, \Pi_{00}(s)\, \Phi^0(s,s_1) f_0(s_1))$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^{0} d\theta \int_{-a}^{0} d\eta \int_{-a}^{0} d\alpha \, (\Pi_{01}(s,\eta)\, \Phi^1(s+\eta,t,\theta) h(\theta),\ R(s)\, \Pi_{01}(s,\alpha)\, \Phi^0(s+\alpha,s_1) f_0(s_1))$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^{0} d\theta \, (\Phi^1(s,t,\theta) h(\theta),\ Q(s)\, \Phi^0(s,s_1) f_0(s_1))$$

$$= \sum_{i=1}^{N} \int_t^T ds \int_{\theta_i}^{\min(0,\theta_i+s-t)} d\theta \, (\Phi^0(T,t+\theta-\theta_i) A_i(t+\theta-\theta_i) h(\theta),\ F\, \Phi^0(T,s) f_0(s))$$

$$+ \int_t^T ds \int_{-a}^{0} d\theta \int_{\max(-a,\theta-s+t)}^{\theta} d\delta \, (\Phi^0(T,t+\theta-\delta) D_{01}(t+\theta-\delta,\delta) h(\theta),\ F\, \Phi^0(s,t) f_0(s))$$

$$+ \sum_{i=1}^{N} \int_t^T ds \int_{\theta_i}^{\min(0,\theta_i+s-t)} d\theta \, (\Pi_{00}(s)\, \Phi^0(s,t+\theta-\theta_i) A_i(t+\theta-\theta_i) h(\theta),\ R(s)\, g_0(s))$$

$$+ \int_t^T ds \int_{-a}^{0} d\theta \int_{\max(-a,\theta-s+t)}^{\theta} d\delta \, (\Pi_{00}(s)\, \Phi^0(s,t+\theta-\delta) D_{01}(t+\theta-\delta,\delta) h(\theta),\ R(s)\, g_0(s))$$

$$+ \sum_{i=1}^{N} \int_t^T ds \int_{\max(-a,t-s)}^{0} d\eta \int_{\theta_i}^{\min(0,\theta_i+s+\eta-t)} d\theta \, (\Pi_{01}(s,\eta)\, \overset{\circ}{\Phi}(s+\eta, t+\theta-\theta_i)\, A_i(t+\theta-\theta_i)\, h(\theta), R(s)\, g_0(s))$$

$$+ \int_t^T ds \int_{-a}^{0} d\theta \int_{\max(-a,t-s)}^{0} d\eta \int_{\max(-a,\theta-s-\eta+t)}^{0} d\delta \, (\Pi_{01}(s,\eta)\, \overset{\circ}{\Phi}(s+\eta, t+\theta-\delta)\, D_{01}(t+\theta-\delta,\delta)\, h(\theta), R(s)\, g_0(s))$$

$$+ \int_t^T ds \int_{-a}^{\max(-a,t-s)} d\eta \, (\Pi_{01}(s,\eta)\, h(s+\eta-t), R(s)\, g_0(s))$$

$$+ \sum_{i=1}^{N} \int_t^T ds \int_t^s ds_1 \int_{\theta_i}^{\min(0,\theta_i+s-t)} d\theta \, (\Pi_{00}(s)\, \overset{\circ}{\Phi}(s, t+\theta-\theta_i)\, A_i(t+\theta-\theta_i)\, h(\theta), R(s)\, \Pi_{00}(s)\, \overset{\circ}{\Phi}(s,s_1)\, f_0(s_1))$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^{0} d\theta \int_{\max(-a,\theta-s+t)}^{\theta} d\delta \, (\Pi_{00}(s)\, \overset{\circ}{\Phi}(s, t+\theta-\delta)\, D_{01}(t+\theta-\delta,\delta)\, h(\theta), R(s)\, \Pi_{00}(s)\, \overset{\circ}{\Phi}(s,s_1)\, f_0(s_1))$$

$$+ \sum_{i=1}^{N} \int_t^T ds \int_t^s ds_1 \int_{\theta_i}^{\min(0,\theta_i+s-t)} d\theta \int_{\max(-a,s_1-s)}^{0} d\alpha \, (\Pi_{00}(s)\, \overset{\circ}{\Phi}(s, t+\theta-\theta_i)\, A_i(t+\theta-\theta_i)\, h(\theta), R(s)\, \Pi_{01}(s,\alpha)\, \overset{\circ}{\Phi}(s+\alpha,s_1)\, f_0(s_1))$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^{0} d\theta \int_{\max(-a,\theta-s+t)}^{\theta} d\delta \int_{\max(-a,s_1-s)}^{0} d\alpha \, (\Pi_{00}(s)\, \overset{\circ}{\Phi}(s, t+\theta-\delta)\, D_{01}(t+\theta-\delta,\delta)\, h(\theta), R(s)\, \Pi_{01}(s,\alpha)\, \overset{\circ}{\Phi}(s+\alpha,s_1)\, f_0(s_1))$$

$$+ \sum_{i=1}^{N} \int_t^T ds \int_t^s ds_1 \int_{\max(a,t-s)}^{0} d\eta \int_{\theta_i}^{\min(0,\theta_i+s+\eta-t)} d\theta \, (\Pi_{01}(s,\eta)\, \overset{\circ}{\Phi}(s+\eta, t+\theta-\theta_i)\, A_i(t+\theta-\theta_i)\, h(\theta), R(s)\, \Pi_{00}(s)\, \overset{\circ}{\Phi}(s,s_1)\, f_0(s_1))$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{\max(-a,t-s)}^{0} d\eta \int_{-a}^{0} d\theta \int_{\max(-a,\theta-s-\eta+t)}^{\theta} d\delta \, (\Pi_{01}(s,\eta)\, \overset{\circ}{\Phi}(s+\eta, t+\theta-\delta)\, D_{01}(t+\theta-\delta,\delta)\, h(\theta), R(s)\, \Pi_{00}(s)\, \overset{\circ}{\Phi}(s,s_1)\, f_0(s_1))$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^{\max(-a,t-s)} d\eta \, (\Pi_{01}(s,\eta)\, h(s+\eta-t), R(s)\, \Pi_{00}(s)\, \overset{\circ}{\Phi}(s,s_1)\, f_0(s_1))$$

$$+ \sum_{i=1}^{N} \int_t^T ds \int_t^s ds_1 \int_{\max(-a,t-s)}^{0} d\eta \int_{\theta_i}^{\min(0,\theta_i+s+\eta-t)} d\theta \int_{\max(-a,s_1-s)}^{0} d\alpha \, (\Pi_{01}(s,\eta)\, \overset{\circ}{\Phi}(s+\eta, t+\theta-\theta_i)\, A_i(t+\theta-\theta_i)\, h(\theta), R(s)\, \Pi_{01}(s,\alpha)\, \overset{\circ}{\Phi}(s+\alpha,s_1)\, f_0(s_1))$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{\max(-a,t-s)}^{0} d\eta \int_{-a}^{0} d\theta \int_{\max(-a,\theta-s-\eta+t)}^{\theta} d\delta \int_{\max(-a,s_1-s)}^{0} d\alpha \, (\Pi_{01}(s,\eta)\, \overset{\circ}{\Phi}(s+\eta, t+\theta-\delta)\, D_{01}(t+\theta-\delta,\delta)\, h(\theta), R(s)\, \Pi_{01}(s,\alpha)\, \overset{\circ}{\Phi}(s+\alpha,s_1)\, f_0(s_1))$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^{\max(-a,t-s)} d\eta \int_{\max(-a,s_1-s)}^{0} d\alpha \, (\Pi_{01}(s,\eta)\, h(s+\eta-t), R(s)\, \Pi_{01}(s,\alpha)\, \overset{\circ}{\Phi}(s+\alpha,s_1)\, f_0(s_1))$$

$$+ \sum_{i=1}^{N} \int_t^T ds \int_t^s ds_1 \int_{\theta_i}^{\min(0,\theta_i+s-t)} d\theta \, (\overset{\circ}{\Phi}(s, t+\theta-\theta_i)\, A_i(t+\theta-\theta_i)\, h(\theta), Q(s)\, \overset{\circ}{\Phi}(s,s_1)\, f_0(s_1))$$

$$+ \int_t^T ds \int_t^s ds_1 \int_{-a}^{0} d\theta \int_{\max(-a,\theta-s+t)}^{\theta} d\delta \, (\overset{\circ}{\Phi}(s, t+\theta-\delta)\, D_{01}(t+\theta-\delta,\delta)\, h(\theta), Q(s)\, \overset{\circ}{\Phi}(s,s_1)\, f_0(s_1))$$

Interchanging the order of integration

$$\int_{-a}^{0}(g_1(t,\theta), h(\theta))d\theta$$

$$= \sum_{i=1}^{N} \int_{\theta_i}^{0} d\theta \int_{\theta-\theta_i+t}^{\{T\}} ds \, (\Phi^{0}(T, t+\theta-\theta_i) A_i(t+\theta-\theta_i) h(\theta), F\,\Phi^{0}(T,s) f_0(s))$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\delta \int_{\theta-\delta+t}^{\{T\}} ds \, (\Phi^{0}(T, t+\theta-\delta) D_{01}(t+\theta-\delta,\delta) h(\theta), F\,\Phi^{0}(T,s) f_0(s))$$

$$+ \sum_{i=1}^{N} \int_{\theta_i}^{0} d\theta \int_{\theta-\theta_i+t}^{\{T\}} ds \, (\Pi_{00}(s) \Phi^{0}(s, t+\theta-\theta_i) A_i(t+\theta-\theta_i) h(\theta), R(s)\, g_0(s))$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{\theta} d\delta \int_{\theta-\delta+t}^{\{T\}} ds \, (\Pi_{00}(s) \Phi^{0}(s, t+\theta-\delta) D_{01}(t+\theta-\delta,\delta) h(\theta), R(s)\, g_0(s))$$

$$+ \sum_{i=1}^{N} \int_{\theta_i}^{0} d\theta \int_{-a}^{0} d\eta \int_{\theta-\theta_i+t-\eta}^{\{T\}} ds \, (\Pi_{01}(s,\eta) \Phi^{0}(s+\eta, t+\theta-\theta_i) A_i(t+\theta-\theta_i) h(\theta), R(s)\, g_0(s))$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\eta \int_{-a}^{\theta} d\delta \int_{\theta-\delta-\eta+t}^{\{T\}} ds \, (\Pi_{01}(s,\eta) \Phi^{0}(s+\eta, t+\theta-\delta) D_{01}(t+\theta-\delta,\delta) h(\theta), R(s)\, g_0(s))$$

$$+ \int_{-a}^{0} d\theta \int_{t}^{\{\min(T,t+\theta+a)\}} ds \, (\Pi_{01}(s, \theta+t-s) h(\theta), R(s)\, g_0(s))$$

$$+ \sum_{i=1}^{N} \int_{\theta_i}^{0} d\theta \int_{t}^{T} ds_1 \int_{\max(s_1,\theta-\theta_i+t)}^{\{T\}} ds \, (\Pi_{00}(s) \Phi^{0}(s, t+\theta-\theta_i) A_i(t+\theta-\theta_i) h(\theta), R(s) \Pi_{00}(s) \Phi^{0}(s,s_1) f_0(s_1))$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{\theta} d\delta \int_{t}^{T} ds_1 \int_{\max(s_1,\theta-\delta+t)}^{\{T\}} ds \, (\Pi_{00}(s) \Phi^{0}(s, t+\theta-\delta) D_{01}(t+\theta-\delta,\delta) h(\theta), R(s) \Pi_{00}(s) \Phi^{0}(s,s_1) f_0(s_1))$$

$$+ \sum_{i=1}^{N} \int_{\theta_i}^{0} d\theta \int_{-a}^{0} d\alpha \int_{t}^{T} ds_1 \int_{\max(s_1-\alpha,\theta-\theta_i+t)}^{\{T\}} ds \, (\Pi_{00}(s) \Phi^{0}(s, t+\theta-\theta_i) A_i(t+\theta-\theta_i) h(\theta), R(s) \Pi_{01}(s,\alpha) \Phi^{0}(s+\alpha,s_1) f_0(s_1))$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{\theta} d\delta \int_{-a}^{0} d\alpha \int_{t}^{T} ds_1 \int_{\max(s_1-\alpha,\theta-\delta+t)}^{\{T\}} ds \, (\Pi_{00}(s) \Phi^{0}(s, t+\theta-\delta) D_{01}(t+\theta-\delta,\delta) h(\theta), R(s) \Pi_{01}(s,\alpha) \Phi^{0}(s+\alpha,s_1) f_0(s_1))$$

$$+ \sum_{i=1}^{N} \int_{\theta_i}^{0} d\theta \int_{-a}^{0} d\eta \int_{t}^{T} ds_1 \int_{\max(s_1,\theta-\theta_i+t-\eta)}^{\{T\}} ds \, (\Pi_{01}(s,\eta) \Phi^{0}(s+\eta, t+\theta-\theta_i) A_i(t+\theta-\theta_i) h(\theta), R(s) \Pi_{00}(s) \Phi^{0}(s,s_1) f_0(s_1))$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{\theta} d\delta \int_{-a}^{0} d\eta \int_{t}^{T} ds_1 \int_{\max(s_1,\theta-\delta-\eta+t)}^{\{T\}} ds \, (\Pi_{01}(s,\eta) \Phi^{0}(s+\eta, t+\theta-\delta) D_{01}(t+\theta-\delta,\delta) h(\theta), R(s) \Pi_{00}(s) \Phi^{0}(s,s_1) f_0(s_1))$$

$$+ \int_{-a}^{0} d\theta \int_{t}^{T} ds_1 \int_{s_1}^{\{\min(T, t+\theta+a)\}} ds \, \left(\Pi_{01}(s, \theta + t - s) h(\theta), R(s) \Pi_{00}(s) \Phi^0(s, s_1) f_0(s_1) \right)$$

$$+ \sum_{i=1}^{N} \int_{\theta_i}^{0} d\theta \int_{-a}^{0} d\eta \int_{-a}^{0} d\alpha \int_{t}^{T} ds_1 \int_{\max(s_1-\alpha,\, \theta-\theta_i+t-\eta)}^{\{T\}} ds \left(\Pi_{01}(s,\eta) \Phi^0(s+\eta, t+\theta-\theta_i) A_i(t+\theta-\theta_i) h(\theta), R(s) \Pi_{01}(s,\alpha) \Phi^0(s+\alpha, s) f_0(s_1) \right)$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{\theta} d\delta \int_{-a}^{0} d\eta \int_{-a}^{0} d\alpha \int_{t}^{T} ds_1 \int_{\max(s_1-\alpha,\, \alpha-\delta+t-\eta)}^{\{T\}} ds \left(\Pi_{01}(s,\eta) \Phi^0(s+\eta, t+\theta-\delta) D_{01}(t+\theta-\delta, \delta) h(\theta), R(s) \Pi_{01}(s,\alpha) \Phi^0(s+\alpha, s_1) f_0(s_1) \right)$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \int_{t}^{T} ds_1 \int_{s_1-\alpha}^{\{\min(T, t+\theta+a)\}} ds \left(\Pi_{01}(s, \theta+t-s) h(\theta), R(s) \Pi_{01}(s,\alpha) \Phi^0(s+\alpha, s_1) f_0(s_1) \right)$$

$$+ \sum_{i=1}^{N} \int_{\theta_i}^{0} d\theta \int_{t}^{T} ds_1 \int_{\max(s_1,\, \theta-\theta_i+t)}^{\{T\}} ds \left(\Phi^0(s, t+\theta-\theta_i) A_i(t+\theta-\theta_i) h(\theta), Q(s) \Phi^0(s, s_1) f_0(s_1) \right)$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{\theta} d\delta \int_{t}^{T} ds_1 \int_{\max(s_1,\, \theta-\delta+t)}^{\{T\}} ds \left(\Phi^0(s, t+\theta-\delta) D_{01}(t+\theta-\delta, \delta) h(\theta), Q(s) \Phi^0(s, s_1) f_0(s_1) \right) \qquad (A.3-3)$$

Define $g_1^i(t,\theta)$ by:

$g_1^i(t,\theta) = 0$ for $\theta < \theta_i$

and for $\theta \geq \theta_i$,

$g_1^i(t,\theta)$

$$= \int_{\theta-\theta_i+t}^{\{T\}} ds \; A_i^*(t+\theta-\theta_i) \Phi^{0*}(T, t+\theta-\theta_i) F \Phi^0(T, s) f_0(s)$$

$$+ \int_{\theta-\theta_i+t}^{\{T\}} ds \; A_i^*(t+\theta-\theta_i) \Phi^{0*}(s, t+\theta-\theta_i) \Pi_{00}^*(s) R(s) g_0(s)$$

$$+ \int_{-a}^{0} d\eta \int_{\theta-\theta_i+t-\eta}^{\{T\}} ds \; A_i^*(t+\theta-\theta_i) \Phi^{0*}(s+\eta, t+\theta-\theta_i) \Pi_{01}^*(s,\eta) R(s) g_0(s)$$

$$+ \int_{t}^{T} ds_1 \int_{\max(s_1,\, \theta-\theta_i+t)}^{\{T\}} ds \; A_i^*(t+\theta-\theta_i) \Phi^{0*}(s, t+\theta-\theta_i) \Pi_{00}^*(s) R(s) \Pi_{00}(s) \Phi^0(s, s_1) f_0(s_1)$$

$$+ \int_{-a}^{0} d\alpha \int_{t}^{T} ds_1 \int_{\max(s_1-\alpha,\, \theta-\theta_i+t)}^{\{T\}} ds \; A_i^*(t+\theta-\theta_i) \Phi^{0*}(s, t+\theta-\theta_i) \Pi_{00}^*(s) R(s) \Pi_{01}(s,\alpha) \Phi^0(s+\alpha, s_1) f_0(s_1)$$

$$+ \int_{-a}^{0} d\eta \int_{t}^{T} ds_1 \int_{\max(s_1,\, \theta-\theta_i+t-\eta)}^{\{T\}} ds \; A_i^*(t+\theta-\theta_i) \Phi^{0*}(s+\eta, t+\theta-\theta_i) \Pi_{01}^*(s,\eta) R(s) \Pi_{00}(s) \Phi^0(s, s_1) f_0(s_1)$$

$$+ \int_{-a}^{0} d\eta \int_{-a}^{0} d\alpha \int_{t}^{T} ds_1 \int_{\max(s_1-\alpha,\,\theta-\theta_i+t-\eta)}^{\{T\}} ds \; A_i^*(t+\theta-\theta_i)\, \Phi^{0*}(s, t+\theta-\theta_i)\, \Pi_{01}^*(s,\eta)\, R(s)\, \Pi_{01}(s,\alpha)\, \Phi^{0}(s+\alpha, s_1)\, f_0(s_1)$$

$$+ \int_{t}^{T} ds_1 \int_{\max(s_1,\,\theta-\theta_i+t)}^{\{T\}} ds \; A_i^*(t+\theta-\theta_i)\, \Phi^{0*}(s, t+\theta-\theta_i)\; Q(s)\; \Phi^{0}(s, s_1)\; f_0(s_1)$$

Since (A.3-3) holds for all $h' \in L^2(-a,0;R^n)$, we have

$$g_1(t,\theta) = \sum_{i=1}^{N} g_1^{i}(t,\theta)$$

$$+ \int_{-a}^{\theta} d\delta \int_{\theta-\delta+t}^{\{T\}} ds \; D_{01}^*(t+\theta-\delta, \delta)\, \Phi^{0*}(T, t+\theta-\delta)\, F\, \Phi^{0}(T, s)\, f_0(s)$$

$$+ \int_{-a}^{\theta} d\delta \int_{\theta-\delta+t}^{\{T\}} ds \; D_{01}^*(t+\theta-\delta, \delta)\, \Phi^{0*}(T, t+\theta-\delta)\, \Pi_{00}^*(s)\, R(s)\, g_0(s)$$

$$+ \int_{-a}^{\theta} d\delta \int_{-a}^{0} d\eta \int_{\theta-\delta-\eta+t}^{\{T\}} ds \; D_{01}^*(t+\theta-\delta, \delta)\, \Phi^{0*}(s+\eta, t+\theta-\delta)\, \Pi_{01}^*(s,\eta)\, R(s)\, g_0(s)$$

$$+ \int_{t}^{\{\min(T,\,t+\theta+a)\}} ds \;\; \Pi_{01}^*(s, \theta+t-s)\, R(s)\, g_0(s)$$

$$+ \int_{-a}^{\theta} d\delta \int_{t}^{T} ds_1 \int_{\max(s_1,\,\theta-\delta+t)}^{\{T\}} ds \; D_{01}^*(t+\theta-\delta, \delta)\, \Phi^{0*}(s, t+\theta-\delta)\, \Pi_{00}^*(s)\, R(s)\, \Pi_{00}(s)\, \Phi^{0}(s, s_1)\, f_0(s_1)$$

$$+ \int_{-a}^{\theta} d\delta \int_{-a}^{0} d\alpha \int_{t}^{T} ds_1 \int_{\max(s_1-\alpha,\,\theta-\delta+t)}^{\{T\}} ds \; D_{01}^*(t+\theta-\delta, \delta)\, \Phi^{0*}(s, t+\theta-\delta)\, \Pi_{00}^*(s)\, R(s)\, \Pi_{01}(s,\alpha)\, \Phi^{0}(s+\alpha, s_1)\, f_0(s_1)$$

$$+ \int_{-a}^{\theta} d\delta \int_{-a}^{0} d\eta \int_{t}^{T} ds_1 \int_{\max(s_1,\,\theta-\delta+\eta+t)}^{\{T\}} ds \; D_{01}^*(t+\theta-\delta, \delta)\, \Phi^{0*}(s+\eta, t+\theta-\delta)\, \Pi_{01}^*(s,\eta)\, R(s)\, \Pi_{00}(s)\, \Phi^{0}(s, s_1)\, f_0(s_1)$$

$$+ \int_{t}^{T} ds_1 \int_{s_1}^{\{\min(T,\,t+\theta+a)\}} ds \;\; \Pi_{01}^*(s, \theta+t-s)\, R(s)\, \Pi_{00}(s)\; \Phi^{0}(s, s_1)\, f_0(s_1)$$

$$+ \int_{-a}^{0} d\alpha \int_{t}^{T} ds_1 \int_{s_1-\alpha}^{\{\min(T,\,t+\theta+a)\}} ds \;\; \Pi_{01}^*(s, \theta+t-s)\, R(s)\, \Pi_{01}(s,\alpha)\; \Phi^{0}(s+\alpha, s_1)\; f_0(s_1)$$

$$+ \int_{-a}^{\theta} d\delta \int_{-a}^{0} d\eta \int_{-a}^{0} d\alpha \int_{t}^{T} ds_1 \int_{\max(s_1-\alpha,\,\theta-\delta+t-\eta)}^{\{T\}} ds \; D_{01}^*(t+\theta-\delta, \delta)\, \Phi^{0*}(s+\eta, t+\theta-\delta)\, \Pi_{01}^*(s,\eta)\, R(s)\, \Pi_{01}(s,\alpha)\, \Phi^{0}(s+\alpha, s_1)\, f_0(s_1)$$

$$+ \int_{-a}^{\theta} d\delta \int_{t}^{T} ds_1 \int_{\max(s_1,\,\theta-\delta+t)}^{\{T\}} ds \; D_{01}^*(t+\theta-\delta, \varepsilon)\, \Phi^{0*}(s, t+\theta-\delta)\; Q(s)\; \Phi^{0}(s, s_1)\; f_0(s_1) \qquad \text{(A.3-4)}$$

From (A.3-4), it is clear that $g_1(t,\cdot)$ has jumps at $\theta = \theta_i$ of magnitude

$$g_1^i(t,\theta_i) = A_i(t)\, g_0(t) \qquad (A.3\text{-}5)$$

and that the map

$$\theta \mapsto g_1(t,\theta) \qquad (A.3\text{-}6)$$

for fixed t is piecewise absolutely continuous in the intervals (θ_{i+1}, θ_i) for $i = 0, \dots N-1$

Also from (A.3-4) it follows that for $\theta \in [-a, 0]$, $\theta \neq \theta_i$, $i = 1 \dots N$ that the map

$$t \mapsto g_1(t,\theta) \qquad (A.3\text{-}7)$$

is absolutely continuous.

Appendix 4 - Proof of theorem 3 I

Writing out eqn. (3-132) in full, we obtain

$$\left(\frac{dg_0(t)}{dt}, h(0)\right) + \int_{-a}^{0}\left(\frac{\partial g_1(t,\theta)}{\partial t}, h(\theta)\right)d\theta + (g_0(t), A_{00}(t)h(0)) + \int_{-a}^{0}(g_0(t), A_{01}(t,\theta)h(\theta))d\theta$$

$$+ \sum_{i=1}^{N}(g_0(t), A_i(t)h(\theta_i)) + \int_{-a}^{0}\left(g_1(t,\theta), \frac{dh}{d\theta}\right)d\theta - (g_0(t), R(t)\Pi_{00}(t)h(0))$$

$$- \int_{-a}^{0}(g_1(t,\theta), R(t)\Pi_{01}(t,\theta)h(\theta))d\theta + (\Pi_{00}(t)f(t), h(0)) + \int_{-a}^{0}(\Pi_{01}(t,\theta)f(t), h(\theta))d\theta$$

$$= 0 \qquad (A.4-1)$$

and where $h \in \mathcal{D}(\mathcal{A}(t)) = AC^2(-a,0;R^n)$

Define $h_n \in \mathcal{D}(\mathcal{A})$ for $n \geq n_0 > a/|\theta_1|$ by

$$h_n(\theta) = \begin{cases} h(0)\left(1+\frac{n\theta}{a}\right) & -a/n \leq \theta < 0 \\ 0 & -a \leq \theta < -a/n \end{cases}$$

Substituting into (A.4-1) we have

$$\left(\frac{dg_0(t)}{dt}, h(0)\right) + \int_{-a/n}^{0}\left(\frac{\partial g_1(t,\theta)}{\partial t}, h_n(\theta)\right)d\theta + (g_0(t), A_{00}(t)h(0))$$

$$+ \int_{-a/n}^{0}(g_0(t), A_{01}(t,\theta)h_n(\theta))d\theta + \frac{n}{a}\int_{-a/n}^{0}(g_1(t,\theta), h(0))$$

$$- (g_0(t), R(t)\Pi_{00}(t)h(0)) - \int_{-a/n}^{0}(g_1(t,\theta), R(t)\Pi_{01}(t,\theta)h_n(\theta))d\theta$$

$$+ (\Pi_{00}(t)f(t), h(0)) + \int_{-a/n}^{0}(\Pi_{01}(t,\theta)f(t), h_n(\theta))d\theta$$

$$= 0 \qquad (A.4-2)$$

For any $f \in L^2(-a, 0; R^n)$,

$$\left| \int_{-a/n}^{0} (f(\theta), h_n(\theta))\, d\theta \right| = \left| \int_{-a}^{0} (f(\theta), h_n(\theta))\, d\theta \right|$$

$$\leq \left\{ \int_{-a}^{0} |f(\theta)|^2 d\theta \right\}^{1/2} \left\{ \int_{-a/n}^{0} |h_n(\theta)|^2 d\theta \right\}^{1/2}$$

$$\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

Since $g_1(t, \cdot)$ is (absolutely) continuous on $(\theta_1, 0]$

$$\frac{n}{a} \int_{-a/n}^{0} (g_1(t, \theta), h(0))\, d\theta \rightarrow (g_1(t, 0), h(0)) \quad \text{as} \quad n \rightarrow \infty$$

Letting $n \rightarrow \infty$ in (A.4-2) we obtain

$$\left(\frac{d g_0(t)}{dt}, h(0)\right) + (g_1(t, 0), h(0)) + (g_0(t), A_{00}(t)\, h(0))$$

$$- (g_0(t), R(t)\, \Pi_{00}(t)\, h(0)) + (\Pi_{00}(t)\, f(t), h(0)) \qquad \text{(A.4-3)}$$

$$= 0$$

and since (A.4-3) holds for all $h(0) \in R^n$, we obtain equation (3-133).

Combining (A.4-1) and (A.4-3) we have

$$\int_{-a}^{0} \left(\frac{\partial g_1(t, \theta)}{\partial t}, h(\theta)\right) d\theta + \int_{-a}^{0} (A_{01}^*(t, \theta)\, g_0(t), h(\theta))\, d\theta$$

$$+ \sum_{i=1}^{N} (g_0(t), A_i(t)\, h(\theta_i)) + \int_{-a}^{0} \left(g_1(t, \theta), \frac{dh}{d\theta}\right) d\theta - (g_1(t, 0), h(0))$$

$$- \int_{-a}^{0} (\Pi_{01}^*(t, \theta)\, R(t)\, g_1(t, \theta), h(\theta))\, d\theta + \int_{-a}^{0} (\Pi_{01}(t, \theta)\, f(t), h(\theta))\, d\theta = 0 \qquad \text{(A.4-4)}$$

Now chose h absolutely continuous and such that $\operatorname{supp} h \subset (\theta_{i+1}, \theta_i)$

Since $\theta \to g_1(t,\theta) \; : \; (\theta_{i+1}, \theta_i) \to R^n$

is absolutely continuous, we can integrate by parts in equation (A.4-4) to obtain

$$\int_{\theta_{i+1}}^{\theta_i} \left(\frac{\partial g_1(t,\theta)}{\partial t}, h(\theta)\right) d\theta + \int_{\theta_{i+1}}^{\theta_i} (A_{01}^*(t,\theta)\, g_1(t,\theta), h(\theta))\, d\theta$$

$$- \int_{\theta_{i+1}}^{\theta_i} \left(\frac{\partial g_1(t,\theta)}{\partial \theta}, h(\theta)\right) d\theta - \int_{\theta_{i+1}}^{\theta_i} (\Pi_{01}^*(t,\theta) R(t)\, g_1(t,\theta), h(\theta))\, d\theta$$

$$+ \int_{\theta_{i+1}}^{\theta_i} (\Pi_{01}(t,\theta) f(t), h(\theta))\, d\theta$$

$$= 0 \qquad \text{(A.4-5)}$$

Now equation (3-134) follows from (A.4-5) from the density of the absolutely continuous maps in $L^2(\theta_{i+1}, \theta_i; R^n)$ (and with support in (θ_{i+1}, θ_i))

Now define h_n for $n \geq n_0 > a/|\theta_N - \theta_{N-1}|$ by

$$h_n(\theta) = \begin{cases} h(-a)\left\{-\frac{n\theta}{a} + 1 - n\right\} & -a \leq \theta \leq -a(1-\frac{1}{n}) \\ 0 & -a(1-\frac{1}{n}) < \theta \leq 0 \end{cases}$$

Substituting into (A.4-1) we have

$$\int_{-a}^{-a(1-\frac{1}{n})} d\theta \left(\frac{\partial g_1(t,\theta)}{\partial t}, h_n(\theta)\right) + \int_{-a}^{-a(1-\frac{1}{n})} d\theta\, (g_0(t), A_{01}(t,\theta)\, h_n(\theta))$$

$$+ (g_0(t), A_N(t) h(-a)) - \frac{n}{a} \int_{-a}^{-a(1-\frac{1}{n})} d\theta\, (g_1(t,\theta), h(-a)) + \int_{-a}^{-a(1-\frac{1}{n})} d\theta\, (\Pi_{01}(t,\theta) f(t), h_n(\theta))$$

$$- \int_{-a}^{-a(1-\frac{1}{n})} d\theta\, (g_1(t,\theta), R(t)\Pi_{01}(t,\theta)\, h_n(\theta)) = 0 \qquad \text{(A.4-6)}$$

Taking limit $n \to \infty$, we have

$$(A_N^*(t) g_0(t), h(-a)) = (g_1(t,-a), h(-a)) \qquad (A.4\text{-}7)$$

and since this holds for all $h(-a) \in R^n$ it follows that

$$g_1(t,-a) = A_N^*(t) g_0(t)$$

It follows from the definition of $\tilde{g}(t)$ as well as expressions for $g_0(t)$ and $g_1(t,\theta)$ that $g_0(T) = 0$ and $g_1(T,\theta) = 0$ a.e $\theta \in [-a, 0]$.

Appendix 5 - Proof of theorem 4E

We have

$$(h, \Pi k)_{M^2} = \int_0^\infty \{(\Phi(t)h, Q\Phi(t)k)_{M^2} + (\Pi\, \Phi(t)\, h, \mathcal{R}\Pi\Phi(t)k)_{M^2}\}dt \qquad (A.5-1)$$

where $\Phi(t)$ is a one parameter semigroup of operators with infinitesimal generator $\mathcal{A} - R\Pi$

Writing equation (A.5-1) in full, we obtain

$$(h(0), \Pi_{00}\, k(0)) + \int_{-a}^{0}(h(0), \Pi_{01}(\alpha)k(\alpha))d\alpha \;+ \int_{-a}^{0}(h(\theta), \Pi_{10}(\theta)\, k(0))d\theta$$

$$+ \;\; (h', \Pi_{11}\, k')_{L^2}$$

$$= \int_0^\infty dt\{(\Phi^0(t)\, h(0), Q\, \Phi^0(t)\, k(0)) \;+ (\Pi_{00}\, \Phi^0(t) h(0), R\,\Pi_{00}\, \Phi^0(t) k(0))\}$$

$$+ \int_0^\infty dt \int_{-a}^{0} d\eta \;\; (\Pi_{01}(\eta)\, \Phi^0(t+\eta)\, h(0), R\,\Pi_{00}\; \Phi^0(t)\, k(0))$$

$$+ \int_0^\infty dt \; \int_{-a}^{0} d\beta \;\; (\Pi_{00}\; \Phi^0(t)\; h(0), \; R\,\Pi_{01}(\beta)\; \Phi^0(t+\beta)\, k(0))$$

$$+ \int_0^\infty dt \; \int_{-a}^{0} d\eta \int_{-a}^{0} d\beta \;\; (\Pi_{01}(\eta)\; \Phi^0(t+\eta) h(0), \; R\,\Pi_{01}(\beta)\, \Phi^0(t+\beta)\; k(0))$$

$$+ \int_0^\infty dt \int_{-a}^{0} d\alpha\, (\Phi^0(t)\; h(0), \; Q\; \Phi'(t,\alpha)\;\; k(\alpha))$$

$$+ \int_0^\infty dt \int_{-a}^{0} d\alpha\, (\Pi_{00}\, \Phi^0(t)\;\, h(0), R\,\Pi_{00}\, \Phi'(t,\alpha)\; k(\alpha))$$

$$+ \int_0^\infty dt \int_{-a}^{0} d\beta \int_{-a}^{0} d\alpha \;\; (\Pi_{00}\; \Phi^0(t) h(0), R\,\Pi_{01}(\beta)\;\; \Phi'(t+\beta,\alpha)\; k(\alpha))$$

$$+ \int_0^\infty dt \int_{-a}^{0} d\eta \; \int_{-a}^{0} d\alpha\, (\Pi_{01}(\eta)\;\, \Phi^0(t+\eta)\, h(0), R\,\Pi_{00}\;\; \Phi'(t,\alpha)\;\; k(\alpha))$$

$$+ \int_0^\infty dt \int_{-a}^0 d\eta \int_{-a}^0 d\beta \int_{-a}^0 d\alpha \left(\Pi_{01}(\eta)\, \Phi^0(t+\eta)\, h(0),\ R\Pi_{01}(\beta)\, \Phi'(t+\beta,\alpha)\, k(\alpha)\right)$$

$$+ \int_0^\infty dt \int_{-a}^0 d\theta \left(\Pi_{00}\, \Phi'(t,\theta)\, h(\theta),\ R\Pi_{00}\, \Phi^0(t)\, k(0)\right)$$

$$+ \int_0^\infty dt \int_{-a}^0 d\theta \left(\Phi'(t,\theta)\, h(\theta),\ Q\, \Phi^0(t)\, k(0)\right)$$

$$+ \int_{-a}^0 dt \int_{-a}^0 d\eta \int_{-a}^0 d\theta \left(\Pi_{01}(\eta)\, \Phi'(t+\eta,\theta)\, h(\theta),\ R\Pi_{00}\, \Phi^0(t)\, k(0)\right)$$

$$+ \int_0^\infty dt \int_{-a}^0 d\theta \int_{-a}^0 d\beta \left(\Pi_{00}\, \Phi'(t,\theta)\, h(\theta),\ R\,\Pi_{01}(\beta)\, \Phi^0(t+\beta)\, k(0)\right)$$

$$+ \int_0^\infty dt \int_{-a}^0 d\eta \int_{-a}^0 d\theta \int_{-a}^0 d\beta \left(\Pi_{01}(\eta)\, \Phi'(t+\eta,\theta)\, h(\theta),\ R\,\Pi_{01}(\beta)\, \Phi^0(t+\beta)\, k(0)\right)$$

$$+ \int_0^\infty dt \int_{-a}^0 d\theta \int_{-a}^0 d\alpha \left(\Phi'(t,\theta)\, h(\theta),\ Q\, \Phi'(t,\alpha)\, k(\alpha)\right)$$

$$+ \int_0^\infty dt \int_{-a}^0 d\theta \int_{-a}^0 d\alpha \left(\Pi_{00}\, \Phi'(t,\theta)\, h(\theta),\ R\Pi_{00}\, \Phi'(t,\alpha)\, k(\alpha)\right)$$

$$+ \int_0^\infty dt \int_{-a}^0 d\theta \int_{-a}^0 d\beta \int_{-a}^0 d\alpha \left(\Pi_{00}\, \Phi'(t,\theta)\, h(\theta),\ R\,\Pi_{01}(\beta)\, \Phi'(t+\beta,\alpha)\, k(\alpha)\right)$$

$$+ \int_0^\infty dt \int_{-a}^0 d\eta \int_{-a}^0 d\theta \int_{-a}^0 d\alpha \left(\Pi_{01}(\eta)\, \Phi'(t+\eta,\theta)\, h(\theta),\ R\Pi_{00}\, \Phi'(t,\alpha)\, k(\alpha)\right)$$

$$+ \int_0^\infty dt \int_{-a}^0 d\eta \int_{-a}^0 d\theta \int_{-a}^0 d\beta \int_{-a}^0 d\alpha \left(\Pi_{01}(\eta)\, \Phi'(t+\eta,\theta)\, h(\theta),\ R\Pi_{01}(\beta)\, \Phi'(t+\beta,\alpha)\, k(\alpha)\right) \tag{A.5-2}$$

Putting $h' = k' = 0$ in (A.5-2) we have

$$(h(0), \Pi_{00}\, k(0)) = \int_0^\infty dt \left(h(0),\ \Phi^{0*}(t)\left[Q + \Pi_{00} R \Pi_{00}\right] \Phi^0(t)\, k(0)\right)$$

$$+ \int_0^\infty dt \int_{\max(-a,-t)}^0 d\eta \left(h(0),\ \Phi^{0*}(t+\eta)\, \Pi_{01}^*(\eta)\, R\, \Pi_{00}\, \Phi^0(t)\, k(0)\right)$$

$$+ \int_0^\infty dt \int_{\max(-a,-t)}^0 d\beta \left(h(0),\ \Phi^{0*}(t)\, \Pi_{00}^*\, R\, \Pi_{01}(\beta)\, \Phi^0(t+\beta)\, k(0)\right)$$

$$+ \int_0^\infty dt \int_{\max(-a,-t)}^0 d\eta \int_{\max(-a,-t)}^0 d\beta \left(h(0),\ \Phi^{0*}(t+\eta)\,\Pi_{01}^*(\eta)\,R\,\Pi_{01}(\beta)\,\Phi^0(t+\beta)\,k(0)\right)$$

$$= \int_0^\infty dt \left(h(0),\ \Phi^{0*}(t)\,[Q + \Pi_{00}^* R\, \Pi_{00}]\,\Phi^0(t)\,k(0)\right)$$

$$+ \int_{-a}^0 d\eta \int_{-\eta}^\infty dt \left(h(0),\ \Phi^{0*}(t+\eta)\,\Pi_{01}^*(\eta)\,R\,\Pi_{00}\,\Phi^0(t)\,k(0)\right)$$

$$+ \int_{-a}^0 d\beta \int_{-\beta}^\infty dt \left(h(0),\ \Phi^{0*}(t)\,\Pi_{00}^*\,R\,\Pi_{01}(\beta)\,\Phi^0(t+\beta)\,k(0)\right)$$

$$+ \int_{-a}^0 d\eta \int_{-a}^0 d\beta \int_{\max(-\eta,-\beta)}^\infty dt \left(h(0),\ \Phi^{0*}(t+\eta)\,\Pi_{01}^*(\eta)\,R\,\Pi_{01}(\beta)\,\Phi^0(t+\beta)\,k(0)\right)$$

(A.5-3)

interchanging order of integration by Fubini.

Since (A.5-3) holds for all $h(0), k(0) \in R^n$, we have

$$\Pi_{00} = \int_0^\infty dt\ \Phi^{0*}(t)[Q + \Pi_{00}^* R\, \Pi_{00}]\,\Phi^0(t)$$

$$+ \int_{-a}^0 d\eta \int_{-\eta}^\infty dt\ \Phi^{0*}(t+\eta)\,\Pi_{01}^*(\eta)\,R\,\Pi_{00}\,\Phi^0(t)$$

$$+ \int_{-a}^0 d\beta \int_{-\beta}^\infty dt\ \Phi^{0*}(t)\,\Pi_{00}^*\,R\,\Pi_{01}(\beta)\,\Phi^0(t+\beta)$$

$$+ \int_{-a}^0 d\eta \int_{-a}^0 d\beta \int_{\max(-\eta,-\beta)}^\infty dt\ \Phi^{0*}(t+\eta)\,\Pi_{01}^*(\eta)\,R\,\Pi_{01}(\beta)\,\Phi^0(t+\beta)$$

(A.5-4)

Putting $h^1 = 0$, $k(0) = 0$ in (A.5-2), we have

$$\int_{-a}^0 (h(0), \Pi_{01}(\alpha)\,k(\alpha))\,d\alpha = \int_0^\infty dt \int_{-a}^0 d\alpha \left(\Phi^0(t)\,h(0),\ Q\,\Phi^1(t,\alpha)\,k(\alpha)\right)$$

$$+ \int_0^\infty dt \int_{-a}^0 d\theta \left(\Pi_{00}\,\Phi^0(t)\,h(0),\ R\,\Pi_{00}\,\Phi^1(t,\alpha)\,k(\alpha)\right)$$

$$+ \int_0^\infty dt \int_{-a}^0 d\beta \int_{-a}^0 d\alpha \, (\Pi_{00} \Phi^0(t) h(0), R \Pi_{01}(\beta) \Phi'(t+\beta, \alpha) k(\alpha))$$

$$+ \int_0^\infty dt \int_{-a}^0 d\eta \int_{-a}^0 d\alpha \, (\Pi_{01}(\eta) \Phi^0(t+\eta) h(0), R \Pi_{00} \Phi'(t, \alpha) k(\alpha))$$

$$+ \int_0^\infty dt \int_{-a}^0 d\eta \int_{-a}^0 d\beta \int_{-a}^0 d\alpha \, (\Pi_{01}(\eta) \Phi^0(t+\eta) h(0), R \Pi_{01}(\beta) \Phi'(t+\beta, \alpha) k(\alpha))$$

$$= \sum_{i=1}^N \int_0^\infty dt \int_{\theta_i}^{\min(0, \theta_i + t)} d\alpha \, (h(0), \Phi^{0*}(t) [Q + \Pi_{00}^* R \Pi_{00}] \Phi^0(t - \alpha + \theta_i) A_i k(\alpha))$$

$$+ \int_0^\infty dt \int_{-a}^0 d\alpha \int_{\max(-a, \alpha - t)}^{\alpha} dY \, (h(0), \Phi^{0*}(t) [Q + \Pi_{00}^* R \Pi_{00}] \Phi^0(t - \alpha + Y) A_{01}(Y) k(\alpha))$$

$$+ \sum_{i=1}^N \int_0^\infty dt \int_{\max(-a, -t)}^0 d\beta \int_{\theta_i}^{\min(0, \theta_i + t + \beta)} d\alpha \, (h(0), \Phi^{0*}(t) \Pi_{00}^* R \Pi_{01}(\beta) \Phi^0(t + \beta - \alpha + \theta_i) A_i k(\alpha))$$

$$+ \int_0^\infty dt \int_{\max(-a, -t)}^0 d\beta \int_{-a}^0 d\alpha \int_{\max(-a, \alpha - t - \beta)}^{\alpha} dY \, (h(0), \Phi^{0*}(t) \Pi_{00}^* R \Pi_{01}(\beta) \Phi^0(t + \beta - \alpha + Y) A_{01}(Y) k(\alpha))$$

$$+ \int_0^\infty dt \int_{-a}^{\max(-a, -t)} d\beta \, (h(0), \Phi^{0*}(t) \Pi_{00}^* R \Pi_{01}(\beta) k(t + \beta))$$

$$+ \sum_{i=1}^N \int_0^\infty dt \int_{\max(-a, -t)}^0 d\eta \int_{\theta_i}^{\min(0, \theta_i + t)} d\alpha \, (h(0), \Phi^{0*}(t+\eta) \Pi_{01}^*(\eta) R \Pi_{00} \Phi^0(t - \alpha + \theta_i) A_i k(\alpha))$$

$$+ \int_0^\infty dt \int_{\max(-a, -t)}^0 d\eta \int_{-a}^0 d\alpha \int_{\max(-a, \alpha - t)}^{\alpha} dY \, (h(0), \Phi^{0*}(t+\eta) \Pi_{01}^*(\eta) R \Pi_{00} \Phi^0(t - \alpha + Y) A_{01}(Y) k(\alpha))$$

$$+ \sum_{i=1}^N \int_0^\infty dt \int_{\max(-a, -t)}^0 d\eta \int_{\max(-a, -t)}^0 d\beta \int_{\theta_i}^{\min(0, \theta_i + t + \beta)} d\alpha \, (h(0), \Phi^{0*}(t+\eta) \Pi_{01}^*(\eta) R \Pi_{01}(\beta) \Phi^0(t + \beta - \alpha + \theta_i) A_i k(\alpha)$$

$$+ \int_0^\infty dt \int_{\max(-a, -t)}^0 d\eta \int_{\max(-a, -t)}^0 d\beta \int_{-a}^0 d\alpha \int_{\max(-a, \alpha - t - \beta)}^{\alpha} dY \, (h(0), \Phi^{0*}(t+\eta) \Pi_{01}^*(\eta) R \Pi_{01}(\beta) \Phi^0(t + \beta - \alpha + Y) A_{01}(Y) k(\alpha)$$

$$+ \int_0^\infty dt \int_{\max(-a, -t)}^0 d\eta \int_{-a}^{\max(-a, -t)} d\beta \, (h(0), \Phi^{0*}(t+\eta) \Pi_{01}^*(\eta) R \Pi_{01}(\beta) k(t + \beta))$$

$$= \sum_{i=1}^{N} \int_{\theta_i}^{0} d\alpha \int_{\alpha-\theta_i}^{\infty} dt \, \left(h(0), \, \Phi^{0*}(t)[Q + \Pi_{00}^{*} R \Pi_{00}] \Phi^{0}(t-\alpha+\theta_i) A_i \, k(\alpha)\right)$$

$$+ \sum_{i=1}^{N} \int_{\theta_i}^{0} d\alpha \int_{-a}^{0} d\beta \int_{\alpha-\beta-\theta_i}^{\infty} dt \, \left(h(0), \, \Phi^{0*}(t) \Pi_{00}^{*} R \Pi_{01}(\beta) \Phi^{0}(t+\beta-\alpha+\theta_i) A_i \, k(\alpha)\right)$$

$$+ \sum_{i=1}^{N} \int_{\theta_i}^{0} d\alpha \int_{-a}^{0} d\eta \int_{\max(-\eta,\alpha-\theta_i)}^{\infty} dt \, \left(h(0), \, \Phi^{0*}(t+\eta) \Pi_{01}^{*}(\eta) R \Pi_{00} \Phi^{0}(t-\alpha+\theta_i) A_i \, k(\alpha)\right)$$

$$+ \sum_{i=1}^{N} \int_{\theta_i}^{0} d\alpha \int_{-a}^{0} d\eta \int_{-a}^{0} d\beta \int_{\max(-\eta,\alpha-\beta-\theta_i)}^{\infty} dt \, \left(h(0), \, \Phi^{0*}(t+\eta) \Pi_{01}^{*}(\eta) R \Pi_{01}(\beta) \Phi^{0}(t+\beta-\alpha+\theta_i) A_i \, k(\alpha)\right)$$

$$+ \int_{-a}^{0} d\alpha \int_{-a}^{\alpha} dY \int_{\alpha-Y}^{\infty} dt \, \left(h(0), \, \Phi^{0*}(t)[Q + \Pi_{00}^{*} R \Pi_{00}] \Phi^{0}(t-\alpha+Y) A_{\alpha}(Y) \, k(\alpha)\right)$$

$$+ \int_{-a}^{0} d\alpha \int_{-a}^{\alpha} dY \int_{-a}^{0} d\eta \int_{\max(-\eta,\alpha-Y)}^{\infty} dt \, \left(h(0), \, \Phi^{0*}(t+\eta) \Pi_{01}^{*}(\eta) R \Pi_{00} \Phi^{0}(t-\alpha+Y) A_{01}(Y) \, k(\alpha)\right)$$

$$+ \int_{-a}^{0} d\alpha \int_{-a}^{\alpha} dY \int_{-a}^{0} d\beta \int_{\alpha-Y-\beta}^{\infty} dt \, \left(h(0), \, \Phi^{0*}(t) \Pi_{00}^{*} R \Pi_{01}(\beta) \Phi^{0}(t+\beta-\alpha+Y) A_{01}(Y) \, k(\alpha)\right)$$

$$+ \int_{-a}^{0} d\alpha \int_{-a}^{\alpha} dY \int_{-a}^{0} d\eta \int_{-a}^{0} d\beta \int_{\max(-\eta,\alpha-Y-\beta)}^{\infty} dt \, \left(h(0), \, \Phi^{0*}(t+\eta) \Pi_{01}^{*}(\eta) R \Pi_{01}(\beta) \Phi^{0}(t+\beta-\alpha+Y) A_{01}(Y) \, k(\alpha)\right)$$

$$+ \int_{-a}^{0} d\alpha \int_{0}^{\alpha+a} dt \, \left(h(0), \, \Phi^{0*}(t) \Pi_{00}^{*} R \Pi_{01}(\alpha-t) \, k(\alpha)\right)$$

$$+ \int_{-a}^{0} d\alpha \int_{0}^{\alpha+a} dt \int_{-t}^{0} d\eta \, \left(h(0), \, \Phi^{0*}(t+\eta) \Pi_{01}^{*}(\eta) R \Pi_{01}(\alpha-t) \, k(\alpha)\right) \qquad (A.5-5)$$

interchanging order of integration

Define $\Pi_{01}^{i}(\alpha)$ for $i = 1 \ldots N-1$ by

$$\Pi_{01}^{i}(\alpha) = 0 \qquad \alpha < \theta_i$$

and for $\alpha \geq \theta_i$

$$\Pi_{01}^{i}(\alpha) = \int_{\alpha-\theta_i}^{\infty} dt \, \Phi^{0*}(t)[Q + \Pi_{00}^{*} R \Pi_{00}] \Phi^{0}(t-\alpha+\theta_i) A_i$$

$$+ \int_{-a}^{0} d\beta \int_{\alpha-\beta-\theta_i}^{\infty} dt \, \Phi^{0*}(t) \Pi_{00}^{*} R \Pi_{01}(\beta) \Phi^{0}(t+\beta-\alpha+\theta_i) A_i$$

$$+\int_{-a}^{0} d\eta \int_{\max(-\eta,\alpha-\theta_i)}^{\infty} dt \; \Phi^{0*}(t+\eta)\Pi_{01}^{*}(\eta) R \Pi_{00} \Phi^{0}(t-\alpha+\theta_i) A_i$$

$$+\int_{-a}^{0} d\eta \int_{-a}^{0} d\beta \int_{\max(-\eta,\alpha-\beta-\theta_i)}^{\infty} \Phi^{0*}(t+\eta)\Pi_{01}^{*}(\eta) R \Pi_{01}(\beta) \Phi^{0}(t+\beta-\alpha+\theta_i) A_i$$

Since (A.5-5) holds for all $h(0) \in R^n$, $k' \in L^2(-a,0;R^n)$ we have

$$\Pi_{01}(\alpha) = \sum_{i=1}^{N} \Pi_{01}^{i}(\alpha)$$

$$+\int_{-a}^{\alpha} dY \int_{\alpha-Y}^{\infty} dt \; \Phi^{0*}(t)[Q+\Pi_{00}^{*} R \Pi_{00}] \Phi^{0}(t-\alpha+Y) A_{01}(Y)$$

$$+\int_{-a}^{\alpha} dY \int_{-a}^{0} d\eta \int_{\max(-\eta,\alpha-Y)}^{\infty} dt \; \Phi^{0*}(t+\eta)\Pi_{01}^{*}(\eta) R \Pi_{00} \Phi^{0}(t-\alpha+Y) A_{01}(Y)$$

$$+\int_{-a}^{\alpha} dY \int_{-a}^{0} d\beta \int_{\alpha-Y-\beta}^{\infty} dt \; \Phi^{0*}(t)\Pi_{00} R \Pi_{01}(\beta) \Phi^{0}(t+\beta-\alpha+Y) A_{01}(Y)$$

$$+\int_{-a}^{\alpha} dY \int_{-a}^{0} d\eta \int_{-a}^{0} d\beta \int_{\max(-\eta,\alpha-Y-\beta)}^{\infty} dt \; \Phi^{0*}(t+\eta)\Pi_{01}^{*}(\eta) R \Pi_{01}(\beta) \Phi^{0}(t+\beta-\alpha+Y) A_{a}(Y)$$

$$+\int_{0}^{\alpha+a} dt \; \Phi^{0*}(t)\Pi_{00}^{*} R \Pi_{01}(\alpha-t)\, k(\alpha)$$

$$+\int_{0}^{\alpha+a} dt \int_{-t}^{0} d\eta \; \Phi^{0*}(t+\eta)\Pi_{01}^{*}(\eta) R \Pi_{01}(\alpha-t) \qquad \text{(A.5-6)}$$

$\Pi_{01}(\alpha)$ has jump at $\alpha=\theta_i$ of magnitude

$$\Pi_{01}^{i}(\theta_i) = \Pi_{00} A_i$$

From (A.5-6), it follows that the map

$$\alpha \longmapsto \Pi_{01}(\alpha)$$

is piecewise absolutely continuous and is absolutely continuous in the intervals

(θ_{i+1}, θ_i) $\quad i = 0 \ldots N-1$

Putting $h(0) = k(0) = 0$ in (A.5-2) we obtain

$$(h', \Pi_{11} k')_{L^2} = \int_0^\infty dt \int_{-a}^0 d\theta \int_{-a}^0 d\alpha \, (\Phi'(t,\theta) h(\theta), Q \Phi'(t,\alpha) k(\alpha))$$

$$+ \int_0^\infty dt \int_{-a}^0 d\theta \int_{-a}^0 d\alpha \, (\Pi_{00} \Phi'(t,\theta) h(\theta), R \Pi_{00} \Phi'(t,\alpha) k(\alpha))$$

$$+ \int_0^\infty dt \int_{-a}^0 d\theta \int_{-a}^0 d\beta \int_{-a}^0 d\alpha \, (\Pi_{00} \Phi'(t,\theta) h(\theta), R \Pi_{01}(\beta) \Phi'(t+\beta,\alpha) k(\alpha))$$

$$+ \int_0^\infty dt \int_{-a}^0 d\eta \int_{-a}^0 d\theta \int_{-a}^0 d\alpha \, (\Pi_{01}(\eta) \Phi'(t+\eta,\theta) h(\theta), R \Pi_{00} \Phi'(t,\alpha) k(\alpha))$$

$$+ \int_0^\infty dt \int_{-a}^0 d\gamma \int_{-a}^0 d\theta \int_{-a}^0 d\beta \int_{-a}^0 d\alpha \, (\Pi_{01}(\gamma) \Phi'(t+\eta,\theta) h(\theta), R \Pi_{01}(\beta) \Phi'(t+\beta,\alpha) k(\alpha))$$

$$= \sum_{i=1}^N \sum_{j=1}^N \int_0^\infty dt \int_{\theta_i}^{\min(0,\theta_i+t)} d\theta \int_{\theta_j}^{\min(0,\theta_j+t)} d\alpha \, (\Phi^0(t-\theta+\theta_i) A_i h(\theta), [Q + \Pi_{00}^* R \Pi_{00}] \Phi^0(t-\alpha+\theta_j) A_j k(\alpha))$$

$$+ \sum_{i=1}^N \int_0^\infty dt \int_{\theta_i}^{\min(0,\theta_i+t)} d\theta \int_{-a}^0 d\alpha \int_{\max(-a,\alpha-t)}^{\alpha} d\gamma \, (\Phi^0(t-\theta+\theta_i) A_i h(\theta), [Q + \Pi_{00}^* R \Pi_{00}] \Phi^0(t-\alpha+\gamma) A_{01}(\gamma) k(\alpha))$$

$$+ \sum_{j=1}^N \int_0^\infty dt \int_{-a}^0 d\theta \int_{\max(-a,\theta-t)}^{\theta} d\delta \int_{\theta_i}^{\min(0,\theta_j+t)} d\alpha \, (\Phi^0(t-\theta+\delta) A_{01}(\delta) h(\theta), [Q + \Pi_{00}^* R \Pi_{00}] \Phi^0(t-\alpha+\theta_j) A_j k(\alpha))$$

$$+ \int_0^\infty dt \int_{-a}^0 d\theta \int_{\max(-a,\theta-t)}^{\theta} d\delta \int_{-a}^0 d\alpha \int_{\max(-a,\alpha-t)}^{\alpha} d\gamma \, (\Phi^0(t-\theta+\delta) A_{01}(\delta) h(\theta), [Q + \Pi_{00}^* R \Pi_{00}] \Phi^0(t-\alpha+\gamma) A_{01}(\gamma) k(\alpha))$$

$$+ \sum_{i=1}^N \sum_{j=1}^N \int_0^\infty dt \int_{\theta_i}^{\min(0,\theta_i+t)} d\theta \int_{\max(-a,-t)}^0 d\beta \int_{\theta_j}^{\min(0,\theta_j+t+\beta)} d\alpha \, (\Phi^0(t-\theta+\theta_i) A_i h(\theta), \Pi_{00}^* R \Pi_{01}(\beta) \Phi^0(t+\beta-\alpha+\theta_j) A_j k(\alpha))$$

$$+ \sum_{i=1}^N \int_0^\infty dt \int_{\theta_i}^{\min(0,\theta_i+t)} d\theta \int_{\max(-a,-t)}^0 d\beta \int_{-a}^0 d\alpha \int_{\max(-a,\alpha-t-\beta)}^{\alpha} d\gamma \, (\Phi^0(t-\theta+\theta_i) A_i h(\theta), \Pi_{00}^* R \Pi_{01}(\beta) \Phi^0(t+\beta-\alpha+\gamma) A_{01}(\gamma) k(\alpha))$$

$$+ \sum_{j=1}^N \int_0^\infty dt \int_{-a}^0 d\theta \int_{\max(-a,\theta-t)}^{\theta} d\delta \int_{\max(-a,-t)}^0 d\beta \int_{\theta_j}^{\min(0,\theta_j+t+\beta)} d\gamma \, (\Phi^0(t-\theta+\delta) A_{01}(\delta) h(\theta), \Pi_{00}^* R \Pi_{01}(\beta) \Phi^0(t+\beta-\alpha+\theta_j) A_j k(\alpha))$$

$$+ \int_0^\infty dt \int_{-a}^0 d\theta \int_{\max(-a,\theta-t)}^{\theta} d\delta \int_{\max(-a,-t)}^0 d\beta \int_{-a}^0 d\alpha \int_{\max(-a,\alpha-t-\beta)}^{\alpha} d\gamma \, (\Phi^0(t-\theta+\delta) A_{01}(\delta) h(\theta), \Pi_{00}^* R \Pi_{01}(\beta) \Phi^0(t+\beta-\alpha+\gamma) A_{01}(\gamma) k(\alpha))$$

$$+\sum_{i=1}^{N}\int_0^\infty dt\int_{\theta_i}^{\min(0,\theta_i+t)}d\theta\int_{-a}^{\max(-a,-t)}d\beta\,\left(\Phi^0(t-\theta+\theta_i)A_i h(\theta),\Pi_{00}^* R\Pi_{01}(\beta)k(t+\beta)\right)$$

$$+\int_0^\infty dt\int_{-a}^{0}d\theta\int_{\max(-a,\theta-t)}^{\theta}d\delta\int_{-a}^{\max(-a,-t)}d\beta\,\left(\Phi^0(t-\theta+\delta)A_{01}(\delta)h(\theta),\Pi_{00}^* R\Pi_{01}(\beta)\,k(t+\beta)\right)$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{N}\int_0^\infty dt\int_{\max(-a,-t)}^{0}d\eta\int_{\theta_i}^{\min(0,\theta_i+t+\eta)}d\theta\int_{\theta_j}^{\min(0,\theta_j+t)}d\alpha\,\left(\Phi^0(t+\eta-\theta+\theta_i)A_i h(\theta),\Pi_{01}^*(\eta)R\Pi_{00}\Phi^0(t-\alpha+\theta_j)A_j k(\alpha)\right)$$

$$+\sum_{i=1}^{N}\int_0^\infty dt\int_{\max(-a,-t)}^{0}d\eta\int_{\theta_i}^{\min(0,\theta_i+t+\eta)}d\theta\int_{-a}^{0}d\alpha\int_{\max(-a,\alpha-t)}^{\alpha}d\gamma\,\left(\Phi^0(t+\eta-\theta+\theta_i)A_i h(\theta),\Pi_{01}^*(\eta)R\Pi_{00}\Phi^0(t-\alpha+\gamma)A_{01}(\gamma)k(\alpha)\right)$$

$$+\sum_{j=1}^{N}\int_0^\infty dt\int_{\max(-a,-t)}^{0}d\eta\int_{-a}^{0}d\theta\int_{\max(-a,\theta-t-\eta)}^{\theta}d\delta\int_{\theta_i}^{\min(0,\theta_j+t)}d\alpha\,\left(\Phi^0(t+\eta-\theta+\delta)A_{01}(\delta)h(\theta),\Pi_{01}^*(\eta)R\Pi_{00}\Phi^0(t-\alpha+\theta_j)A_j\,k(\alpha)\right)$$

$$+\int_0^\infty dt\int_{\max(-a,-t)}^{0}d\eta\int_{-a}^{0}d\theta\int_{\max(-a,\theta-t-\eta)}^{\theta}d\delta\int_{-a}^{0}d\alpha\int_{\max(-a,\alpha-t)}^{\alpha}d\gamma\,\left(\Phi^0(t+\eta-\theta+\delta)A_{01}(\delta)h(\theta),\Pi_{01}^*(\eta)R\Pi_{00}\Phi^0(t-\alpha+\gamma)A_{01}(\gamma)k(\alpha)\right)$$

$$+\sum_{j=1}^{N}\int_0^\infty dt\int_{-a}^{\max(-a,-t)}d\eta\int_{\theta_j}^{\min(0,\theta_j+t)}d\alpha\,\left(h(t+\eta),\Pi_{01}^*(\eta)R\Pi_{00}\Phi^0(t-\alpha+\theta_j)A_j\,k(\alpha)\right)$$

$$+\int_0^\infty dt\int_{-a}^{\max(-a,-t)}d\eta\int_{-a}^{0}d\alpha\int_{\max(-a,\theta-t)}^{\alpha}d\gamma\,\left(h(t+\eta),\Pi_{01}^*(\eta)R\Pi_{00}\Phi^0(t-\alpha+\gamma)A_{01}(\gamma)k(\alpha)\right)$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{N}\int_0^\infty dt\int_{\max(-a,-t)}^{0}d\eta\int_{\theta_i}^{\min(0,\theta_i+t+\eta)}d\theta\int_{\max(-a,-t)}^{0}d\beta\int_{\theta_j}^{\min(0,\theta_j+t+\beta)}d\alpha\,\left(\Phi^0(t+\eta-\theta+\theta_i)A_i h(\theta),\Pi_{01}^*(\eta)R\Pi_{01}(\beta)\Phi^0(t+\beta-\alpha+\theta_j)A_j k(\alpha)\right)$$

$$+\sum_{i=1}^{N}\int_0^\infty dt\int_{\max(-a,-t)}^{0}d\eta\int_{\theta_i}^{\min(0,\theta_i+t+\eta)}d\theta\int_{\max(-a,-t)}^{0}d\beta\int_{-a}^{0}d\alpha\int_{\max(-a,\alpha-t-\beta)}^{\alpha}d\gamma\,\left(\Phi^0(t+\eta-\theta+\theta_i)A_i h(\theta),\Pi_{01}^*(\eta)R\Pi_{01}(\beta)\Phi^0(t+\beta-\alpha+\gamma)A_{01}(\gamma)k(\alpha)\right)$$

$$+\sum_{j=1}^{N}\int_0^\infty dt\int_{\max(-a,-t)}^{0}d\eta\int_{-a}^{0}d\theta\int_{\max(-a,\theta-t-\eta)}^{\theta}d\delta\int_{\max(-a,-t)}^{0}d\beta\int_{\theta_j}^{\min(0,\theta_j+t+\beta)}d\alpha\,\left(\Phi^0(t+\eta-\theta+\delta)A_{01}(\delta)h(\theta),\Pi_{01}^*(\eta)R\Pi_{01}(\beta)\Phi^0(t+\beta-\alpha+\theta_j)A_j k(\alpha)\right)$$

$$+\int_0^\infty dt\int_{\max(-a,-t)}^{0}d\eta\int_{-a}^{0}d\theta\int_{\max(-a,\theta-t-\eta)}^{\theta}d\delta\int_{\max(-a,-t)}^{0}d\beta\int_{-a}^{0}d\alpha\int_{\max(-a,\alpha-t-\beta)}^{\alpha}d\gamma\,\left(\Phi^0(t+\eta-\theta+\delta)A_{01}(\delta)h(\theta),\Pi_{01}^*(\eta)R\Pi_{01}(\beta)\Phi^0(t+\beta-\alpha+\gamma)A_{01}(\gamma)k(\alpha)\right)$$

$$+\sum_{i=1}^{N}\int_0^\infty dt\int_{\max(-a,-t)}^{0}d\eta\int_{\theta_i}^{\min(0,\theta_i+t+\eta)}d\theta\int_{-a}^{\max(-a,-t)}d\beta\,\left(\Phi^0(t+\eta-\theta+\theta_i)A_i h(\theta),\Pi_{01}^*(\eta)R\Pi_{01}(\beta)k(t+\beta)\right)$$

$$+\int_0^\infty dt\int_{\max(-a,-t)}^{0}d\eta\int_{-a}^{0}d\theta\int_{\max(-a,\theta-t-\eta)}^{\theta}d\delta\int_{-a}^{\max(-a,-t)}d\beta\,\left(\Phi^0(t+\eta-\theta+\delta)A_{01}(\delta)h(\theta),\Pi_{01}^*(\eta)R\Pi_{01}(\beta)\,k(t+\beta)\right)$$

$$+\sum_{j=1}^{N}\int_{0}^{\infty}dt\int_{-a}^{\max(-a,-t)}d\eta\int_{\max(-a,-t)}^{0}d\beta\int_{\theta_j}^{\min(0,\theta_j+t+\beta)}d\alpha\,\left(h(t+\eta),\Pi_{01}^{*}(\eta)R\Pi_{01}(\beta)\Phi^{0}(t+\beta-\alpha+\theta_j)A_j\,k(\alpha)\right)$$

$$+\int_{0}^{\infty}dt\int_{-a}^{\max(-a,-t)}d\eta\int_{\max(-a,-t)}^{0}d\beta\int_{-a}^{0}d\alpha\int_{\max(-a,\alpha-t-\beta)}^{\alpha}d\gamma\,\left(h(t+\eta),\Pi_{01}^{*}(\eta)R\Pi_{01}(\beta)\Phi^{0}(t+\beta-\alpha+\gamma)A_{01}(\gamma)k(\alpha)\right)$$

$$+\int_{0}^{\infty}dt\int_{-a}^{\max(-a,-t)}d\eta\int_{-a}^{\max(-a,-t)}d\beta\,\left(h(t+\eta),\Pi_{01}^{*}(\eta)R\Pi_{01}(\beta)k(t+\beta)\right)$$

$$=\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\theta_i}^{0}d\theta\int_{\theta_j}^{0}d\alpha\int_{\max(\theta-\theta_i,\alpha-\theta_j)}^{\infty}dt\,\left(h(\theta),A_i^{*}\Phi^{0*}(t-\theta+\theta_i)[Q+\Pi_{00}^{*}R\Pi_{00}]\Phi^{0}(t-\alpha+\theta_j)A_j\,k(\alpha)\right)$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\theta_i}^{0}d\theta\int_{\theta_j}^{0}d\alpha\int_{-a}^{0}d\beta\int_{\max(\theta-\theta_i,\alpha-\theta_j+\beta)}^{\infty}dt\,\left(h(\theta),A_i^{*}\Phi^{0*}(t-\theta+\theta_i)\Pi_{00}^{*}R\Pi_{01}(\beta)\Phi^{0}(t+\beta-\alpha+\theta_j)A_j\,k(\alpha)\right)$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\theta_i}^{0}d\theta\int_{\theta_j}^{0}d\alpha\int_{-a}^{0}d\eta\int_{\max(\theta-\theta_i-\eta,\alpha-\theta_j)}^{\infty}dt\,\left(h(\theta),A_i^{*}\Phi^{0*}(t+\eta-\theta+\theta_i)\Pi_{01}^{*}(\eta)R\Pi_{00}\Phi^{0}(t-\alpha+\theta_j)A_j\,k(\alpha)\right)$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\theta_i}^{0}d\theta\int_{\theta_j}^{0}d\alpha\int_{-a}^{0}d\eta\int_{-a}^{0}d\beta\int_{\max(\theta-\theta_i-\eta,\alpha-\theta_j-\beta)}^{\infty}dt\,\left(h(\theta),A_i^{*}\Phi^{0*}(t+\eta-\theta+\theta_i)\Pi_{01}^{*}(\eta)R\Pi_{01}(\beta)\Phi^{0}(t+\beta-\alpha+\theta_j)A_j\,k(\alpha)\right)$$

$$+\sum_{i=1}^{N}\int_{\theta_i}^{0}d\theta\int_{-a}^{0}d\alpha\int_{-a}^{\alpha}d\gamma\int_{\max(\theta-\theta_i,\alpha-\gamma)}^{\infty}dt\,\left(h(\theta),A_i^{*}\Phi^{0*}(t-\theta+\theta_i)[Q+\Pi_{00}^{*}R\Pi_{00}]\Phi^{0}(t-\alpha+\gamma)A_{01}(\gamma)k(\alpha)\right)$$

$$+\sum_{i=1}^{N}\int_{\theta_i}^{0}d\theta\int_{-a}^{0}d\alpha\int_{-a}^{0}d\eta\int_{-a}^{\alpha}d\gamma\int_{\max(\theta-\theta_i-\eta,\alpha-\gamma)}^{\infty}dt\,\left(h(\theta),A_i^{*}\Phi^{0*}(t+\eta-\theta+\theta_i)\Pi_{01}^{*}(\eta)R\Pi_{00}\Phi^{0}(t-\alpha+\gamma)A_{01}(\gamma)k(\alpha)\right)$$

$$+\sum_{i=1}^{N}\int_{\theta_i}^{0}d\theta\int_{-a}^{0}d\alpha\int_{-a}^{0}d\beta\int_{-a}^{\alpha}d\gamma\int_{\max(\theta-\theta_i,\alpha-\gamma-\beta)}^{\infty}dt\,\left(h(\theta),A_i^{*}\Phi^{0*}(t-\theta+\theta_i)\Pi_{00}^{*}R\Pi_{01}(\beta)\Phi^{0}(t+\beta-\alpha+\gamma)A_{01}(\gamma)k(\alpha)\right)$$

$$+\sum_{i=1}^{N}\int_{\theta_i}^{0}d\theta\int_{-a}^{0}d\alpha\int_{-a}^{\alpha}d\gamma\int_{-a}^{0}d\beta\int_{-a}^{0}d\eta\int_{\max(\theta-\theta_i-\eta,\alpha-\gamma-\beta)}^{\infty}dt\,\left(h(\theta),A_i^{*}\Phi^{0*}(t+\eta-\theta+\theta_i)\Pi_{01}^{*}(\eta)R\Pi_{01}(\beta)\Phi^{0}(t+\beta-\alpha+\gamma)A_{01}(\gamma)k(\alpha)\right)$$

$$+\sum_{i=1}^{N}\int_{\theta_i}^{0}d\theta\int_{-a}^{0}d\alpha\int_{0}^{a+\alpha}dt\int_{\theta-\theta_i-\eta}^{0}d\eta\,\left(h(\theta),A_i^{*}\Phi^{0*}(t+\eta-\theta+\theta_i)\Pi_{01}^{*}(\eta)R\Pi_{01}(\alpha-t)k(\alpha)\right)$$

$$+\sum_{i=1}^{N}\int_{\theta_i}^{0}d\theta\int_{-a}^{0}d\alpha\int_{\theta-\theta_i}^{\max(\theta-\theta_i,\alpha+a)}dt\,\left(h(\theta),A_i^{*}\Phi^{0*}(t-\theta+\theta_i)\Pi_{00}^{*}R\Pi_{01}(\alpha-t)\,k(\alpha)\right)$$

$$+\sum_{j=1}^{N}\int_{-a}^{0}d\theta\int_{\theta_j}^{0}d\alpha\int_{-a}^{\theta}d\delta\int_{\max(\theta-\delta,\alpha-\theta_j)}^{\infty}dt\,\left(h(\theta),A_{01}^{*}(\delta)\Phi^{0*}(t-\theta+\delta)[Q+\Pi_{00}^{*}R\Pi_{00}]\Phi^{0}(t-\alpha+\theta_j)A_j\,k(\alpha)\right)$$

$$+\sum_{j=1}^{N}\int_{-a}^{0}d\theta\int_{\theta_j}^{0}d\alpha\int_{-a}^{\theta}d\delta\int_{-a}^{0}d\beta\int_{\max(\theta-\delta,\alpha-\theta_j-\beta)}^{\infty}dt\,\left(h(\theta),A_{01}^{*}(\delta)\Phi^{0}(t-\theta+\delta)\Pi_{00}^{*}R\Pi_{01}(\beta)\Phi^{0}(t+\beta-\alpha+\theta_j)A_j\,k(\alpha)\right)$$

$$+ \sum_{j=1}^{N} \int_{-a}^{0} d\theta \int_{\theta_j}^{0} d\alpha \int_{-a}^{0} d\eta \int_{-a}^{\theta} d\delta \int_{\max(\theta-\delta-\eta,\, \alpha-\theta_j)}^{\infty} dt\, (h(\theta), A_{01}^*(\delta)\, \Phi^{0*}(t+\eta-\theta+\delta) \Pi_{01}^*(\eta) R \Pi_{00} \Phi^{0}(t-\alpha+\theta_j) A_j k(\alpha))$$

$$+ \sum_{j=1}^{N} \int_{-a}^{0} d\theta \int_{\theta_j}^{0} d\alpha \int_{-a}^{0} d\eta \int_{-a}^{0} d\beta \int_{-a}^{\theta} d\delta \int_{\max(\theta-\delta-\eta,\, \alpha-\theta_j-\beta)}^{\infty} dt\, (h(\theta), A_{01}^*(\delta)\, \Phi^{0*}(t+\eta-\theta+\delta) \Pi_{01}^*(\eta) R \Pi_{01}(\beta) \Phi^{0}(t+\beta-\alpha+\theta_j) A_j k(\alpha))$$

$$+ \sum_{j=1}^{N} \int_{-a}^{0} d\theta \int_{\theta_j}^{0} d\alpha \int_{0}^{a+\theta} dt \int_{\alpha-\theta_j-t}^{0} d\beta\, (h(\theta), \Pi_{01}^*(\theta-t) R \Pi_{01}(\beta) \Phi^{0}(t+\beta-\alpha+\theta_j) A_j k(\alpha))$$

$$+ \sum_{j=1}^{N} \int_{-a}^{0} d\theta \int_{\theta_j}^{0} d\alpha \int_{\alpha-\theta_j}^{\max(\alpha-\theta_j,\, \theta+a)} dt\, (h(\theta), \Pi_{01}^*(\theta-t) R \Pi_{00} \Phi^{0}(t-\alpha+\theta_j) A_j k(\alpha))$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \int_{-a}^{\theta} d\delta \int_{-a}^{\alpha} d\gamma \int_{\max(\theta-\delta,\, \alpha-\gamma)}^{\infty} dt\, (h(\theta), A_{01}^*(\delta)\, \Phi^{0*}(t-\theta+\delta)[Q+\Pi_{00}^* R \Pi_{00}] \Phi^{0}(t-\alpha+\gamma) A_{01}(\gamma) k(\alpha))$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \int_{-a}^{0} d\beta \int_{-a}^{\theta} d\delta \int_{-a}^{\alpha} d\gamma \int_{\max(\theta-\delta,\, \alpha-\gamma-\beta)}^{\infty} dt\, (h(\theta), A_{01}^*(\delta)\, \Phi^{0*}(t-\theta+\delta) \Pi_{00}^* R \Pi_{01}(\beta) \Phi^{0}(t+\beta-\alpha+\gamma) A_{01}(\gamma) k(\alpha))$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \int_{-a}^{0} d\eta \int_{-a}^{\theta} d\delta \int_{-a}^{\alpha} d\gamma \int_{\max(\theta-\delta-\eta,\, \alpha-\gamma)}^{\infty} dt\, (h(\theta), A_{01}^*(\delta)\, \Phi^{0*}(t+\eta-\theta+\delta) \Pi_{01}^*(\eta) R \Pi_{00} \Phi^{0}(t-\alpha+\gamma) A_{01}(\gamma) k(\alpha))$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \int_{-a}^{0} d\eta \int_{-a}^{0} d\beta \int_{-a}^{\theta} d\delta \int_{-a}^{\alpha} d\gamma \int_{\max(\theta-\delta-\eta,\, \alpha-\gamma-\beta)}^{\infty} dt\, (h(\theta), A_{01}^*(\delta)\, \Phi^{0}(t+\eta-\theta+\delta) \Pi_{01}^*(\eta) R \Pi_{01}(\beta) \Phi^{0}(t+\beta-\alpha+\gamma) A_{01}(\gamma) k(\alpha))$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \int_{0}^{\min(\theta+a,\, \alpha+a)} dt\, (h(\theta), \Pi_{01}^*(\theta-t) R \Pi_{01}(\alpha-t) k(\alpha))$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \int_{-a}^{\theta} d\delta \int_{\theta-\delta}^{\max(\theta-\delta,\, \alpha+a)} dt\, (h(\theta), A_{01}^*(\delta)\, \Phi^{0*}(t-\theta+\delta) \Pi_{00}^* R \Pi_{01}(\alpha-t) k(\alpha))$$

$$+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \int_{-a}^{\alpha} d\gamma \int_{\alpha-\gamma}^{\max(\alpha-\gamma,\, \theta+a)} dt\, (h(\theta), \Pi_{01}^*(\theta-t) R \Pi_{00} \Phi^{0}(t-\alpha+\gamma) A_{01}(\gamma) k(\alpha))$$

(interchanging order of integration)

$$= \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha\, (h(\theta), \Pi_{11}(\theta, \alpha) k(\alpha)) \qquad \text{(A.5-7)}$$

where we define $\Pi_{11}^{ij}(\theta, \alpha)$ for $i = 1 \ldots N-1$, $j = 1 \ldots N-1$
by $\Pi_{11}^{ij}(\theta, \alpha) = 0$ for $\theta < \theta_i$, $\alpha < \theta_j$
and for $\theta \geq \theta_i$, $\alpha \geq \alpha_j$

$$\Pi_{11}^{ij}(\theta, \alpha) = \int_{\max(\theta-\theta_i,\, \alpha-\theta_j)}^{\infty} dt\; A_i^* \Phi^{0*}(t-\theta+\theta_i)[Q+\Pi_{00}^* R \Pi_{00}] \Phi^{0}(t-\alpha+\theta_j) A_j$$

$$+ \int_{-a}^{0} d\beta \int_{\max(\theta-\theta_i,\, \alpha-\theta_j-\beta)}^{\infty} dt \; A_i^* \Phi^{0*}(t-\theta+\theta_i) \Pi_{00}^* R \Pi_{01}(\beta) \Phi^0(t+\beta-\alpha+\theta_j) A_j$$

$$+ \int_{-a}^{0} d\eta \int_{\max(\theta-\theta_i-\eta,\, \alpha-\theta_j)}^{\infty} dt \; A_i^* \Phi^{0*}(t+\eta-\theta+\theta_i) \Pi_{01}^*(\eta) R \Pi_{00} \Phi^0(t-\alpha+\theta_j) A_j$$

$$+ \int_{-a}^{0} d\eta \int_{-a}^{0} d\beta \int_{\max(\theta-\theta_i-\eta,\, \alpha-\theta_j-\beta)}^{\infty} dt \; A_i^* \Phi^{0*}(t+\eta-\theta+\theta_i) \Pi_{01}^*(\eta) R \Pi_{01}(\beta) \Phi^0(t+\beta-\alpha+\theta_j) A_j \tag{A.5-8}$$

and we define $\Pi_{11}^{i,0}(\theta,\alpha)$ for $i = 1 \ldots N-1$
by $\Pi_{11}^{i,0}(\theta,\alpha) = 0$ for $\theta < \theta_i$
and for $\theta \geq \theta_i$

$$\Pi_{11}^{i,0}(\theta,\alpha) = \int_{-a}^{\alpha} d\gamma \int_{\max(\theta-\theta_i,\, \alpha-\gamma)}^{\infty} dt \; A_i^* \Phi^{0*}(t-\theta+\theta_i) [Q + \Pi_{00}^* R \Pi_{00}] \Phi^0(t-\alpha+\gamma) A_{01}(\gamma)$$

$$+ \int_{-a}^{\alpha} d\gamma \int_{-a}^{0} d\eta \int_{\max(\theta-\theta_i-\eta,\, \alpha-\gamma)}^{\infty} dt \; A_i^* \Phi^{0*}(t+\eta-\theta+\theta_i) \Pi_{01}^*(\eta) R \Pi_{00} \Phi^0(t-\alpha+\gamma) A_{01}(\gamma)$$

$$+ \int_{-a}^{\alpha} d\gamma \int_{-a}^{0} d\beta \int_{\max(\theta-\theta_i,\, \alpha-\beta-\gamma)}^{\infty} dt \; A_i^* \Phi^{0*}(t-\theta+\theta_i) \Pi_{00}^* R \Pi_{01}(\beta) \Phi^0(t+\beta-\alpha+\gamma) A_{01}(\gamma)$$

$$+ \int_{-a}^{\alpha} d\gamma \int_{-a}^{0} d\eta \int_{-a}^{0} d\beta \int_{\max(\theta-\theta_i-\eta,\, \alpha-\beta-\gamma)}^{\infty} dt \; A_i^* \Phi^{0*}(t+\eta-\theta+\theta_i) \Pi_{01}^*(\eta) R \Pi_{01}(\beta) \Phi^0(t+\beta-\alpha+\gamma) A_{01}(\gamma)$$

$$+ \int_{0}^{\alpha+a} dt \int_{\theta-\theta_i-t}^{0} d\eta \; A_i^* \Phi^0(t+\eta-\theta+\theta_i) \Pi_{01}^*(\eta) R \Pi_{10}(\alpha-t)$$

$$+ \int_{\theta-\theta_i}^{\max(\theta-\theta_i,\, \alpha+a)} dt \; A_i^* \Phi^0(t-\theta+\theta_i) \Pi_{00}^* R \Pi_{10}(\alpha-t) \tag{A.5-9}$$

$\Pi_{11}^{0,j}(\theta,\alpha)$ is defined analogously

$$\Pi_{11}(t,\theta) = \sum_{i=1}^{N} \sum_{j=1}^{N} \Pi_{11}^{i,j}(\theta,\alpha) + \sum_{i=1}^{N} \Pi_{11}^{i,0}(\theta,\alpha) + \sum_{j=1}^{N} \Pi_{11}^{0,j}(\theta,\alpha)$$

$$+ \int_{-a}^{\theta} d\delta \int_{-a}^{\alpha} d\gamma \int_{\max(\theta-\delta,\, \alpha-\gamma)}^{\infty} dt \; A_{01}^*(\delta) \Phi^0(t-\theta+\delta) [Q + \Pi_{00} R \Pi_{00}] \Phi^0(t-\alpha+\gamma) A_{01}(\gamma)$$

$$+ \int_{-a}^{\theta} d\delta \int_{-a}^{\alpha} d\gamma \int_{-a}^{0} d\beta \int_{\max(\theta-\delta,\, \alpha-\gamma-\beta)}^{\infty} dt \; A_{01}^*(\delta) \Phi^{0*}(t-\theta+\delta) \Pi_{00}^* R \Pi_{01}(\beta) \Phi^0(t+\beta-\alpha+\gamma) A_{01}(\gamma)$$

$$
\begin{aligned}
&+ \int_{-a}^{\theta} d\delta \int_{-a}^{\alpha} d\gamma \int_{-a}^{0} d\eta \int_{\max(\theta-\delta-\eta,\,\alpha-\gamma)}^{\infty} dt \;\; A_{01}^{*}(\delta)\, \Phi^{0*}(t+\eta-\theta+\delta)\, \Pi_{01}^{*}(\eta)\, R\, \Pi_{00}\, \Phi^{0}(t-\alpha+\gamma)\, A_{01}(\gamma) \\
&+ \int_{-a}^{\theta} d\delta \int_{-a}^{\alpha} d\gamma \int_{-a}^{0} d\eta \int_{-a}^{0} d\beta \int_{\max(\theta-\delta-\eta,\,\alpha-\gamma-\beta)}^{\infty} dt \;\; A_{01}^{*}(\delta)\, \Phi^{0}(t+\eta-\theta+\delta)\, \Pi_{01}^{*}(\eta)\, R\, \Pi_{01}(\beta)\, \Phi^{0}(t-\alpha+\gamma)\, A_{01}(\gamma) \\
&+ \int_{-a}^{\theta} d\delta \int_{\theta-\delta}^{\max(\theta-\delta,\,\alpha+a)} dt \;\; A_{01}^{*}(\delta)\, \Phi^{0}(t-\theta+\delta)\, \Pi_{00}^{*}\, R\, \Pi_{01}(\alpha-t) \\
&+ \int_{-a}^{\alpha} d\gamma \int_{\alpha-\gamma}^{\max(\alpha-\gamma,\,\theta+a)} dt \;\; \Pi_{01}^{*}(\theta-t)\, R\, \Pi_{00}\, \Phi^{0}(t-\alpha+\gamma)\, A_{01}(\gamma) \\
&+ \int_{0}^{\min(\theta+a,\,\alpha+a)} dt \;\; \Pi_{01}^{*}(\theta-t)\, R\, \Pi_{01}(\alpha-t).
\end{aligned}
$$

(A.5-10).

$\Pi_{11}(\theta,\alpha)$ has jump at $\theta=\theta_i$ of magnitude

$$
\sum_{j=1}^{N} \Pi_{11}^{i,j}(\theta_i,\alpha) + \Pi_{11}^{i,0}(\theta_i,\alpha)
$$

$$
= A_i^{*}\, \Pi_{01}(\alpha) \quad \text{for } i = 1 \ldots N-1
$$

Similarly $\Pi_{11}(\theta,\alpha)$ has jump at $\alpha=\theta_j$ of magnitude
$\Pi_{01}^{*}(\theta)\, A_i$ for $j = 1, \ldots N-1$

From equation (A.5-10), it is clear that the map

$$
(\theta,\alpha) \mapsto \Pi_{11}(\theta,\alpha)
$$

is piecewise absolutely continuous in each variable and that it is absolutely continuous in each variable in every rectangle of form

$$
(\theta_{i+1}, \theta_i) \times (\theta_{j+1}, \theta_j)
$$

for $i = 0, \ldots N-1$, $j = 0, \ldots N-1$

Writing out equation (4-56) in full we have

$$
\begin{aligned}
&(A_{00}h(0), \Pi_{00}k(0)) + \int_{-a}^{0} d\alpha\, (A_{00}h(0), \Pi_{01}(\alpha)k(\alpha)) + \int_{-a}^{0} d\theta\, (A_{01}(\theta)h(\theta), \Pi_{00}k(0)) \\
&+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha\, (A_{01}(\theta)h(\theta), \Pi_{01}(\alpha)k(\alpha)) \quad + \sum_{i=1}^{N} (A_i h(\theta_i), \Pi_{00}k(0)) \\
&+ \sum_{i=1}^{N} \int_{-a}^{0} d\alpha\, (A_i h(\theta_i), \Pi_{01}(\alpha)k(\alpha)) \quad + \int_{-a}^{0} d\theta \left(\frac{dh}{d\theta}, \Pi_{01}^{*}(\theta)k(0)\right) \\
&+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \left(\frac{dh}{d\theta}, \Pi_{11}(\theta,\alpha)k(\alpha)\right) + (h(0), \Pi_{00}A_{00}k(0)) \\
&+ \int_{-a}^{0} d\alpha\, (h(0), \Pi_{00}A_{01}(\alpha)k(\alpha)) + \sum_{i=1}^{N} (h(0), A_i k(\theta_i)) + \int_{-a}^{0} d\alpha \left(h(0), \Pi_{01}(\alpha)\frac{dk}{d\alpha}\right) \\
&+ \int_{-a}^{0} d\theta\, (h(\theta), \Pi_{01}^{*}(\theta)A_{00}k(0)) + \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha\, (h(\theta), \Pi_{01}^{*}(\theta)A_{01}(\alpha)k(\alpha)) \\
&+ \sum_{i=1}^{N} \int_{-a}^{0} d\theta\, (h(\theta), \Pi_{01}^{*}(\theta)A_i k(\theta_i)) + \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \left(h(\theta), \Pi_{11}(\theta,\alpha)\frac{dk}{d\alpha}\right) \\
&- (h(0), \Pi_{00}R\Pi_{00}k(0)) - \int_{-a}^{0} d\alpha\, (h(0), \Pi_{00}R\Pi_{01}(\alpha)k(\alpha)) \\
&- \int_{-a}^{0} d\theta\, (h(\theta), \Pi_{01}^{*}(\theta)R\Pi_{00}k(0)) - \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha\, (h(\theta), \Pi_{01}^{*}(\theta)R\Pi_{01}(\alpha)k(\alpha)) \\
&+ (h(0), Q\,k(0)) \\
&= 0 \qquad\qquad (A.5-11)
\end{aligned}
$$

for $h, k \in \mathcal{D}(A) = AC^2(-a,0;R^n)$

Now for $n_0 \geq a/|\theta_1|$, define for $n \geq n_0$, $m \geq n_0$

$$h_n(\theta) = \begin{cases} h(0)\left(1 + \frac{n\theta}{a}\right) & -a/n \leq \theta \leq 0 \\ 0 & -a \leq \theta < -a/n \end{cases}$$

$$k_m(\alpha) = \begin{cases} k(0)\left(1 + \frac{m\alpha}{a}\right) & -a/m \leq \alpha \leq 0 \\ 0 & -a \leq \alpha < -a/m \end{cases}$$

Substituting into (A.5-11) we have

$$
\begin{aligned}
&(A_{00}h(0), \Pi_{00}k(0)) + \int_{-a/m}^{0}(A_{00}h(0), \Pi_{01}(\alpha)k_m(\alpha)) + \int_{-a/n}^{0}d\theta\,(A_{01}(\theta)h_n(\theta), \Pi_{00}k(0)) \\
&+ \int_{-a/n}^{0}d\theta\int_{-a/m}^{0}d\alpha\,(A_{01}(\theta)h_n(\theta), \Pi_{01}(\alpha)k_m(\alpha)) + \frac{n}{a}\int_{-a/n}^{0}d\theta(h(0), \Pi_{01}^{*}(\theta)\,k(0)) \\
&+ \frac{n}{a}\int_{-a/n}^{0}d\theta\int_{-a/m}^{0}d\alpha\,(h(0), \Pi_{11}(\theta,\alpha)k_m(\alpha)) + (h(0), \Pi_{00}A_{00}k(0)) \\
&+ \int_{-a/m}^{0}d\alpha\,(h(0), \Pi_{00}A_{01}(\alpha)k_m(\alpha)) + \frac{m}{a}\int_{-a/m}^{0}d\alpha\,(h(0), \Pi_{01}(\alpha)k(0)) \\
&+ \int_{-a/n}^{0}d\theta\,(h_n(\theta), \Pi_{01}^{*}(\theta)A_{00}k(0)) + \int_{-a/n}^{0}d\theta\int_{-a/m}^{0}d\alpha\,(h_n(\theta), \Pi_{01}^{*}(\theta)A_{01}(\alpha)k_m(\alpha)) \\
&+ \int_{-a/n}^{0}d\theta\,\frac{m}{a}\int_{-a/m}^{0}d\alpha\,(h_n(\theta), \Pi_{11}(\theta,\alpha)k(0)) - (h(0), \Pi_{00}R\Pi_{00}k(0)) \\
&- \int_{-a/m}^{0}d\alpha(h(0), \Pi_{00}R\Pi_{01}(\alpha)k_m(\alpha)) - \int_{-a/n}^{0}d\theta\,(h_n(\theta), \Pi_{01}^{*}(\theta)R\Pi_{00}k(0)) \\
&- \int_{-a/n}^{0}d\theta\int_{-a/m}^{0}d\alpha\,(h_n(\theta), \Pi_{01}^{*}(\theta)R\Pi_{01}(\alpha)k_n(\alpha)) + (h(0), Q\,k(0)) \\
&= 0 \qquad\qquad (A.5\text{-}12)
\end{aligned}
$$

Letting $m, n \to \infty$ in (A.5-12) we have

$$
\begin{aligned}
&(h(0), A_{00}^{*}\Pi_{00}k(0)) + (h(0), \Pi_{01}^{*}(0)\,k(0)) + (h(0), \Pi_{00}A_{00}k(0)) \\
&+ (h(0), \Pi_{01}(0)k(0)) - (h(0), \Pi_{00}R\Pi_{00}k(0)) + (h(0), Q\,k(0)) \\
&= 0 \qquad\qquad (A.5\text{-}13)
\end{aligned}
$$

and since (A.5-13) holds for all $h(0), k(0) \in R^n$; equation (4-66) follows.

Combining (A.5-11) and (A.5-13) we have

$$
\begin{aligned}
&\int_{-a}^{0} d\alpha \, (A_{00}\, h(0), \Pi_{01}(\alpha)\, k(\alpha)) + \int_{-a}^{0} d\theta (A_{01}(\theta) h(\theta), \Pi_{00}\, k(0)) \\
&+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \, (A_{01}(\theta)\, h(\theta), \Pi_{01}(\alpha)\, k(\alpha)) + \sum_{i=1}^{N} (A_i\, h(\theta_i), \Pi_{00}\, k(0)) \\
&+ \sum_{i=1}^{N} \int_{-a}^{0} d\alpha \, (A_i\, h(\theta_i), \Pi_{01}(\alpha)\, k(\alpha)) + \int_{-a}^{0} d\theta \left(\frac{dh}{d\theta}, \Pi_{01}^{*}(\theta)\, k(0)\right) \\
&- (h(0), \Pi_{01}^{*}(0)\, k(0)) + \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \left(\frac{dh}{d\theta}, \Pi_{11}(\theta,\alpha) k(\alpha)\right) \\
&+ \int_{-a}^{0} d\alpha \, (h(0), \Pi_{00} A_{01}(\alpha)\, k(\alpha)) + \sum_{i=1}^{N} (h(0), A_i k(\theta_i)) + \int_{-a}^{0} d\alpha \left(h(0), \Pi_{01}(\alpha) \frac{dk}{d\alpha}\right) \\
&- (h(0), \Pi_{01}(0)\, k(0)) + \int_{-a}^{0} d\theta \, (h(\theta), \Pi_{01}^{*}(\theta)\, A_{00}\, k(0)) + \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \, (h(\theta), \Pi_{01}^{*}(\theta) A_{01}(\alpha) k(\alpha)) \\
&+ \sum_{i=1}^{N} \int_{-a}^{0} d\theta \, (h(\theta), \Pi_{01}^{*}(\theta) A_i\, k(\theta_i)) + \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \left(h(\theta), \Pi_{11}(\theta,\alpha) \frac{dk}{d\alpha}\right) \\
&- \int_{-a}^{0} d\alpha \, (h(0), \Pi_{00} R \Pi_{01}(\alpha)\, k(\alpha)) - \int_{-a}^{0} d\theta \, (h(\theta), \Pi_{01}^{*}(\theta)\, R \Pi_{00}\, k(0)) \\
&- \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \, (h(\theta), \Pi_{01}^{*}(\theta)\, R\, \Pi_{01}(\alpha)\, k(\alpha)) \\
&= 0 \qquad\qquad (A.5\text{-}14)
\end{aligned}
$$

As before take
$$
h_n(\theta) = \begin{cases} h(0)\left(1 + \dfrac{n\theta}{a}\right) & -a/n \le \theta \le 0 \\ 0 & -a \le \theta < -a/n \end{cases}
$$

and chose k absolutely continuous
$\operatorname{supp} k \subset (\theta_{i+1}, \theta_i)$

Substituting into (A.5-14) we have

$$\int_{\theta_{i+1}}^{\theta_i} d\alpha\,(A_{00}\,h(0),\Pi_{01}(\alpha)\,k(\alpha)) + \int_{-a/n}^{0} d\theta \int_{\theta_{i+1}}^{\theta_i} d\alpha\,(A_{01}(\theta)\,h_n(\theta),\Pi_{01}(\alpha)\,k(\alpha))$$

$$+\ \frac{n}{a}\int_{-a/n}^{0} d\theta\ (h(0),\Pi_{01}^*(\theta)\,k(0)) - (h(0),\Pi_{01}^*(0)\,k(0))$$

$$+\ \frac{n}{a}\int_{-a/n}^{0} d\theta \int_{\theta_{i+1}}^{\theta_i} d\alpha\,(h(0),\Pi_{11}(\theta,\alpha)\,k(\alpha)) + \int_{\theta_{i+1}}^{\theta_i} d\alpha\,(h(0),\Pi_{00}\,A_{01}(\alpha)\,k(\alpha))$$

$$-\int_{\theta_{i+1}}^{\theta_i} d\alpha\,\left(h(0),\frac{d\Pi_{01}(\alpha)}{d\alpha}\,k(\alpha)\right) + \int_{-a/n}^{0} d\theta \int_{\theta_{i+1}}^{\theta_i} d\alpha\,(h_n(\theta),\Pi_{01}^*(\theta)\,A_{01}(\alpha)\,k(\alpha))$$

$$-\int_{-a/n}^{0} d\theta \int_{\theta_{i+1}}^{\theta_i} d\alpha\,\left(h_n(\theta),\frac{\partial\Pi_{11}(\theta,\alpha)}{\partial\alpha}\,k(\alpha)\right) - \int_{\theta_{i+1}}^{\theta_i} d\alpha\,(h(0),\Pi_{00}\,R\,\Pi_{01}(\alpha)\,k(\alpha))$$

$$-\int_{-a/n}^{0} d\theta \int_{\theta_{i+1}}^{\theta_i} d\alpha\,(h_n(\theta),\Pi_{01}^*(\theta)\,R\,\Pi_{01}(\alpha)\,k(\alpha))$$

$$= 0 \qquad (A.5-15)$$

Taking limit $n \to \infty$ in (A.5-15) we obtain

$$\int_{\theta_{i+1}}^{\theta_i} d\alpha\,(h(0),A_{00}^*\,\Pi_{01}(\alpha)\,k(\alpha)) + \int_{\theta_{i+1}}^{\theta_i} d\alpha\,(h(0),\Pi_{11}(0,\alpha)\,k(\alpha))$$

$$+\int_{\theta_{i+1}}^{\theta_i} d\alpha\,(h(0),\Pi_{00}\,A_{01}(\alpha)\,k(\alpha)) - \int_{\theta_{i+1}}^{\theta_i} d\alpha\,\left(h(0),\frac{d\Pi_{01}(\alpha)}{d\alpha}\,k(\alpha)\right)$$

$$-\int_{\theta_{i+1}}^{\theta_i} d\alpha\,(h(0),\Pi_{00}\,R\,\Pi_{01}(\alpha)\,k(\alpha))$$

$$= 0 \qquad (A.5-16)$$

Equation (4-67) follows from (A.5-16) because of the density of the absolutely continuous functions in $L^2(\theta_{i+1},\theta_i;R^n)$ (and with support in (θ_{i+1},θ_i))

Combining (A.5-14) and (A.5-16) we obtain

$$
\begin{aligned}
&\int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \left(A_{01}(\theta) h(\theta), \Pi_{01}(\alpha) k(\alpha)\right) + \sum_{i=1}^{N} \left(A_i h(\theta_i), \Pi_{00} k(0)\right) \\
&+ \sum_{i=1}^{N} \int_{-a}^{0} d\alpha \left(A_i h(\theta_i), \Pi_{01}(\alpha) k(\alpha)\right) + \int_{-a}^{0} d\theta \left(\frac{dh}{d\theta}, \Pi_{01}^{*}(\theta) k(0)\right) \\
&- \left(h(0), \Pi_{01}^{*}(0) k(0)\right) + \int_{-a}^{0} d\theta \left(h(\theta), \frac{d\Pi_{01}^{*}(\theta)}{d\theta} k(0)\right) \\
&+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \left(\frac{dh}{d\theta}, \Pi_{11}(\theta,\alpha) k(\alpha)\right) + \sum_{i=1}^{N} \left(h(0), A_i k(\theta_i)\right) \\
&+ \int_{-a}^{0} d\alpha \left(h(0), \Pi_{01}(\alpha) \frac{dk}{d\alpha}\right) - \left(h(0), \Pi_{01}(0) k(0)\right) + \int_{-a}^{0} d\alpha \left(h(0), \frac{d\Pi_{01}(\alpha)}{d\alpha} k(\alpha)\right) \\
&+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \left(h(\theta), \Pi_{01}^{*}(\theta) A_{01}(\alpha) k(\alpha)\right) + \sum_{i=1}^{N} \int_{-a}^{0} d\theta \left(h(\theta), \Pi_{01}^{*}(\theta) A_i k(\theta_i)\right) \\
&+ \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \left(h(\theta), \Pi_{11}(\theta,\alpha) \frac{dk}{d\alpha}\right) - \int_{-a}^{0} d\theta \int_{-a}^{0} d\alpha \left(h(\theta), \Pi_{01}^{*}(\theta) R \Pi_{01}(\alpha) k(\alpha)\right) \\
&= 0 \qquad\qquad (A.5\text{-}17)
\end{aligned}
$$

Now take h, k absolutely continuous,

$\operatorname{supp} h \subset (\theta_{i+1}, \theta_i)$

$\operatorname{supp} k \subset (\theta_{j+1}, \theta_j)$

Substituting into (A.5-17) we obtain

$$
\begin{aligned}
&\int_{\theta_{i+1}}^{\theta_i} d\theta \int_{\theta_{j+1}}^{\theta_j} d\alpha \left(A_{01}(\theta) h(\theta), \Pi_{01}(\alpha) k(\alpha)\right) - \int_{\theta_{i+1}}^{\theta_i} d\theta \int_{\theta_{j+1}}^{\theta_j} d\alpha \left(h(\theta), \frac{\partial \Pi_{11}(\theta,\alpha)}{\partial \theta} k(\alpha)\right) \\
&+ \int_{\theta_{i+1}}^{\theta_i} d\theta \int_{\theta_{j+1}}^{\theta_j} d\alpha \left(h(\theta), \Pi_{01}^{*}(\theta) A_{01}(\alpha) k(\alpha)\right) - \int_{\theta_{i+1}}^{\theta_i} d\theta \int_{\theta_{j+1}}^{\theta_j} d\alpha \left(h(\theta), \frac{\partial \Pi_{11}(\theta,\alpha)}{\partial \alpha} k(\alpha)\right) \\
&- \int_{\theta_{i+1}}^{\theta_i} d\theta \int_{\theta_{j+1}}^{\theta_j} d\alpha \left(h(\theta), \Pi_{01}^{*}(\theta) R \Pi_{01}(\alpha) k(\alpha)\right) = 0 \qquad\qquad (A.5\text{-}18)
\end{aligned}
$$

Equation (4-68) now follows from (A.5-18) because of the density of the absolutely continuous functions with support in (θ_{i+1}, θ_i) in $L^2(\theta_{i+1}, \theta_i ; R^n)$.

To obtain the boundary condition

$$\Pi_{01}(-a) = \Pi_{00} A_N$$

we substitute

$$h_n(\theta) = \begin{cases} h(-a)\{-\frac{n\theta}{a} + 1 - n\} & -a \leq \theta \leq -a(1-\frac{1}{n}) \\ 0 & -a(1-\frac{1}{n}) < \theta \leq 0 \end{cases}$$

defined for $n \geq n_0 > a/|\theta_N - \theta_{N-1}|$

and $$k_m(\alpha) = \begin{cases} k(0)(1 + \frac{m\alpha}{a}) & -a/m \leq \alpha \leq 0 \\ 0 & -a \leq \alpha < -a/m \end{cases}$$

defined for $m \geq m_0 > a/|\theta_1|$

into (A.5-11) and take the limit $m, n \to \infty$.

To obtain the boundary condition

$$\Pi_{11}(-a, \alpha) = A_N^* \Pi_{01}(\alpha) \quad \text{a.e}$$

we take h_n defined as before and k absolutely continuous with support in (θ_{i+1}, θ_i) and take the limit $n \to \infty$.

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