Information Percolation in Segmented Markets*

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Abstract

We study equilibria of dynamic over-the-counter markets in which agents are distinguished by their preferences and information. Over time, agents are privately informed by bids and offers. Investors differ with respect to information quality, including initial information precision, and also in terms of market "connectivity," the expected frequency of their bilateral trading opportunities. We characterize endogenous information acquisition and show how learning externalities affect information gathering incentives. More "liquid" markets lead to higher equilibrium information acquisition when the gains from trade and market duration are sufficiently large. On the other hand, for a small market duration, the opposite may occur if agents vary sufficiently in terms of their market connectivity.

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1 Introduction

We study equilibria of dynamic over-the-counter double-auction markets in which agents are distinguished by their preferences, information, and market "connectivity." Agents collect information privately over time from the bids and offers of their counterparties. We characterize the effect of market segmentation associated with the limited "connectivity" of investors caused by search frictions. Agents can choose how much information to gather on their own, or invest in improvements in their connectivity, allowing them to more easily trade with specific classes of agents. This limited connectivity can affect some classes of investors more than others, with endogenous consequences for equilibrium information gathering incentives.

We show that if there are sufficiently many rounds of trade before pay-off relevant information is revealed, then there is strategic complementarity in information gathering and in connectivity. That is, when faced by counterparties that increase their information gathering or connectivity, a given agent will respond likewise. For a short-lived market, however, the opposite can occur, in that an agent may cut back on information gathering or investments in connectivity when other agents are found to increase their information gathering or connectivity.

In our model, the various classes of agents are distinguished by their preferences for the asset to be auctioned, by the expected frequency of their trading opportunities with each of the other classes of agents, and by the quality of their initial information about a random variable Y that determines the ultimate payoff of the asset.

At each time period, any agent of class i has a trading encounter with probability λ_i. At each such encounter, a counterparty of class j is selected with probability κ_{ij}. Any two agents that meet are given the opportunity to trade one unit of the asset in a double auction. Based on their initial information and on the information they have gathered from bids in prior auctions with other agents, the two agents typically have different conditional expectations of Y. Because the two agents' preference parameters are commonly observed, it is common knowledge which of them is the prospective buyer and which is the prospective seller. Trade occurs in the event that the price β bid by the buyer is above the seller's offer price σ, in which case the buyer pays σ to the seller. This double-auction format, chosen for tractability, is known as the "seller's price auction."

We provide technical conditions under which these double auctions have a unique equilibrium in bidding strategies that are strictly increasing in the agents' conditional expectations of the asset payoff. We show how to compute the offer price σ and the bid price β, state by state, by solving an ordinary differential equation that depends on the quality of the information of the buyer and that of the seller at the time of their trading encounter. Because the bid and offer are strictly increasing with respect to the seller's and buyer's conditional expectations of the asset payoff, they reveal these conditional expectations to the respective counterparties, who then use them to update their priors for the purposes of subsequent trading opportunities. The technical conditions that we impose in order to guarantee the existence of such an equilibrium also imply that this equilibrium uniquely maximizes expected gains from trade in each auction and, consequently, total welfare, given the market structure. We also endogenize the market structure to account for learning and trade incentives.

Because our strictly monotone double-auction equilibrium fully reveals the bidders' conditional beliefs for Y, we are able to explicitly calculate the evolution over time of the cross-sectional distribution of posterior beliefs of the population of agents by extending the results of Duffie and Manso (2007) and Duffie, Giroux, and Manso (2010) to multiple classes of investors. In order to characterize the solutions, we extend the Wild summation method of Duffie, Giroux, and Manso (2010) to directly solve the evolution equation for the cross-sectional distribution of conditional beliefs.

The double-auction equilibrium characterization, together with the characterization of the dynamics of the cross-sectional distribution of posterior beliefs of each class of agents, permits a calculation of the expected lifetime utility of each class of agents, including the manner in which utility depends on the class characteristics determining information quality, namely the precision of the initially acquired information and the connectivity of that agent.

This in turn allows us to characterize the endogenous costly acquisition of information. We also characterize, under conditions, the endogenous formation of "trading channels," by which agents invest more heavily in the ability to locate agents are more heavily endowed with information or are known to have themselves invested more heavily in "trading connectivity" with other agents.

2 Related Literature

A large literature in economics and finance addresses learning from market prices of transactions that take place in centralized exchanges.[1] Less attention, however, has

[1]See, for example, Grossman (1976), Grossman and Stiglitz (1980), Wilson (1977), Milgrom (1981), Vives (1993), Pesendorfer and Swinkels (1997), and Reny and Perry (2006).

been paid to information transmission in over-the-counter markets. Private information sharing is typical in functioning over-the-counter markets for many types of financial assets, including bonds and derivatives. In these markets, trades occur at private meetings in which counterparties offer prices that reveal information to each other, but not to other market participants.

Wolinsky (1990), Blouin and Serrano (2001), Duffie and Manso (2007), Duffie, Giroux, and Manso (2010), and Duffie, Malamud, and Manso (2009, 2010), and Golosov, Lorenzoni, and Tsyvinski (2013), are among the few studies that have investigated learning in over-the-counter (OTC) markets. The models of search and random matching used in these studies are unsuitable for the analysis of the effects of segmentation of investors into groups that differ by connectivity or initial information quality. Further, this prior work has not considered the implication of endogenous information acquisition. Here, we are able to study these effects by allowing for classes of investors with distinct preferences, initial information quality, and market connectivity.

Our finding, that in an OTC market information acquisition can have natural strategic complementarities, need not apply in a centralized market because the centrally announced price reveals "for free" some of the information that has been gathered by others, diluting the incentive to gather information at a cost. For example, in the standard setting of Grossman and Stiglitz (1980), there is never complementarity in information acquisition. Manzano and Vives (2011) extend Grossman and Stiglitz's model to allow for private information about asset payoffs with common and private components as well as exposure to an aggregate risk factor. They show that such equilibria are unstable whenever there is strategic complementarity in information acquisition. Breon-Drish (2010), however, shows that a relaxation of the normality assumption of Grossman and Stiglitz can, under some additional conditions, lead to strategic complementarity in information acquisition in a standard central-market rational expectations equilibrium. The underlying incentive to gather information in such a model depends on the cooperation of "noise traders," who contribute all of the expected gains from trade obtained through gathering information.[2]

In our model, whenever two agents meet, they have the opportunity to participate in a double auction. Chatterjee and Samuelson (1983) are among the first to study double auctions. The case of independent private values has been extensively analyzed by Williams (1987), Satterthwaite and Williams (1989), and Leininger, Linhart, and Radner (1989). Kadan (2007) studies the case of correlated private values. We extend

[2]See, also, Barlevi and Veronesi (2000) and other sources cited by Breon-Drish (2010).

these results by providing conditions for the existence of a unique strictly monotone equilibrium in undominated strategies, for a double auction with common values. Bid monotonicity is natural in our setting given the strict monotone dependence on the asset payoff of each agent's ex-post utility for a unit of the asset. Strictly monotone equilibria are not typically available, however, in more general double auctions with a common value component, as indicated by Reny and Perry (2006).

The conditions provided here for fully-revealing double auctions carry over to a setting in which the transactions prices of a finite sample of trades are publicly revealed, as is often the case in functioning over-the-counter markets. With this mixture of private and public information sharing, information dynamics can be analyzed by the methods[3] of Duffie, Malamud, and Manso (2009).

Beyond the application to information transmission in over-the-counter markets, our model can be applied in other settings in which learning occurs through successive local interactions, such as bank runs, knowledge spillovers, social learning, and technology diffusion. For example, Banerjee and Fudenberg (2004) and Duffie, Malamud, and Manso (2009) study social learning through word-of-mouth communication, but do not consider situations in which agents differ with respect to connectivity. In social networks, agents naturally differ with respect to connectivity. DeMarzo, Vayanos, and Zwiebel (2003), Gale and Kariv (2003), Acemoglu, Dahleh, Lobel, and Ozdaglar (2008), and Golub and Jackson (2010) study learning in social networks. Our model provides an alternative tractable framework to study the dynamics of social learning when different groups of agents in the population differ in connectivity with other groups of agents.

3 The Model

This section specifies the economy and solves for the dynamics of information transmission and the cross-sectional distribution of beliefs, fixing the initial distribution of information and assuming that bids and offers are fully revealing. The following section characterizes equilibrium bidding behavior, providing conditions for the existence and uniqueness of a fully revealing equilibrium, again taking as given the initial allocation of information to agents. Finally, Section 5 characterizes the costly endogenous acquisition of initial information. Appendix K extends the model so as to allow the endogenous acquisition of information by any agent not only before the first round of trade, but at

[3] One obtains an evolution equation for the cross-sectional distribution of beliefs that is studied by Duffie, Malamud, and Manso (2010) for the case $M = 1$, and easily extended to the case of general M.

any time period, based on the outcome of the agent's conditional beliefs at that time. Appendix Table 1 is a directory of the locations of proofs.

3.1 The Double Auctions

A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is fixed. An economy is populated by a continuum (a non-atomic measure space) of risk-neutral agents who are randomly paired over time for trade, in a manner that will be described. There are M different classes of agents that differ according to the quality of their initial information, their preferences for the asset to be traded, and the likelihoods with which they meet each of other classes of agents for trade, period by period. At some future time T, the economy ends and the utility realized by an agent of class i for each additional unit of the asset is

$$U_i = v_i Y + v^H(1 - Y),$$

measured in units of consumption, for strictly positive constants v^H and $v_i < v^H$, where Y is a non-degenerate 0-or-1 random variable whose outcome is revealed immediately after time T.

Whenever two agents meet, they are given the opportunity to trade one unit of the asset in a double auction. The auction format allows (but does not require) the agents to submit a bid or an offer price for a unit of the asset. That agents trade at most one unit of the asset at each encounter is an artificial restriction designed to simplify the model. One could suppose, alternatively, that the agents bid for the opportunity to produce a particular service for their counterparty.

Any bid and offer is observed by both agents participating in the auction, and not by other agents. If an agent submits a bid price that is higher than the offer price submitted by the other agent, then one unit of the asset is assigned to that agent submitting the bid price, in exchange for an amount of consumption equal to the ask price. Certain other auction formats would be satisfactory for our purposes. We chose this format, known as the "seller's price auction," for simplicity.

When a class-i and a class-j agent meet, their respective classes i and j are observable to both.[4] Based on their initial information and on the information that they have received from prior auctions held with other agents, the two agents typically assign different conditional expectations to Y. From the no-speculative-trade theorem of Milgrom

[4]That is, all of the primitive characteristics, $\psi_{i0}, \lambda_i, \kappa_i$, and v_i of each agent are common knowledge to them. In a prior version of this paper, we considered variants of the model in which the initial type density ψ_{i0} and the per-period trading probabilities $\lambda_i\kappa_{i1}, \ldots, \lambda_i\kappa_{iM}$ need not be observable.

and Stokey (1982), as extended by Serrano-Padial (2007) to our setting of risk-neutral investors,[5] the two counterparties decline the opportunity to bid if they have identical preferences, that is, if $v_i = v_j$. If $v_i \neq v_j$, then it is common knowledge which of the two agents is the prospective buyer ("the buyer") and which is the prospective seller ("the seller"). The buyer is of class j whenever $v_j > v_i$.

The seller has an information set $\mathcal{F}_S$ that consists of his initially endowed signals relevant to the conditional distribution of Y, as well any bids and offers that he has observed at his previous auctions. The seller's offer price σ must be based only on (must be measurable with respect to) the information set $\mathcal{F}_S$. The buyer, likewise, makes a bid β that is measurable with respect to her information set $\mathcal{F}_B$.

The bid-offer pair (β, σ) constitute an equilibrium for a seller of class i and a buyer of class j provided that, fixing β, the offer σ maximizes[6] the seller's conditional expected gain,

$$E\left[(\sigma - E(U_i \,|\, \mathcal{F}_S \cup \{\beta\}))1_{\{\sigma<\beta\}} \,|\, \mathcal{F}_S\right], \tag{1}$$

and fixing σ, the bid β maximizes the buyer's conditional expected gain

$$E\left[(E(U_j \,|\, \mathcal{F}_B \cup \{\sigma\}) - \sigma)1_{\{\sigma<\beta\}} \,|\, \mathcal{F}_B\right]. \tag{2}$$

The seller's conditional expected utility for the asset, $E(U_i \,|\, \mathcal{F}_S \cup \{\beta\})$, once having conducted a trade, incorporates the information $\mathcal{F}_S$ that the seller held before the auction as well as the bid β of the buyer. Similarly, the buyer's utility is affected by the information contained in the seller's offer. The information gained from more frequent participation in auctions with well informed bidders is a key focus here.

In Section 4.1, we demonstrate technical conditions under which there are equilibria in which the offer price σ and bid price β can be computed, state by state, by solving an ordinary differential equation corresponding to the first-order conditions for optimality. The offer and bid are strictly monotonically decreasing with respect to $E(Y \,|\, \mathcal{F}_S)$ and $E(Y \,|\, \mathcal{F}_B)$, respectively. Bid monotonicity is natural given the strictly monotone decreasing dependence on Y of U_i and U_j. Strictly monotone equilibria are not typically available, however, in more general settings explored in the double-auctions literature, as indicated by Reny and Perry (2006). Because our strictly monotone equilibria fully

[5] Milgrom and Stokey (1982) assume strictly risk-averse investors. Serrano-Padial (2007) shows that for investors with identical preferences, even if risk-neutral, if the distributions of counterparties' posteriors have a density, as here, then there is no equilibrium with a strictly positive probability of trade in our common-value environment.

[6] Here, to "maximize" means, as usual, to achieve, almost surely, the essential supremum of the conditional expectation.

reveal the bidders' conditional beliefs for Y, we are able to calculate the evolution over time of the cross-sectional distribution of posterior beliefs of the population of agents. For this, we extend results from Duffie and Manso (2007) and Duffie, Giroux, and Manso (2008). This, in turn, permits a characterization of the expected lifetime utility of each class of agents, including the manner in which utility depends on the quality of the initial information endowment and the "market connectivity" of that agent. This will also allow us to examine equilibrium incentives to gather information.

3.2 Information Setting

Agents are initially informed by signals drawn from an infinite pool of 0-or-1 random variables. Conditional on Y, almost every pair of these signals is independent.[7] Each signal is received by at most one agent. Each agent is initially allocated a randomly selected finite subset of these signals. For almost every pair of agents, the sets of signals that they receive are independently chosen.[8] The random allocation of signals to agents is also independent of the signals themselves. (The allocation of signals to an agent is allowed to be deterministic.) The signals need not have the same probability distributions.

Whenever it is finite, we define the "information type" of an arbitrary finite set K of random variables to be

$$\log \frac{\mathbb{P}(Y=0 \,|\, K)}{\mathbb{P}(Y=1 \,|\, K)} - \log \frac{\mathbb{P}(Y=0)}{\mathbb{P}(Y=1)}, \tag{3}$$

the difference between the conditional and unconditional log-likelihood ratios. The conditional probability that $Y=0$ given signals with information type θ is thus

$$P(\theta) = \frac{Re^{\theta}}{1+Re^{\theta}}, \tag{4}$$

where $R = \mathbb{P}(Y=0)/\mathbb{P}(Y=1)$. Thus, the information type of a collection of signals is one-to-one with the conditional probability that $Y=0$ given the signals. Proposition 3 of Duffie and Manso (2007) implies that whenever a collection of signals of type θ is combined with a disjoint collection of signals of type ϕ, the type of the combined set of signals is $\theta+\phi$. More generally, we will use the following result from Duffie and Manso (2007).

[7] More precisely, there is a continuum of signals, indexed by a non-atomic measure space, say $[0,1]$. For almost every signal Z, almost every other signal is conditionally independent of Z given Y.

[8] Letting $\mathcal{S}_F$ denote the set of finite subsets of signals, this means that for almost all pairs (i,j) of agents, the mappings $S_i : \Omega \to \mathcal{S}_F$ and $S_j : \Omega \to \mathcal{S}_F$ determining the signals they receive are independent random variables.

Lemma 3.1 *Let $S_1, \ldots, S_n$ be disjoint sets of signals with respective types $\theta_1, \ldots, \theta_n$. Then the union $S_1 \cup \cdots \cup S_n$ of the signals has type $\theta_1 + \cdots + \theta_n$. Moreover, the type of the information set $\{\theta_1, \theta_2, \ldots, \theta_n\}$ is also $\theta_1 + \theta_2 + \cdots + \theta_n$.*

The Lemma has two key implications for our analysis. First, if two agents meet and reveal all of their endowed signals, they both achieve posterior types equal to the sum of their respective prior types. Second, for the purpose of determining posterior types, revealing one's prior type (or any random variable such as a bid that is strictly monotone with respect to that type) is payoff-equivalent to revealing all of one's signals.

For each time $t \in \{0, 1, \ldots, T\}$, an agent of class i is randomly matched with some other agent with probability $\lambda_i \in [0, 1)$. This counterparty is of class-j with probability κ_{ij}. Upon meeting, the two agents are given the opportunity to trade one unit of the asset in a double auction. Without loss of generality for the purposes of analyzing the evolution of information, we take $\kappa_{ij} = 0$ whenever $v_i = v_j$, because of the no-trade result for agents with the same preferences.[9]

As is standard in search-based models of markets, we assume that, for almost every pair of agents, the matching times and the counterparties of one agent are independent of those of almost every other agent. Duffie and Sun (2007) and Duffie, Qiao, and Sun (2014) show the existence of a model with this random matching property, as well as the associated law of large numbers for random matching (with directed probability) on which we rely.[10] There are algebraic consistency restrictions on the random matching parameters λ_i, κ_{ij} and the population masses $m_1, \ldots, m_M$ of the respective classes. Specifically, the exact law of large numbers for random matching implies that the total quantity of matches of agents of a given class i with the agents of a given class j is almost surely $m_i \lambda_i \kappa_{ij} = m_j \lambda_j \kappa_{ji}$.

[9]If the primitive parameters do not satisfy this property, they can without effect on the results be adjusted so as to satisfy this property by conditioning, case by case, on the event that the agents matched have $v_i \neq v_j$.

[10]Taking G to be the set of agents, we assume throughout the joint measurability of agents' type processes $\{\theta_{it} : i \in G\}$ with respect to a σ-algebra $\mathcal{B}$ on $\Omega \times G$ that allows the Fubini property that, for any measurable subset A of types,

$$\int_G \mathbb{P}(\theta_{\alpha t} \in A)\, d\gamma(\alpha) = E\left(\int_G 1_{\theta_{\alpha t} \in A}\, d\gamma(\alpha)\right),$$

where γ is the measure on the agent space. We rely on a "richness" condition on $\mathcal{B}$ that allows an application of the exact law of large numbers. In our setting, because almost every pair of types from $\{\theta_{\alpha t} : \alpha \in G\}$ is independent, this law implies that $E\left(\int_G 1_{\theta_{\alpha t} \in A}\, d\gamma(\alpha)\right) = \int_G 1_{\theta_{\alpha t} \in A}\, d\gamma(\alpha)$ almost surely. Duffie and Sun (2007) prove the existence of a model with this property. Xiang Sun (2012) has extended these results to cases in which the types of agents can be drawn from a Polish space. In our setting, information types play no role in the random matching model.

In this random-matching setting, with probability one, a given pair of agents that have been matched will almost surely never be matched again nor will their respective lifetime sets of trading counterparties overlap, nor will the counterparties of those counterparties overlap, and so on. Thus, equilibrium bidding behavior in the multi-period setting is characterized by equilibrium bidding behavior in each individual auction, as described above. Later, we will provide primitive technical conditions on the preference parameters v^H and v_i, as well as the cross-sectional distribution of initially endowed information types, that imply the existence of an equilibrium with strictly monotone bidding strategies. In this setting, bids therefore reveal types. Lemma 3.1 and induction thus imply that agents' types add up from auction to auction. Specifically, an agent leaves any auction with a type that is the sum of his or her type immediately before the auction and the type of the other agent bidding at the auction. This fact now allows us to characterize the dynamics of the cross-sectional evolution of posterior types.

3.3 Evolution of Type Distributions

For each class i, we suppose that the initial cross-sectional distribution of types of the class-i agents has some density ψ_{i0}. This initial density may have been endogenously determined through pre-trade information acquisition decisions, which we analyze in Section 5. We do not require that the individual class-i agents have types with the same probability distribution. Nevertheless, our independence and measurability assumptions imply the exact law of large numbers, by which the density function ψ_{i0} has two deterministic outcomes, almost surely, one on the event that $Y = 0$, denoted ψ_{i0}^H, the other on the event that $Y = 1$, denoted ψ_{i0}^L. That is, for any real interval (a, b), the fraction of class-i agents whose type is initially between a and b is almost surely $\int_a^b \psi_{i0}^H(\theta)\, d\theta$ on the event that $Y = 0$, and is almost surely $\int_a^b \psi_{i0}^L(\theta)\, d\theta$ on the event that $Y = 1$. We make the further assumption that ψ_{i0}^H and ψ_{i0}^L have moment-generating functions that are finite on a neighborhood of zero.

The initial cross-sectional type densities in the high and low states, ψ_{i0}^H and ψ_{i0}^L, are related by the following result, proved in Appendix A.

Proposition 3.2 *For all x,*

$$\psi_{i0}^H(x) \;=\; e^x\, \psi_{i0}^L(x)\,. \tag{5}$$

Appendix Lemma A.2 implies that *any* pair of probability density functions on the real line satisfying (5) can be realized as the outcomes ψ_{i0}^H and ψ_{i0}^L of the cross-sectional density of beliefs resulting from some allocation of signals to agents. Thus, (5) is a necessary and sufficient condition for such a pair of densities. In order to create model examples, moreover, it suffices to pick an arbitrary density function for ψ_{i0}^L satisfying $\int_{\mathbb{R}} e^x \psi_{i0}^L(x)\, dx = 1$ and then let ψ_{i0}^H be given by (5). Thus, one can safely skip the step of specifying the set of signals and their allocation to agents.

Our objective now is to calculate, for any time t, the cross-sectional density ψ_{it} of the types of class-i agents. Again by the law of large numbers, this cross-sectional density has (almost surely) only two outcomes, one on the event $Y = 0$ and one on the event $Y = 1$, denoted ψ_{it}^H and ψ_{it}^L, respectively.

Assuming that the equilibrium bids and offers are fully revealing, which will be confirmed in the next section under explicit technical conditions on model primitives, the evolution equation for the cross-sectional densities is

$$\psi_{i,t+1} = \ (1-\lambda_i\,)\psi_{it} \ + \ \lambda_i\,\psi_{it} * \sum_{j=1}^{M} \kappa_{ij}\,\psi_{jt}, \qquad i \in \{1,\ldots,M\}, \tag{6}$$

where $*$ denotes convolution.

We offer a brief explanation of this evolution equation. The first term on the righthand side reflects the fact that, with probability $1-\lambda_i$, an agent of class i does not meet anybody at time $t+1$. Because of the exact law of large numbers, the first term on the righthand side is therefore, almost surely, the cross-sectional density of the information types of class-i investors who are not matched. The second term is the cross-sectional density function of class-i agents that are matched and whose types are thereby changed by observing bids at auctions. The second term is understood by noting that auctions with class-j counterparties occur with probability $\lambda_i\kappa_{ij}$. At such an encounter, in a fully revealing equilibrium, bids reveal the types of both agents, which are then added to get the posterior types of each. A class-i agent of type θ is thus created if a class-i agent of some type ϕ meets a class-j agent of type $\theta-\phi$. Because this is true for any possible ϕ, we integrate over ϕ with respect to the population densities. Thus, the total density of class-i agents of type θ that is generated by the information released at auctions with class-j agents is

$$\lambda_i\kappa_{ij}\int_{-\infty}^{+\infty} \psi_{it}(\phi)\psi_{jt}(\theta-\phi)\, d\phi = \lambda_i\kappa_{ij}(\psi_{it} * \psi_{jt})(\theta).$$

Adding over j gives the second term on the righthand side of the evolution equation (6). For the case $M = 1$, a continuous-time analog of this evolution model is motivated and solved by Duffie and Manso (2007) and Duffie, Giroux, and Manso (2010).

The multi-dimensional evolution equation (6) can be solved explicitly by an inductive procedure, a discrete-time multi-dimensional analogue of the Wild summation method of Duffie, Giroux, and Manso (2010), which is based on continuous-time random matching and on a single class of investors.

In order to calculate the Wild-sum representation of type densities solving the evolution equation (6), we proceed as follows. For an M-tuple $k = (k_1, \ldots, k_M)$ of nonnegative integers, let $a_{it}(k)$ denote the fraction of class-i agents who by time t have collected (directly, or indirectly through auctions) the originally endowed signal information of k_1 class-1 agents, of k_2 class-2 agents, and so on, including themselves. This means that $|k| = k_1 + \cdots + k_M$ is the number of agents whose originally endowed information has been collected by such an agent. To illustrate, consider an example agent of class 1 who, by a particular time t has met one agent of class 2, and nobody else, with that agent of class 2 having beforehand met 3 agents of class 4 and nobody else, and with those class-4 agents not having met anyone before they met the class-2 agent. The class-1 agents with this precise scenario of meeting circumstances would contribute to $a_{1t}(k)$ for $k = (1, 1, 0, 3, 0, 0, \ldots, 0)$. We can view a_{it} as a measure on $\mathbb{Z}_+^M$, the set of M-tuples of nonnegative integers. By essentially the same reasoning used to explain the evolution equation (6), we have

$$a_{i,t+1} \;=\; (1-\lambda_i)\, a_{it} \;+\; \lambda_i\, a_{it} * \sum_{j=1}^{M} \kappa_{ij}\, a_{jt}, \qquad a_{i0} = \delta_{e_i}, \tag{7}$$

where

$$(a_{it} * a_{jt})(k) \;=\; \sum_{\{l \in \mathbb{Z}_+^M,\; |l| \le |k|\}} a_{it}(l)\, a_{jt}(k-l).$$

Here, δ_{e_i} is the dirac measure placing all mass on e_i, the unit vector whose i-th coordinate is 1. The definition of $a_{it}(k)$ and Lemma 3.1 now imply the following solution for the dynamic evolution of cross-sectional type densities.

Theorem 3.3 *There is a unique solution of (6), given by*

$$\psi_{it} \;=\; \sum_{k \in \mathbb{Z}_+^M} a_{it}(k)\, \psi_{10}^{*k_1} * \cdots * \psi_{M0}^{*k_M}, \tag{8}$$

where ψ_{i0}^{*n} denotes n-fold convolution.

This representation captures the percolation of information through "intermediation chains" between classes that are only indirectly connected with each other. Lemma

C.3 provides the asymptotic (long horizon) properties of this evolution of cross-sectional information distribution in terms of the characteristics of the "network connectivity matrix" $(\lambda_i \kappa_{ij})_{i,j=1}^M$.

4 The Double Auction Properties

We turn in this section to the equilibrium characterization of bidding behavior. For simplicity, we assume that there are only two individual private asset valuations, that of a prospective seller, v_s, and that of a buyer, $v_b > v_s$. That is, $v_i \in \{v_b, v_s\}$ for all i.

4.1 Double Auction Solution

Fixing a time t, we suppose that a class-i agent and a class-j agent have met, and that the prospective buyer is of class i. We now calculate their equilibrium bidding strategies. Naturally, we look for equilibria in which the outcome of the offer σ for a seller of type θ is $S(\theta)$ and the outcome of the bid β of a buyer of type ϕ is $B(\phi)$, where $S(\,\cdot\,)$ and $B(\,\cdot\,)$ are some strictly monotone increasing functions on the real line. In this case, if (σ, β) is an equilibrium, we also say that (S, B) is an equilibrium.

Given a candidate pair (S, B) of such bidding policies, a seller of type θ who offers the price s has an expected increase in utility, defined by (1), of

$$v_{jit}(\theta; B, S) = \int_{B^{-1}(s)}^{+\infty} (s - v_s - \Delta_s P(\theta + \phi)) \; \Psi_b(P(\theta), \phi)\, d\phi, \tag{9}$$

where $\Delta_s = v^H - v_s$ and where $\Psi_b(P(\theta), \,\cdot\,)$ is the seller's conditional probability density for the unknown type of the buyer, defined by

$$\Psi_b(p, \phi) \;=\; p\, \psi_{it}^H(\phi) \,+\, (1 - p)\, \psi_{it}^L(\phi). \tag{10}$$

Likewise, from (2), a buyer of type ϕ who bids b has an expected increase in utility for the auction of

$$v_{ijt}(\phi; B, S) = \int_{-\infty}^{S^{-1}(b)} \big(v_b \,+\, \Delta_b P(\theta + \phi) \,-\, S(\theta)\big)\, \Psi_s(P(\phi), \theta)\, d\theta. \tag{11}$$

The pair (S, B) therefore constitutes an equilibrium if, for almost every ϕ and θ, these gains from trade are maximized with respect to b and s by $B(\phi)$ and $S(\theta)$, respectively.

It is convenient for further analysis to recall that the the hazard rate $h^L_{it}(\theta)$ associated with ψ^L_{it} is defined by

$$h^L_{it}(\theta) \;=\; \frac{\psi^L_{it}(\theta)}{G^L_{it}(\theta)},$$

where $G^L_{it}(\theta) = \int_\theta^\infty \psi^L_{it}(x)\,dx$. That is, given $Y = 1$, $h^L_{it}(\theta)$ is the probability density for the type θ of a randomly selected buyer, conditional on this type being at least θ. We likewise define the hazard rate $h^H_{it}(\theta)$ associated with ψ^H_{it}. We say that ψ_{it} satisfies the hazard-rate ordering if, for all θ, we have $h^H_{it}(\theta) \leq h^L_{it}(\theta)$.

Appendix A confirms that property (5) is maintained under mixtures and convolutions. The same property therefore applies to the type densities at any time $t \geq 0$. The likelihood ratio $\psi^H_{it}(x)/\psi^L_{it}(x) = e^x$ is therefore always increasing. Appendix A also provides a proof of the following.

Lemma 4.1 *For each agent class i and time t, the type density ψ_{it} satisfies the hazard-rate ordering, $h^H_{it}(\theta) \leq h^L_{it}(\theta)$.*

We will exploit the following technical regularity condition on initial type densities.

Standing Assumption: The initial type densities are strictly positive and twice continuously differentiable.

The calculation of an equilibrium is based on the ordinary differential equation (ODE) stated in the following result for the type $V_b(b)$ of a buyer who optimally bids b. That is, V_b is the inverse B^{-1} of the candidate equilibrium bid policy function B.

Lemma 4.2 *For any $V_0 \in \mathbb{R}$, there exists a unique solution $V_b(\,\cdot\,)$ on $[v_b, v^H)$ to the ODE*

$$V_b'(z) \;=\; \frac{1}{v_b - v_s}\left(\frac{z - v_b}{v^H - z}\,\frac{1}{h^H_{it}(V_b(z))} + \frac{1}{h^L_{it}(V_b(z))}\right), \qquad V_b(v_b) \;=\; V_0. \tag{12}$$

This solution, also denoted $V_b(V_0, z)$, is monotone increasing in both z and V_0. Further, $\lim_{z\to v^H} V_b(V_0, z) \;=\; +\infty$, and the limit $V_b(-\infty, z) \;=\; \lim_{V_0\to-\infty} V_b(V_0, z)$ exists. Finally, $z \mapsto V_b(-\infty, z)$ is continuously differentiable.

As shown in the proof of the next proposition, found in Appendix B, the ODE (12) arises from the first-order optimality conditions for the buyer and seller. The solution of the ODE can be used to characterize all continuous nondecreasing equilibria in the double auction, as follows.

Proposition 4.3 *Suppose that* (S, B) *is a continuous, nondecreasing equilibrium for which* $S(\theta) \leq v^H$ *for all* $\theta \in \mathbb{R}$. *Let* $V_0 = \sup\{B^{-1}(v_b)\} \geq -\infty$. *Then*

$$B(\phi) \;=\; V_b^{-1}(\phi), \quad \phi \,>\, V_0.$$

Further, $S(-\infty) \equiv \lim_{\theta \to -\infty} S(\theta) \;=\; v_b$ *and* $S(+\infty) \equiv \lim_{\theta \to -\infty} S(\theta) \;=\; v^H$. *For any* θ, *we have* $S(\theta) \;=\; V_s^{-1}(\theta)$, *where*

$$V_s(z) \;=\; \log \frac{z - v_b}{v^H - z} \;-\; V_b(z) \,-\, \log R\,, \quad z \,\in\, (v_b, v^H), \tag{13}$$

recalling that $R = \mathbb{P}(Y = 0)/\mathbb{P}(Y = 1)$. *If the buyer has type* $\phi < V_0$, *then no trade will occur. The bidding policy* B *is not uniquely determined at types below* V_0.

In our double-auction setting, welfare is increasing in the probability of trade conditional on $Y = 1$, because the buyer and seller both have a strictly positive expected gain from any trade conditional on $Y = 1$. We are therefore able to rank the equilibria of our model in terms of welfare, because, from the following corollary of Proposition 4.3, we can rank the equilibria in terms of the probability of trade conditional on $Y = 1$.

Corollary 4.4 *Let* (S, B) *be a continuous, nondecreasing equilibrium with* $V_0 = \sup\{B^{-1}(v_b)\}$. *Then* $S(\phi)$ *is strictly increasing in* V_0 *for all* ϕ, *while* $B(\phi)$ *is strictly decreasing in* V_0 *for all* $\phi > V_0$. *Consequently, the probability of trade conditional on* $Y = 1$ *is strictly decreasing in* V_0.

We turn to the study of particular equilibria, providing conditions for the existence of equilibria in strictly monotone undominated strategies. We also give sufficient conditions for the *failure* of such equilibria to exist. These welfare-maximizing equilibria will be our focus of attention in the subsequent sections of the paper.

From Proposition 4.3, the bidding policy B is not uniquely determined at types below $\sup\{B^{-1}(v_b)\}$, because agents with these types do not trade in equilibrium. Nevertheless, the equilibrium bidding policy B satisfying $B(\phi) = v_b$ whenever $\phi < V_0$ weakly dominates any other equilibrium bidding policy. That is, an agent whose type is below V_0 and who bids less than v_b can increase his bid to v_b, thereby increasing the probability of buying the asset, without affecting the price, which will be at most the lowest valuation v_b of the bidder. An equilibrium in strictly monotone undominated strategies is therefore only possible for the limit case, $V_0 = -\infty$. We now provide technical conditions supporting the existence of such a welfare-maximizing equilibrium.

We say that a function $g(\cdot)$ on the real line or the integers is of exponential type α at $-\infty$ if, for some constants $c > 0$ and $\gamma > -1$,

$$\lim_{x \to -\infty} \frac{g(x)}{|x|^{\gamma} e^{\alpha x}} = c. \tag{14}$$

In this case, we write $g(x) \sim \text{Exp}_{-\infty}(c, \gamma, \alpha)$. We use the notation $g(x) \sim \text{Exp}_{+\infty}(c, \gamma, \alpha)$ analogously for the case of $x \to +\infty$. Our results regarding strictly monotone bidding strategies, and thus full revealing equilibria, rely on the following technical regularity condition on the tails of the initial cross-sectional distribution of beliefs.

Condition 1 *There exists an $\alpha_- \geq 2.4$ and an $\alpha_+ > 0$ such that, for each class i, the initial type densities satisfy*

$$\frac{d}{dx}\psi_{i0}^{H}(x) \sim \text{Exp}_{-\infty}(c_{i,-}, \gamma_{i,-}, \alpha_-)$$

and

$$\frac{d}{dx}\psi_{i0}^{H}(x) \sim \text{Exp}_{+\infty}(c_{i,+}, \gamma_{i,+}, \alpha_+),$$

for some constants $c_{i,\pm} > 0$ and $\gamma_{i,\pm} \geq 0$.

In order to analyze the propagation of information, it is crucial that the exponential tail behavior of type densities is maintained at each successive exchange of information between two agents, when their type densities are both replaced by the convolution of their respective pre-trade type densities. This is ensured by the following lemma, demonstrated in Appendix C.

Lemma 4.5 *Suppose g_1 and g_2 are densities with $g_1(x) \sim \text{Exp}_{+\infty}(c_1, \gamma_1, -\alpha)$ and $g_2(x) \sim \text{Exp}_{+\infty}(c_2, \gamma_2, -\alpha)$. Then*

$$g_1 * g_2 \sim \text{Exp}_{+\infty}(c, \gamma, -\alpha),$$

where $\gamma = \gamma_1 + \gamma_2 + 1$ and

$$c = \frac{c_1\, c_2\, \Gamma(\gamma_1 + 1)\, \Gamma(\gamma_2 + 1)}{\Gamma(\gamma_1 + \gamma_2 + 2)}.$$

The analogous result applies to the asymptotic behavior of $g_1 * g_2$ at $-\infty$.

The following proposition follows.

Proposition 4.6 *Under Condition 1, for any* $t \geq 0$, *there exist* $\gamma_{it} > 0$ *and* $c_{it} > 0$ *such that* $\psi_{it}^H \sim \text{Exp}_{+\infty}(c_{it}, \gamma_{it}, -\alpha)$.

Proposition 4.6 follows from a straightforward application of Lemma 4.5 to the equilibrium type dynamics given by (6). Using Lemma 4.5, the tail parameters (c_{it}, γ_{it}) of the type densities can be characterized explicitly by a recursive procedure described in detail in Appendix C. As we show in Appendices E-F, for cases in which gains from trade are sufficiently large, equilibrium quantities can be approximated by relatively simple expressions involving only the parameters (c_{it}, γ_{it}), allowing us to analyze the percolation of information at this level of generality. This follows from the fact that when gains from trade are sufficiently large, an agent's equilibrium utility is dominated by terms corresponding to trades with "extreme" (very large and very small) types. There are two reasons for the important role of extreme types. First, because equilibrium bids and asks are increasing in type, trading with agents of an "opposite" type leads to the greatest realized trading gain. Second, trading with an agent whose type is larger in absolute value type has a greater impact on the post-trade type of the agent. The expected utility contributions of these extreme types are determined by the tail behavior of the type distribution, which are characterized by the coefficients $\gamma_{it} > 0$ and $c_{it} > 0$.

We now provide for the existence of a unique fully-revealing equilibrium, provided that the proportional gain from trade,

$$\overline{G} = \frac{v_b - v_s}{v^H - v_b}$$

is sufficiently high.

Theorem 4.7 *Suppose that the initial type densities satisfy Condition 1. Then there exists some* $\overline{g}$ *such that for any proportional gain from trade* $\overline{G} > \overline{g}$, *whenever a buyer of class* i *and a seller of class* j *meet at time* t, *there exists a unique strictly increasing continuous equilibrium, denoted* $(S_{ijt}(\,\cdot\,), B_{ijt}(\,\cdot\,))$. *This equilibrium is that characterized by Proposition 4.3 for the limit case* $V_0 = -\infty$. *In contrast, if* $\overline{G} < \alpha_+^{-1}$, *then a strictly increasing equilibrium does not exist.*

5 Endogenous Information Acquisition

A recurrent theme in the study of rational-expectations equilibria in centralized markets is that, absent noise in supply or demand, prices fully reveal the payoff relevant information held by investors. This leads to the well-known paradoxes of Grossman (1976) and

Beja (1976). If prices are fully revealing, then investors would avoid any costly gathering of private information, raising the question of how private information was ever incorporated into prices.

In our decentralized-market setting, because trade is private, it takes time for information to become incorporated into prices. Informed investors may therefore profitably invest in gathering information. Furthermore, incentives to gather information may improve if agents anticipate trading with more informed counterparties.

5.1 Setup

We suppose from this point that agents are endowed with, or endogenously acquire, information in the form of disjoint "packets" (subsets) of signals. These packets have a common type density[11] $\overline{\psi}^H$ conditional on $\{Y = 0\}$ and a common type density $\overline{\psi}^L$ conditional on $\{Y = 1\}$. We will be relying for simplicity on the following sufficient condition for Condition 1.

Condition 2 *For some* $\alpha \geq 1.4$ *and for some* $c_0 > 0$, $\frac{d}{dx}\overline{\psi}^H(x) \;\sim\; \mathrm{Exp}_{-\infty}(c_0, 0, \alpha + 1)$ *and* $\frac{d}{dx}\overline{\psi}^H(x) \;\sim\; \mathrm{Exp}_{+\infty}(c_0, 0, -\alpha)$.

All agents are initially endowed with $N_{\min} > 0$ signal packets. Before trade begins at time 0, each agent has the option to acquire up to $\overline{n}$ additional packets, at a cost of π per packet. Given the initial information acquisitions, whenever agents of classes i and j are in contact at some time t, trade is according to the unique fully-revealing bidding equilibria (B_{ijt}, S_{ijt}) characterized by Theorem 4.7.

We focus on symmetric equilibria, those in which agents of the same class acquire the same number of signal packets. Appendix H analyzes asymmetric equilibria. If agents of class i each acquire N_i signal packets, then the exact law of large numbers implies that the initial cross-sectional type density of this class is the $(N_{\min} + N_i)$-fold

[11]A deterministic finite set of signals has a discretely supported type distribution, so cannot have a type distribution with a density. Thus, the existence of density in this case implies that a packet must have a random selection of signals, which allows for a differentiable cumulative type distribution. For example, we can suppose that the set S of all signals is a non-atomic measure space, and that the cross-sectional distribution of types of S has the type density $\overline{\psi}$. Then we can suppose that a packet consists of one signal drawn randomly from S with the uniform distribution. By this construction, a packet has a random type with probability density $\overline{\psi}^H$ conditional on $Y = 0$ and probability density $\overline{\psi}^L$ conditional on $Y = 1$. This implies, by the exact law of large numbers, that the cross sectional distribution of the types of the set of agents who have received a total of k packets of information, for any integer $k \geq 1$, is almost surely the k-fold convolution $\overline{\psi}^{*k}$, which suffices for our application.

convolution of $\overline{\psi}^H$ on the event $\{Y = 0\}$, and the $(N_{\min} + N_i)$-fold convolution of $\overline{\psi}^L$ on the event $\{Y = 1\}$. This initial cross-sectional density, with two outcomes, is as usual denoted $\overline{\psi}^{*(N_{\min}+N_i)}$.

Consider an agent of class i who initially acquires n signal packets, and who assumes a given vector $N = (N_1, \ldots, N_M)$ of signal-packet acquisition quantities of each of the M classes. As a result, this agent has an information-type process denoted $\Theta_{n,N,t}$ at time t, and has the initial expected utility

$$u_i(n, N) = E\left(-\pi n + \sum_{t=1}^{T} \lambda_i \sum_j \kappa_{ij}\, v_{ijt}(\Theta_{n,N,t}; B_{ijt}, S_{ijt})\right), \tag{15}$$

where the gain v_{ijt} associated with a given sort of trading encounter is as defined by (9) or (11), depending on whether class-i agents are sellers or buyers, respectively.

In equilibrium, given the information acquisition decisions of other agents, each agent chooses a number of signal packets that maximizes this initial utility. We formalize this equilibrium concept as follows.

Definition 5.1 *A (symmetric) rational expectations equilibrium is: for each class i, a number N_i of acquired signal packets; for each time t and seller-buyer pair (i, j), a pair (S_{ijt}, B_{ijt}) of bid and ask functions; and for each class i and time t, a cross-sectional type density ψ_{it} such that:*

(1) The cross-sectional type density ψ_{it} is initially $\psi_{i0} = \overline{\psi}^{*(N_{\min}+N_i)}$ and satisfies the evolution equation (6).

(2) The bid and ask functions (S_{ijt}, B_{ijt}) form the equilibrium uniquely defined by Theorem 4.7.

(3) The number N_i of signal packets acquired by class i solves $\max_{n \in \{0,\ldots,\overline{n}\}} u_i(n, N)$.

We are now ready to characterize endogenous information acquisition in this setting.

5.2 Strategic Complementarity in Information Acquisition

We say that *information acquisition is a strategic complement* if, for any agent class i and any numbers n and $n' > n$ of signal packets that could be acquired, the utility gain $u_i(n', N) - u_i(n, N)$ is increasing in the assumed amounts $(N_1, \ldots, N_{i-1}, N_{i+1}, \ldots, N_M)$

of signal packets acquired by other classes of agents. The main result of this section is the following.

Theorem 5.2 *Suppose Condition 2 holds. There exist* $\overline{g}$ *and* $\widetilde{T} > 1$ *such that for any proportional gain from trade* $\overline{G} > \overline{g}$ *and market duration* $T > \widetilde{T}$*, information acquisition is a strategic complement.*

This theorem describes the effects of information acquisition by others on their bidding behavior and is responsible for most of the phenomena on which we will now focus.

In order to understand the general complementary result for large T, we note that the incentives to gather information in an OTC market are determined by two basic effects: a "learning effect" and an opposing "pricing effect." Specifically, for any class i of agents, a change in the information acquisition policy of some other class agents influences the gain $u_i(n', N) - u_i(n, N)$ from information acquisition via two channels: *(i)* the profit from trade with a counterparty of any given type changes because the double-auction equilibrium bid and ask prices change (the pricing effect), and *(ii)* the distribution of types of future trading counterparties changes (the learning effect).

As we show in Appendix G, the pricing effect always reduces the information gathering incentives of individual sellers and buyers as their counterparties become better informed. The intuition is that, in order to avoid missing unconditional expected gains from trade with buyers that are increasingly well informed, sellers find it optimal to reduce their ask prices. Therefore, with lower ask prices, a seller's expected gain from trade decreases, whereas buyers know that they will get a good price anyway, even without acquiring additional information. This distinction between buyers and sellers is a side effect of the seller's-price double auction.

The learning effect is more subtle. As agents acquire more information, the average quality of information is improved, but there is also an increased risk of receiving an unusually misleading bid. That is, the tails of the cross-sectional type densities become fatter *at both ends,* for some period of time. This is evident from Lemma 4.5. Specifically, as an agent gathers more and more signal packets, his or her type density is repeatedly convolved with the type density of a signal packet. By Lemma 4.5, the power γ correspondingly increases, reflecting greater fatness of both tails of the type density. In order to mitigate the cost of extreme, albeit unlikely, adverse selection, associated with ending up on the wrong tail of the type distribution, the learning effect leads agents to respond by gathering more information.

Because the percolation of information through bilateral trade occurs at an exponential rate, whereas the pricing effect does not depend as strongly on time, the learning effect dominates the pricing effect if there are sufficiently many trading rounds, thus providing enough opportunities for agents to exploit any information that they have acquired. For a sufficiently small number of trading rounds, the pricing effect dominates the learning effect.

The failure of strategic complementarity for small T arises from within-class externalities in the information acquisition game. If a given (non-trivial) subset of class-i agents acquire a lot of information, then their counterparties (agents from opposite classes) will eventually learn a lot from them. However, because information percolates at a finite speed, this learning effect is weak when T is small, and dominates only when T is sufficiently large. As Theorem 5.2 shows, this provides an incentive for any class-i agent to acquire more information. By contrast, if T is small enough, this learning effect is weak, and information acquisition by agents within the same class i leads only to stronger adverse selection in future trades (because, for any given agent of class i, his counterparties, anticipating that he is better informed, offer him worse prices). This makes within-class information acquisition a strategic substitute if T is small enough, and may lead to the absence of any equilibria.

Under strategic complementarity, it follows that the amount of information optimally acquired by a given agent is increasing in the amount of information that is conjectured to be acquired by other agents. These complementarity effects are responsible for the existence or non-existence[12] of equilibria for market durations that are above or below some market-duration threshold $\widetilde{T}$. The Tarski (1955) fixed point theorem implies the following result.

Corollary 5.3 *Suppose Condition 2 holds. For any proportional gain from trade $\overline{G} > \overline{g}$ and market duration $T > \widetilde{T}$, there exists a symmetric equilibrium. Furthermore, the set of equilibria is a lattice with respect to the natural partial order on $\mathbb{Z}_+^M$, corresponding to the number of packets acquired by the M agent classes.*

Thus, we can select a maximal and a minimal element of the set of equilibria. These equilibria can be easily constructed by means of a standard iteration procedure. Namely, consider the map from conjectured information acquisition policies to the corresponding optimal responses. Since, by Theorem 5.2, this map is monotone increasing, iterations

[12] It is well known that, in a game in which players' choice are discrete and are strategic substitutes, pure strategy equilibria may fail to exist.

of this map applied to the initial condition $n_i = \bar{n}$ for all i are monotone decreasing in the number of iterations and converge to the maximal equilibrium. Similarly, iterations of this map applied to the initial condition $n_i = 0$ for all i converge to the minimal equilibrium.[13]

Definition 5.4 *We say that equilibrium information acquisition is increasing in a given ordered set of parameters if, for any parameters α_1 and $\alpha_2 > \alpha_1$, the following are true:*

- *For any equilibrium information acquisition policy vector $N(\alpha_1)$ corresponding to α_2 there exists an equilibrium policy vector $N(\alpha_2)$ corresponding to α_2 and such that $N(\alpha_1) \leq N(\alpha_2)$.*
- *For any equilibrium information acquisition policy vector $N(\alpha_2)$ corresponding to α_1 there exists an equilibrium policy vector $N(\alpha_1)$ corresponding to α_1 and such that $N(\alpha_1) \leq N(\alpha_2)$.*

The following result characterizes equilibrium incentives for information gathering.

Proposition 5.5 *Suppose Condition 2 holds. Suppose also that $\kappa_{ij} > 0$ for all pairs of buyer and seller classes. There exist $\overline{g}$ and $\widetilde{T}$ such that for any proportional gain from trade $\overline{G} > \overline{g}$ and market duration $T > \widetilde{T}$, the following are true.*

1. *Equilibrium information acquisition is increasing in each of $N_{\min}$, $\bar{n}$, and a common rescaling of the matching probabilities $(\lambda_1, \ldots, \lambda_M)$.*
2. *There exists a threshold $\bar{T} > \widetilde{T}$ such that, for any market duration $T > \bar{T}$, equilibrium information acquisition is increasing in the matching probability λ_i, for any class i.*

The intuition here is analogous to that behind Theorem 5.2. Namely, an increase in meeting intensities increases the speed of information percolation and hence makes

[13] We have established complementarity only with respect to the information-acquisition policies of other classes. Within-class information acquisition may be a strategic substitute. Consider a map that, for each class i, takes as given the information acquisition policies $\{N_j : j \neq i\}$ of the other classes, and picks the largest "within-class equilibrium" response, which would be an equilibrium within class i, fixing these choices of other classes. We show in Appendix G that the "within-class equilibrium" always exists. Hence, the map is well defined and has the monotonicity properties on which we rely. Iterations converge to the largest equilibrium. Similarly, choosing the map that picks the smallest "within-class equilibrium" and iterates leads to the smallest equilibrium.

traders better informed. In order for this learning effect to dominate, we need sufficiently many ($T > \bar{T}$) trading rounds.

This result illustrates the role of cross-class externalities. Even if a class j does not trade with some class i, an increase in the trading intensity of class i increases the information acquisition incentives for class j. This is a "pure" learning externality in that, if class i trades more frequently, the additional information acquired will eventually percolate to the trading counterparties of class j. This encourages class j to acquire more information. This requires an ordered path $\{k_1, \ldots, k_m\}$ of classes connecting class $i = k_1$ with class $j = k_m$ via the property that the counterparty selection probability $\kappa_{k_\ell k_{\ell+1}}$ is non-zero for all $\ell < m$.

If the market duration is moderate in the sense of this Theorem (that is, $T \in (\widetilde{T}, \bar{T})$), then the learning effect may not be strong enough. That is, improving the connectivity of some investor classes may asymmetrically affect the information gathering incentives of other classes of investors, leading to a lower amount of equilibrium information acquisition. We explore this potential in the next sub-section.

5.3 An Illustrative 3-Class Example

We now illustrate our general results with a three-class example. Classes 1 and 2 are sellers. Class 3 consists of buyers. The seller classes have contact probabilities λ_1 and λ_2, respectively. Without loss of generality, $\lambda_2 \geq \lambda_1$. The evolution equations (6) for the cross-sectional distributions of information types are then entirely determined by the fractions of the populations that are sellers of each class.[14] We assume that these fractions are the same, $\bar{m} \in (0, 1)$. As explained in Section 3.2, this implies that the contact probability of a buyer is $(\lambda_1 + \lambda_2)\bar{m}/(1 - 2\bar{m})$. We take $\bar{m} = 0.25$, so that the buyer contact probability is the simple average of the seller contact probabilities.[15] In a symmetric equilibrium, sellers of classes 1 and 2 acquire N_1 and N_2 packets of signals, respectively. Buyers acquire N_b signal packets.

[14] That is, $\psi_{i,t+1} = (1 - \lambda_i)\,\psi_{it} + \lambda_i\,\psi_{it} * \psi_{bt}, \quad i \in \{1, 2\}$, where "$b$" denotes buyers. The buyers' contact probabilities and the evolution equation of the cross-sectional distribution of the buyers' information types are then determined by the population masses of the two seller classes. Specifically, letting m_i denote the mass of the population consisting of class-i sellers, the exact law of large numbers for random matching implies that the total quantity of contacts of buyers by sellers in a given period is $\lambda_1 m_1 + \lambda_2 m_2$. Thus, the probability that a given buyer is contacted is $\lambda_b = (\lambda_1 m_1 + \lambda_2 m_2)/(1 - m_1 - m_2)$. Likewise, because $\lambda_i m_i = \lambda_b \kappa_{bi} m_b$, we know that, conditional on the event that a buyer contacts a seller, the probability that the seller is of type i must be $\kappa_{bi} = \lambda_i m_i/(\lambda_1 m_1 + \lambda_2 m_2)$.

[15] This implies that $\kappa_{bi} = \lambda_i/(\lambda_1 + \lambda_2)$. Thus, $\psi_{b,t+1} = (1 - 0.5(\lambda_1 + \lambda_2))\,\psi_{bt} + 0.5(\lambda_1\,\psi_{1t} + \lambda_2\,\psi_{2t}) * \psi_{bt}$.

Proposition 5.6 *Suppose Condition 2 holds, $T > \widetilde{T}$, and $\overline{G} > \overline{g}$, for the time $\widetilde{T}$ and the proportional-gain-from-trade threshold $\overline{g}$ of Theorem 5.2. Then symmetric equilibria exist. There are at most three distinct such equilibria. In any symmetric equilibrium, $N_1 \leq N_2$.*

The fact that better connected sellers acquire more information ($N_1 \leq N_2$) follows from the fact that they can "amortize" the cost of the information over a higher expected number of trading opportunities. This result extends to the general class of segmented market models studied in this paper.

Our final result states that raising market contact rates can *reduce* equilibrium information gathering. This can be explained as follows. As class-2 sellers become more active, buyers learn at a faster rate. The impact of this on the incentive of the "slower" class-1 sellers to gather information is determined by a "learning effect" and an opposing "pricing effect."

The pricing effect works in this three-class example as follows. Buyers become increasingly well informed by class-2 sellers as λ_2 increases, and therefore class-1 sellers find it optimal to reduce their ask prices. This reduces the information gathering incentives of individual sellers and buyers, given the information acquisition decisions of other agents.

Furthermore, as λ_2 rises, class-2 sellers transport more information through the market, giving rise to a learning effect. As we have explained, the pricing effect is stronger for small T, whereas the learning effect eventually dominates for $T > \bar{T}$. This leads to the following result.

Proposition 5.7 *Suppose Condition 2 holds and let $\overline{g}$ and $\widetilde{T}$ be chosen as in Theorem 5.2. If the proportional gain $\overline{G}$ from trade is larger than $\overline{g}$, then there is some time $\hat{T} \in (\widetilde{T}, \bar{T})$ such that, if $\widetilde{T} < T < \hat{T}$, the following is true. There exists an information-cost threshold $\mathcal{K}$, depending on only the primitive model parameters λ_1, λ_2, $N_{\min}$, $\bar{n}$, and $\overline{\psi}$, such that there is an equilibrium in which a non-zero fraction of agents acquire a non-zero number of signal packets if and only if the cost π per signal packet is less than or equal to $\mathcal{K}$. This cost threshold $\mathcal{K}$ is strictly monotone decreasing in λ_2. Thus, if $\pi < \mathcal{K}$ is sufficiently close to $\mathcal{K}$, then increasing λ_2 leads to a full collapse of information acquisition, in the sense that the fraction of agents choosing to acquire signals is zero.*

Proposition 5.7 implies that increasing market contact rates does not necessarily lead to more informative markets. It is instructive to compare with the case of a static double auction, corresponding to $T = 0$. With only one round of trade, the learning effect is absent and the expected gain from acquiring information for class-1 sellers is proportional to λ_1 and does not depend on λ_2. Similarly, the gain from information acquisition for buyers is linear and increasing in λ_2. Consequently, in the static case, an increase in λ_2 never leads to less information acquisition in equilibrium, in contrast to the conclusion of Proposition 5.7.

This result is not an artifact of the artificially fixed amount of the asset exchanged at each auction, which might suggest a role for increasing T through increasing the total amount of the asset traded. Indeed, our result holds even if we scale the amount of the asset exchanged by the number T of rounds of trade.

5.4 Further Remarks on Endogenous Information Gathering

Appendix K considers a setting in which information can be gathered dynamically, based on learning over time. Under natural conditions, in equilibrium an agent acquires an additional signal packet at some time t provided that the agent's current information is sufficiently imprecise, meaning that the agent's type is neither high enough (above some deterministic upper threshold depending on t), nor low enough (below some deterministic threshold depending on t). At intermediate types, the value of more information (through the net effect of price and probability of trade) exceeds the cost.

Given our results, it is natural to address the impact on information gathering incentives of public transactions reporting, as currently proposed for U.S. over-the-counter derivatives markets. The results of Duffie, Manso, Malamud (2011) imply that, under natural conditions, the public reporting of a (random) selection of individual bids and offers would speed the rate of convergence of the cross-sectional distribution of beliefs to perfect information by the mean rate at which these quotes are published. Public quote reporting would generally reduce the number of welfare-improving trades that do not occur because of adverse selection. Public reporting could, however, depending on the parameters of our model, either raise or lower the incentives to gather costly information. For the limit case in which all quotes are reported publicly, all agents would instantly learn all payoff-relevant information, and the incentive to gather fundamental information would completely disappear, returning us to the setting of the Beja-Grossman-Stiglitz "paradox." It also bears mentioning that in a setting with risk aversion, the "Hirshleifer effect" implies that delaying or suppressing the publication of quotes may also benefit risk sharing.[16] In light of the above discussion, an interesting direction for future research is to study the welfare implications of public information releases in decentralized markets.[17]

In Appendix I we endogenize information gathering by instead allowing the agents to increase, at a cost, their matching probabilities with other classes of agents. This can be viewed as "endogenous network formation." As we show, in equilibrium agents decide to target their search efforts toward classes of counterparties that are expected to be better informed. These better informed classes of agents thus endogenously arise as "hubs" in the matching structure, and, as consequence, accumulate even more information.

[16]See Hirshleifer (1971).

[17]In centralized markets, Vives (2011) finds that releasing public information on the common component of value improves total surplus. In other social learning settings, previous research has found that public information releases may crowd out learning (e.g. Burguet and Vives, 2000; Morris and Shin, 2005; Duffie, Malamud, and Manso, 2009; Amador and Weill, 2010; Colombo, Femminis, and Pavan, 2014).

Appendices

The appendices are organized as follows. Appendices D through L are found in a supplement, Duffie, Manso, and Malamud (2013).

A. A characterization of all possible initial type distributions.

B. A derivation of the ordinary differential equation for the equilibria of the seller's-price double auction.

C. A demonstration of the exponential tail behavior of the cross-sectional type distributions, as necessary to prove the existence of double-auction equilibria..

D. A proof of the existence and uniqueness of a strictly monotone equilibrium of the double auction.

E. An application of results from Appendix D to obtain approximations for the double-auction equilibrium for the case of large gains from trade.

F. An application of the results of Appendix E to derive approximations for the expected trading profits for the case of large gains from trade.

G. An application of the results of Appendix F to approximate the expected gains from information acquisition for the case of large gains from trade. From this, a derivation of general properties of equilibrium information acquisition. Appendices G.1 and G.2 apply these results to special cases of one and two classes of sellers, respectively.

H. An analysis of mixed-strategy equilibria for the special case of two classes.

I. Results on endogenous investment in matching technology.

J. Proofs of results in Section I, on endogenous investment in matching technology.

K. Results for the case of dynamic information acquisition.

L. Proofs of results in Appendix K.

Table 1 indicates the locations of proofs of all major results from the text of the paper.

Table 1: Directory of Locations of Proofs of Main Results. Appendices D through L are located in an online supplement, Duffie, Manso, and Malamud (2013).

Result	Proof location
Proposition 3.2	Appendix A
Lemma 4.1	Appendix A, Lemma A.1
Lemma 4.2	Appendix B
Proposition 4.3	Appendix B
Theorem 4.7	Appendix D
Theorem 5.2	Appendix G
Proposition 5.5	Appendix G
Proposition 5.6	Appendix G, Proposition G.12
Proposition 5.7	Appendix G
Theorem I.2	Appendix J
Proposition I.4	Appendix J
Theorem I.5	Appendix J

A Amenable Cross-Sectional Information Type Densities

This appendix provides for the existence and properties of cross-sectional densities of information types that are obtained from signals that are Y-conditionally independent.

Proof of Proposition 3.2. First, we say that a pair (F^H, F^L) of cumulative distribution functions (CDFs) on the real line is *amenable* if

$$dF^H(y) \;=\; e^y\, dF^L(y)\,. \tag{16}$$

It follows directly from (20) and (21) below that if F^H and F^L correspond to the conditional type distributions associated with a single fixed signal, then they are amenable. The claim then follows from Fact 1, stated and proved below. ■

The claim of Lemma 4.1 is contained in the following lemma.

Lemma A.1 *For each agent class i and time t, the type density ψ_{it} satisfies the hazard-rate ordering, $h^H_{it}(\theta) \geq h^L_{it}(\theta)$, and $\psi^H_{it}(x) \;=\; e^x\,\psi^L_{it}(x)$. If, in addition, each signal Z satisfies*

$$\mathbb{P}(Z = 1 \,|\, Y = 0) + \mathbb{P}(Z = 1 \,|\, Y = 1) = 1, \tag{17}$$

then

$$\psi^H_{it}(x) = e^x \psi^H_{it}(-x), \qquad \psi^L_{it}(x) \;=\; \psi^H_{it}(-x), \qquad x \in \mathbb{R}. \tag{18}$$

Proof of Lemma A.1. First, we say that a pair (F^H, F^L) of cumulative distribution functions (CDFs) on the real line is *symmetric amenable* if

$$dF^L(y) = dF^H(-y) = e^{-y}\, dF^H(y), \tag{19}$$

that is, if for any bounded measurable function g,

$$\int_{-\infty}^{+\infty} g(y)\, dF^L(y) = \int_{-\infty}^{+\infty} g(-y)\, dF^H(y) = \int_{-\infty}^{+\infty} e^{-y} g(y)\, dF^H(y).$$

It is immediate that the sets of amenable and symmetric amenable pairs of CDFs is closed under mixtures, in the following sense.

Fact 1. *Suppose $(A, \mathcal{A}, \eta)$ is a probability space and $F^H : \mathbb{R}\times A \to [0,1]$ and $F^L : \mathbb{R}\times A \to [0,1]$ are jointly measurable functions such that, for each α in A, $(F^H(\,\cdot\,,\alpha), F^L(\,\cdot\,,\alpha))$ is an amenable (symmetric amenable) pair of CDFs. Then an amenable (symmetric amenable) pair of CDFs is defined by $(\overline{F}^H, \overline{F}^L)$, where*

$$\overline{F}^H(y) = \int_A F^H(y,\alpha)\, d\eta(\alpha), \qquad \overline{F}^L(y) = \int_A F^L(y,\alpha)\, d\eta(\alpha).$$

The set of amenable (symmetric amenable) pairs of CDFs is also closed under finite convolutions.

Fact 2. *Suppose that $X_1, \ldots, X_n$ are independent random variables and $Y_1, \ldots, Y_n$ are independent random variables such that, for each i, the CDFs of X_i and Y_i are amenable (symmetric amenable). Then the CDFs of $X_1 + \cdots + X_n$ and $Y_1 + \cdots + Y_n$ are amenable (symmetric amenable).*

For a particular signal Z with type θ_Z, let F_Z^H be the CDF of θ_Z conditional on $Y = 0$, and let F_Z^L be the CDF of θ_Z conditional on $Y = 1$.

Fact 3. *If (F_Z^H, F_Z^L) is an amenable pair of CDFs and if Z satisfies (17), then (F_Z^H, F_Z^L) is a symmetric amenable pair of CDFs.*

In order to verify Fact 3, we let θ be the outcome of the type θ_Z on the event $\{Z = 1\}$, so that

$$\theta = \log \frac{\mathbb{P}(Y = 0 \mid Z = 1)}{\mathbb{P}(Y = 1 \mid Z = 1)} - \log \frac{\mathbb{P}(Y = 0)}{\mathbb{P}(Y = 1)} = \log \frac{\mathbb{P}(Z = 1 \mid Y = 0)}{\mathbb{P}(Z = 1 \mid Y = 1)}.$$

Further, let

$$\tilde{\theta} \;=\; \log \frac{\mathbb{P}(Y=0 \mid Z=0)}{\mathbb{P}(Y=1 \mid Z=0)} - \log \frac{\mathbb{P}(Y=0)}{\mathbb{P}(Y=1)} = \log \frac{\mathbb{P}(Z=0 \mid Y=0)}{\mathbb{P}(Z=0 \mid Y=1)}$$

be the outcome of the type θ_Z on the event $\{Z=0\}$. Then,

$$F_Z^H \;=\; \frac{e^{\theta} - e^{\theta+\tilde{\theta}}}{e^{\theta} - e^{\tilde{\theta}}}\, 1_{\theta \le y} \;+\; \frac{e^{\theta+\tilde{\theta}} - e^{\tilde{\theta}}}{e^{\theta} - e^{\tilde{\theta}}}\, 1_{\tilde{\theta} \le y} \tag{20}$$

and

$$F_Z^L \;=\; \frac{1 - e^{\tilde{\theta}}}{e^{\theta} - e^{\tilde{\theta}}}\, 1_{\theta \le y} \;+\; \frac{e^{\theta} - 1}{e^{\theta} - e^{\tilde{\theta}}}\, 1_{\tilde{\theta} \le y}. \tag{21}$$

The amenable property (16) is thus satisfied.

If Z satisfies (17), then $-\theta$ is the outcome of θ_Z associated with observing $Z=0$, so

$$F_Z^H(y) = \frac{e^{\theta}}{1+e^{\theta}}\, 1_{\{\theta \le y\}} \;+\; \frac{1}{1+e^{\theta}}\, 1_{\{-\theta \le y\}}$$

and

$$F_Z^L(y) = \frac{1}{1+e^{\theta}}\, 1_{\{\theta \le y\}} \;+\; \frac{e^{\theta}}{1+e^{\theta}}\, 1_{\{-\theta \le y\}}.$$

These CDFs are each piece-wise constant, and jump only twice, at $y=-\theta$ and $y=\theta$. We let $\Delta F(y) = F(y) - \lim_{z \uparrow y} F(z)$. At $y=-\theta$ and $y=\theta$, we have $\Delta F_Z^H(-y) = e^{-y} \Delta F_Z^H(y)$ and $\Delta F_Z^L(y) = \Delta F_Z^H(-y)$, completing the proof of Fact 3.

Now, we recall that a particular agent receives at time 0 a random finite set, say N, of signals, where N is independent of all else, and can depend on the agent. The type of the set of signals received by the agent is, by Lemma 3.1, the sum of the types of the individual signals. Thus, conditional on N, the type θ of this agent's signal set has a CDF conditional on $Y=0$, denoted F_N^H, and a CDF conditional on $Y=1$, denoted F_N^L, that are the convolutions of the conditional distributions of the underlying set N of signals given $Y=0$ and given $Y=1$, respectively. Thus, by Fact 2, conditional on N, (F_N^H, F_N^L) is an amenable pair of CDFs. Now, we can average these CDFs over the distribution of the outcome of N to see by Fact 1 that this agent's type has CDFs given $Y=0$ and $Y=1$, respectively, that are amenable. Under (17), we likewise have symmetric amenability, using also Fact 3.

Now, let us consider the cross-sectional distribution of agent types of a given class i at time 0, across the population. Recall that the agent space is the measure space $(G, \mathcal{G}, \gamma)$. Let γ_i denote the restriction of γ to the subset of class-i agents, normalized by the total mass of this subset. Because of the exact law of large numbers of Sun (2006), we have, almost surely, that on the event $Y = 0$, the fraction $\gamma_i(\{\alpha : \theta_{\alpha 0} \leq y\})$ of class-i agents whose types are less than a given number y is

$$F^H(y) \equiv \int_G F^H_\alpha(y)\, d\gamma_i(\alpha),$$

where F^H_α is the conditional CDF of the type $\theta_{\alpha 0}$ of agent α given $Y = 0$. We similarly define F^L as the cross-sectional distribution of types on the event $Y = 1$. Now, by Fact 1, (F^H, F^L) is an amenable pair of CDFs. By assumption, these CDFs have densities denoted ψ^H_{i0} and ψ^L_{i0}, respectively, for class i. The definition (19) of symmetric amenability also implies, under (17), that

$$\psi^L_{i0}(y) \;=\; \psi^H_{i0}(-y) \;=\; \psi^H_{i0}(y)\, e^{-y}\,,$$

as was to be demonstrated. That the densities $(\psi^H_{it}, \psi^L_{it})$ correspond for any t to amenable CDFs, and symmetric amenable CDFs under (17), follows from the facts that amenability and symmetric amenability are preserved under convolutions (Fact 2) and mixtures (Fact 1). That the hazard-rate ordering property is satisfied for any density satisfying (5) follows from the calculation (suppressing subscripts for notational simplicity):

$$\frac{G^L(x)}{\psi^L(x)} \;=\; \frac{\int_x^{+\infty} \psi^L(y)\, dy}{\psi^L(x)} \;=\; \frac{\int_x^{+\infty} \psi^H(y)\, e^{(x-y)}\, dy}{\psi^H(x)} \;\leq\; \frac{\int_x^{+\infty} \psi^H(y)\, dy}{\psi^H(x)} \;=\; \frac{G^H(x)}{\psi^H(x)}\,.$$

This completes the proof of the lemma. ■

Lemma A.2 *For any amenable pair (F^H, F^L) of CDFs, there exists some initial allocation of signals such that the initial cross-sectional type distribution is F^H almost surely on the event $H = \{Y = 0\}$ and F^L almost surely on the event $L = \{Y = 1\}$.*

Proof. Since

$$1 \;=\; \int_{\mathbb{R}} dF^L(x) \;=\; \int_{\mathbb{R}} e^{-x}\, dF^H(x),$$

it suffices to show that any CDF F^H satisfying

$$\int_{\mathbb{R}} e^{-x}\, dF^H(x) = 1 \tag{22}$$

can be realized from some initial allocation of signals.

Suppose that initially each agent is endowed with one signal Z, but $X_1 = P(Z = 1 \,|\, Y = 0)$ and $X_2 = P(Z = 1 \,|\, L)$ are distributed across the population according to a joint probability distribution $d\nu(x_1, x_2)$ on $(0,1) \times (0,1)$. We denote by $F^H_{d\nu}$ the corresponding type distribution conditioned on state H. The case when ν is supported on one point corresponds to the case of identical signal characteristics across agents, in which case $F^H = F^H_{\theta,\tilde{\theta}}$ is given by (20). Furthermore, any distribution supported on two points $\theta, \tilde{\theta}$ and satisfying (22) is given by (20). We will now show that any distribution F^H supported on a finite number of points can be realized. To this end, we will show that any such distribution can be written down as a convex combination of distributions of $F^H_{\theta,\tilde{\theta}}$,

$$F^H = \sum_i \alpha_i \, F^H_{\theta_i,\tilde{\theta}_i}\,.$$

In this case, picking

$$d\nu = \sum_i \alpha_i \, \delta_{(x_1^i, x_2^i)}$$

to be a convex combination of delta-functions with

$$x_1^i = \frac{e^{\theta_i} - e^{\theta_i + \tilde{\theta}_i}}{e^{\theta_i} - e^{\tilde{\theta}_i}}\,, \qquad x_2^i = \frac{1 - e^{\tilde{\theta}}}{e^{\theta} - e^{\tilde{\theta}}}\,,$$

we get the required result.

Fix a finite set $S = \{\theta_1, \ldots, \theta_K\}$ and consider the set $\mathcal{L}$ of probability distributions with support S that satisfies (22). If we identify a distribution with the probabilities $p_1, \ldots, p_K$ assigned to the respective points in S, then $\mathcal{L}$ is isomorphic to the compact subset of $(p_1, \ldots, p_K) \in \mathbb{R}_+^K$, satisfying

$$\sum_i p_i = 1, \qquad \sum_i e^{-\theta_i} p_i = 1\,.$$

Because this compact set is convex, the Krein-Milman Theorem (see Krein and Milman (1940)) implies that it coincides with the convex hull of its extreme points. Thus, it suffices to show that the extreme points of this set coincide with the measures, supported on two points. Indeed, pick a measure $\pi = (p_1, \ldots, p_K)$, supported on at least three points. Without loss of generality, we may assume that p_1, p_2, $p_3 > 0$. Then, we can pick an $\varepsilon > 0$ such that

$$p_1 \pm \varepsilon > 0\ ,\ p_2 \pm \varepsilon \frac{e^{-\theta_1} - e^{-\theta_3}}{e^{-\theta_3} - e^{-\theta_2}} > 0\,,\ p_3 \pm \varepsilon \frac{e^{-\theta_1} - e^{-\theta_2}}{e^{-\theta_3} - e^{-\theta_2}} > 0\,.$$

Then, clearly,

$$\pi = \frac{1}{2}(\pi^+ + \pi^-)$$

with

$$\pi^{\pm} = \left(p_1 \pm \varepsilon\,,\ p_2 \pm \varepsilon \frac{e^{-\theta_1} - e^{-\theta_3}}{e^{-\theta_3} - e^{-\theta_2}}\,,\ p_3 \pm \varepsilon \frac{e^{-\theta_1} - e^{-\theta_2}}{e^{-\theta_3} - e^{-\theta_2}}, p_4, \ldots, p_K\right)\,.$$

By direct calculation, π^+ and π^- correspond to measures in $\mathcal{L}$. Thus, all extreme points of $\mathcal{L}$ coincide with measures, supported on two points and the claim follows.

Now, clearly, for any measure F^H satisfying (22) there exists a sequence F_n^H of measures, supported on a finite number of points, converging weakly to F^H. By the just proved result, for each F_i^H there exists a measure $d\nu_i$ on $(0,1)\times(0,1)$, such that $F_i^H = F_{d\nu_i}^H$. By Helly's Selection Theorem (Gut (2005), p. 232, Theorem 8.1), the set of probability measures on $(0,1)\times(0,1)$ is weakly compact and therefore there exists a subsequence of ν_i converging weakly to some measure ν. Clearly, $F^H = F_\nu^H$ and the proof is complete. ■

B The ODE for the Double-Auction Equilibrium

Everywhere in the sequel we will for simplicity assume that the tail characteristics from Condition 1 satisfy $c_{0,\pm} \equiv c_0$, $\gamma_\pm \equiv \gamma$ and $\alpha_- = \alpha_+ + 1 \equiv \alpha + 1$.

This appendix analyzes the ordinary differential equation determining equilibrium bidding strategies for the seller's price double auction.

Proof of Lemma 4.2. By the assumptions made, the right-hand side of equation (12) is Lipschitz-continuous, so local existence and uniqueness follow from standard results. To prove the claim for finite V_0, it remains to show that the solution does not blow up for $z < v^H$. By Lemma 4.1,

$$\frac{1}{h_{bt}^H(V_b(z))} \;\geq\; \frac{1}{h_{bt}^L(V_b(z))},$$

and therefore

$$\begin{aligned} V_b'(z) \;&=\; \frac{1}{v_b - v_s}\left(\frac{z - v_b}{v^H - z}\,\frac{1}{h_{bt}^H(V_b(z))} + \frac{1}{h_{bt}^L(V_b(z))}\right) \\ &\leq\; \frac{1}{h_{bt}^H(V_b(z))}\,\frac{v^H - v_b}{(v_b - v_s)\,(v^H - z)}\,. \end{aligned} \tag{23}$$

For notational parsimony, in the remainder of this proof we write "G_H" and "G_L" for G_{bt}^H and G_{bt}^L respectively. Thus we have

$$\frac{d}{dz}\left(-\log G_H(V_b(z))\right) \;\leq\; \frac{v^H - v_b}{(v_b - v_s)\,(v^H - z)}.$$

Integrating this inequality, we get

$$\log\left(\frac{G_H(V_0)}{G_H(V_b(z))}\right) \;\leq\; \frac{v^H - v_b}{v_b - v_s}\log\frac{v^H - v_b}{v^H - z}.$$

That is,

$$G_H(V_b(z)) \;\geq\; G_H(V_0)\,\left(\frac{v^H - z}{v^H - v_b}\right)^{\frac{v^H - v_b}{v_b - v_s}},$$

or equivalently,

$$V_b(V_0, z) \;\leq\; G_H^{-1}\left(G_H(V_0)\,\left(\frac{v^H - z}{v^H - v_b}\right)^{\frac{v^H - v_b}{v_b - v_s}}\right).$$

Similarly, we get a lower bound

$$V_b(V_0, z) \;\geq\; G_L^{-1}\left(G_L(V_0)\,\left(\frac{v^H - z}{v^H - v_b}\right)^{\frac{v^H - v_b}{v_b - v_s}}\right). \tag{24}$$

The fact that V_b is monotone increasing in V_0 follows from a standard comparison theorem for ODEs (for example, (Hartman (1982), Theorem 4.1, p. 26). Furthermore, as $V_0 \to -\infty$, the lower bound (24) for V_b converges to

$$G_L^{-1}\left(\left(\frac{v^H - z}{v^H - v_b}\right)^{\frac{v^H - v_b}{v_b - v_s}}\right).$$

Hence, V_b stays bounded from below and, consequently, converges to some function $V_b(-\infty, z)$. Since $V_b(V_0, z)$ solves the ODE (12) for each V_0 and the right-hand side of (12) is continuous, $V_b(-\infty, z)$ is also continuously differentiable and solves the same ODE (12). ■

Proof of Proposition 4.3. Suppose that (S, B) is a non-decreasing continuous equilibrium and let $V_s(z)$, $V_b(z)$ be the corresponding (strictly increasing and right-continuous) inverse functions defined on the intervals (a_1, A_1) and (a_2, A_2) respectively, where one or both ends of the intervals may be infinite.

The optimization problems for auction participants are

$$\max_s f_S(s) \equiv \max_s \int_{V_b(s)}^{+\infty} \left(s - v_s - \Delta_s P(\theta + \phi)\right)\, \Psi_b(P(\theta), \phi)\, d\phi \tag{25}$$

and

$$\max_b f_B(b) \;\equiv\; \max_b \int_{-\infty}^{V_s(b)} \big(v_b \,+\, \Delta_b P(\theta + \phi) \,-\, S(\theta)\big)\, \Psi_s(P(\phi), \theta)\, d\theta. \tag{26}$$

First, we note that the assumption that $A_1 \leq v^H$ implies a positive trading volume. Indeed, by strict monotonicity of S, there is a positive probability that the selling price is below v^H. Therefore, for buyers of sufficiently high type, it is optimal to participate in trade.

In equilibrium, it can never happen that the seller trades with buyers of all types. Indeed, if that were the case, the seller's utility would be

$$\int_{\mathbb{R}} (s - v_s - \Delta_s P(\theta + \phi)) \; \Psi_b(P(\theta), \phi) \, d\phi,$$

which is impossible because the seller can then attain a larger utility by increasing s slightly. Thus, $a_1 \;\geq\; a_2$. Furthermore, given the assumption $S \leq v^H$, buyers of sufficiently high types find it optimal to trade with sellers of arbitrarily high types. That is, $A_2 \;=\; \sup_\theta B(\theta) \;\geq\; \sup_\theta S(\theta) \;=\; A_1$. Thus,

$$A_2 \;\geq\; A_1 \;>\; a_1 \;\geq\; a_2.$$

Let $\theta_l \;=\; V_b(a_1),\; \theta_h \;=\; V_b(A_1)$. (Each of these numbers might be infinite if either $A_2 = A_1$ or $a_2 = a_1$.) By definition, $V_s(a_1) = -\infty,\; V_s(A_1) = +\infty$. Furthermore, $f_B(b)$ is locally monotone increasing in b for all b such that

$$v_b \;+\; \Delta_b P(V_s(b) + \phi) \;-\; S(V_s(b)) \;>\; 0.$$

Further, $f_B(b)$ is locally monotone decreasing in b if

$$v_b \;+\; \Delta_b P(V_s(b) + \phi) \;-\; S(V_s(b)) \;<\; 0.$$

Hence, for any type $\phi \in (\theta_l, \theta_h)$, $B(\phi)$ solves the equation

$$v_b \;+\; \Delta_b \, P(V_s(B(\phi)) \;+\; \phi)) \;=\; B(\phi).$$

Letting $B(\phi) = z \in (a_1, A_1)$, we get that

$$v_b \;+\; \Delta_b \, P(V_s(z) \;+\; V_b(z)) \;=\; z\,. \tag{27}$$

Now, as $\phi \uparrow \theta_h$, we have $B(\phi) \uparrow A_1$ and therefore $V_s(B(\phi)) \uparrow +\infty$. Thus,

$$A_1 \;=\; \lim_{\phi \uparrow \theta_h} B(\phi) \;=\; \lim_{\phi \uparrow \theta_h} (v_b \;+\; \Delta_b \, P(V_s(B(\phi)) \;+\; \phi))) \;=\; v^H,$$

and similarly, $a_1 \;=\; v_b$

We now turn to the first-order condition of the seller. Because V_b is strictly increasing, it is differentiable Lebesgue-almost everywhere by the Lebesgue Theorem (for example, Theorem 7.2 of Knapp (2005), p. 359). Let $X \subset (a_2, A_2)$ be the set on which V_b'

exists and is finite. Then, for all $\theta \in V_s(X)$ the first-order condition holds for the seller. For a seller of type θ, because the offer price s affects the limit of the integral defining the seller's utility (9) as well as the integrand, there are two sources of marginal utility associated with increasing the offer s: (i) losing the gains from trade with the marginal buyers, who are of type $B^{-1}(s)$), and (ii) increasing the gain from every infra-marginal buyer type ϕ. At an optimal offer $S(\theta)$, these marginal effects are equal in magnitude. This leaves the seller's first-order condition

$$\Gamma_b(P(\theta), V_b(S(\theta))) \;=\; V_b'(S(\theta)) \left(S(\theta) - v_s - \Delta_s\, P(\theta + V_b(S(\theta)))\right) \Psi_b(P(\theta)\,,\, S(\theta)), \quad (28)$$

where

$$\Gamma_b(p, x) \;=\; \int_x^{+\infty} \Psi_b(p, y)\, dy.$$

Letting $z = S(\theta)$, we have $\theta = V_s(z)$ and hence

$$\frac{\Gamma_b(P(V_s(z)), V_b(z))}{\Psi_b(P(V_s(z)), V_b(z))} \;=\; V_b'(z) \left(z \,-\, v_s \,-\, \Delta_s\, P(V_s(z) \,+\, V_b(z))\right). \quad (29)$$

Now, if $V_b(z)$ were not absolutely continuous, it would have a singular component and therefore, by the de la Valée Poussin Theorem (Saks (1937), p.127) there would be a point z_0 where $V_b'(z_0) \;=\; +\infty$. Let $\theta \;=\; V_s(z_0)$. Then, $S(\theta)$ could not be optimal because the first order condition (28) could not hold, and there would be a strict incentive to deviate. Thus, $V_b(z)$ is absolutely continuous and, since the right-hand side of (29) is continuous and (29) holds almost everywhere in (a_2, A_2), identity (29) actually holds for all $z \in (a_2, A_2)$.

Now, using the first order condition (27) for the buyer, we have

$$z \,-\, v_s \,-\, \Delta_s\, P(V_s(z) \,+\, V_b(z)) \;=\; z \,-\, v_s \,-\, \frac{\Delta_s}{\Delta_b}\,(z \,-\, v_b) \;=\; \frac{v_b \,-\, v_s}{v^H - v_b}\,(v^H \,-\, z). \quad (30)$$

Furthermore, (27) implies that

$$P(V_s(z) + V_b(z)) \;=\; \frac{R\, e^{V_s(z)+V_b(z)}}{1 \,+\, R\, e^{V_s(z)+V_b(z)}} \;=\; \frac{z - v_b}{v^H - v_b}.$$

That is,

$$V_s(z) + V_b(z) \;=\; \log \frac{z - v_b}{v^H - z} \,-\, \log R,$$

or equivalently,

$$V_s(z) \;=\; \log \frac{z - v_b}{v^H - z} \,-\, V_b(z) \,-\, \log R\,.$$

Therefore,

$$P(V_s(z)) \;=\; \frac{e^{-V_b(z)}\,\frac{z-v_b}{v^H-z}}{1\,+\,e^{-V_b(z)}\,\frac{z-v_b}{v^H-z}} \;=\; \frac{(z-v_b)e^{-V_b(z)}}{v^H-z\,+\,e^{-V_b(z)}\,(z-v_b)}.$$

Using the fact that $\Psi_b^L(V_b(z)) \;=\; e^{-V_b(z)}\,\Psi_b^H(V_b(z))$, we get

$$\begin{aligned}
\Psi_b(P(V_s(z)),V_b(z)) \;&=\; P(V_s(z))\,\Psi_b^H(V_b(z))\,+\,(1-P(V_s(z)))\,\Psi_b^L(V_b(z))\\
&=\; \frac{(z-v_b)\,e^{-V_b(z)}}{v^H-z\,+\,e^{-V_b(z)}\,(z-v_b)}\,\Psi_b^H(V_b(z))\\
&\qquad +\frac{(v^H-z)\,e^{-V_b(z)}}{v^H-z\,+\,e^{-V_b(z)}\,(z-v_b)}\,\Psi_b^H(V_b(z))\\
&=\; \frac{v^H-v_b}{v^H-z\,+\,e^{-V_b(z)}\,(z-v_b)}\,\Psi_b^L(V_b(z))\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Gamma_b(P(V_s(z)),V_b(z)) \;&=\; P(V_s(z))\,G_H(V_b(z))\,+\,(1-P(V_s(z)))\,G_L(V_b(z))\\
&=\; \frac{(z-v_b)e^{-V_b(z)}\,G_H(V_b(z))\,+\,(v^H-z)\,G_L(V_b(z))}{v^H-z\,+\,e^{-V_b(z)}\,(z-v_b)}\,.
\end{aligned} \tag{31}$$

Consequently,

$$\begin{aligned}
\frac{\Gamma_b(P(V_s(z)),V_b(z))}{\Psi_b(P(V_s(z)),V_b(z))} \;&=\; \frac{P(V_s(z))\,G_H(V_b(z))\,+\,(1-P(V_s(z)))\,G_L(V_b(z))}{P(V_s(z))\,\Psi_b^H(V_b(z))\,+\,(1-P(V_s(z)))\,\Psi_b^L(V_b(z))}\\
&=\; \frac{(z-v_b)e^{-V_b(z)}\,G_H(V_b(z))\,+\,(v^H-z)\,G_L(V_b(z))}{(v^H-v_b)\,\Psi_b^L(V_b(z))}\\
&=\; (v^H-v_b)^{-1}\left((z-v_b)\,\frac{1}{h_b^H(V_b(z))}\,+\,(v^H-z)\,\frac{1}{h_b^L(V_b(z))}\right)\,.
\end{aligned}$$

Thus, by (30), the ODE (29) takes the form

$$\begin{aligned}
V_b'(z) \;&=\; \frac{\Gamma_b(P(V_s(z)),V_b(z))}{\Psi_b(P(V_s(z)),V_b(z))\,\big(z\,-\,v_s\,-\,\Delta_s\,P(V_s(z)\,+\,V_b(z))\big)}\\
&=\; (v^H-v_b)^{-1}\left((z-v_b)\,\frac{1}{h_b^H(V_b(z))}\,+\,(v^H-z)\,\frac{1}{h_b^L(V_b(z))}\right)\,\frac{1}{\frac{v_b-v_s}{v^H-v_b}\,(v^H\,-\,z)}\\
&=\; \frac{1}{v_b-v_s}\left(\frac{z-v_b}{v^H-z}\,\frac{1}{h_b^H(V_b(z))}\,+\,\frac{1}{h_b^L(V_b(z))}\right),\quad z\in(a_1,A_1)\;=\;(v_b,v^H).
\end{aligned}$$

Consequently, $V_b(z)$ solves (12). By Lemma 4.2, $V_b(v^H)=+\infty$. Thus $A_2=v^H$ and the proof is complete. ■

Proof of Corollary 4.4. By Proposition 4.3, $V_b(V_0, z)$ is monotone increasing in V_0. Consequently, $B = V_b^{-1}$ is monotone decreasing in V_0. Similarly,

$$V_s(V_0, z) = \log \frac{z - v_b}{v^H - z} - V_b(V_0, z) - \log R$$

is monotone decreasing in V_0 and therefore $S = V_s^{-1}$ is monotone increasing in V_0. ■

C Exponential Tails of the Type Distribution

This appendix proves the dynamic preservation of our exponential tail condition on type densities.

Proof of Lemma 4.5. We will use the decomposition

$$(\psi_1 * \psi_2)(x) = \left(\int_{-\infty}^{A} + \int_{A}^{+\infty} \right) \psi_1(x-y)\, \psi_2(y)\, dy.$$

Now, we fix an $\varepsilon > 0$ and pick some constant A so large that

$$\frac{\psi_2(y)}{c_2\, e^{-\alpha y} y^{\gamma_2}} \in (1-\varepsilon, 1+\varepsilon)$$

for all $y > A$. Then,

$$\frac{\int_A^{+\infty} \psi_1(x-y)\, \psi_2(y)\, dy}{c \int_A^{+\infty} \psi_1(x-y)\, e^{-\alpha y}\, y^{\gamma_2} dy} \in (1-\varepsilon, 1+\varepsilon)$$

for all x. Changing variables, applying L'Hôpital's rule, and using the induction hypothesis, we get

$$\lim_{x \to +\infty} \frac{\int_A^{+\infty} \psi_1(x-y)\, e^{-\alpha y}\, y^{\gamma_2}\, dy}{x^{\gamma_1+\gamma_2+1}\, e^{-\alpha x}} = \lim_{x\to+\infty} \frac{\int_{-\infty}^{x-A} \psi_1(z)\, e^{-\alpha(x-z)}\, (x-z)^{\gamma_2}\, dz}{x^k\, e^{-\alpha x}}. \tag{32}$$

Now, using the same asymptotic argument, we conclude that, for any fixed $B > 0$, the contribution from the integral coming from values of z below B is negligible and therefore we can replace $\int_{-\infty}^{x-A}$ by $\int_C^{x-A}$ and replace $\psi_1(z)$ by $c_1 z^{\gamma_1} e^{\alpha z}$. Thus, we need to calculate the asymptotic, using the change of variables from z/x to y, of

$$\begin{aligned} \int_B^{x-A} (z/x)^{\gamma_1}\, (1-z/x)^{\gamma_2}\, d(z/x) = & \int_{B/x}^{1-A/x} (1-y)^{\gamma_1} y^{\gamma_2}\, dy \\ & \to \int_0^1 y^{\gamma_1}\, (1-y)^{\gamma_2}\, dy = B(\gamma_1+1, \gamma_2+1)\,. \end{aligned} \tag{33}$$

Finally, using the Lebesgue dominated convergence theorem we get that the part $\int_{-\infty}^{A} \psi_1(x-y)\psi_2(y)\, dy$ is also asymptotically negligible. The claim follows. ■

Using these asymptotic results, we now show that, for the case of sufficiently large gains from trade $\overline{G}$, the dependence of equilibrium utilities on the density of types can be characterized in terms of the asymptotic parameters c_{it}, γ_{it} of Proposition 4.6. In the setting of Section 5, let $\bar{N}_s$ be the largest number of signal packets acquired in equilibrium by a seller class, and, similarly, $\bar{N}_b$ the largest number of packets acquired by a buyer class. Let

$$\mathbf{s} = \{s_1, \ldots, s_{\nu_s}\}\ ,\ \mathbf{b} = \{b_1, \ldots, b_{\nu_b}\}$$

be the set of seller classes with $\bar{N}_i = \bar{N}_s$ and the set of buyers with $\bar{N}_i = \bar{N}_b$, respectively. Let also $\lambda_{kl} \equiv \lambda_k \, \kappa_{kl}$ and let $\Lambda_s \in \mathbb{R}^{\nu_s \times \nu_b}$ and $\Lambda_b \in \mathbb{R}^{\nu_b \times \nu_s}$ be the matrices $\{\lambda_{s_i, b_j}\}$ and $\{\lambda_{b_i, s_j}\}$ of matching probabilities between the respective classes. Then, the consistency conditions $m_{s_i} \lambda_{ij} = m_j \lambda_{ji}$ imply that $\mathcal{M}_s \, \Lambda_s \mathcal{M}_b^{-1} = \Lambda_b^\top$, where $\Lambda_b^\top$ denotes the transposed matrix. Let also $c_t^{\mathbf{s}} = \{c_{s_i,t}\}_{i=1}^{\nu_s}$ and $c_t^{\mathbf{b}} = \{c_{b_i,t}\}_{i=1}^{\nu_b}$.

We will need the following assumption.

Assumption C.1 $\kappa_{ij} > 0$ *for all pairs of buyer and seller classes.*

The following lemma is a direct consequence of Lemma 4.5.

Lemma C.2 *We have*

$$c_t^{\mathbf{s}} \;=\; K(t,N) \begin{cases} (\Lambda_s \Lambda_b)^{t-1} \mathbf{1}\ , & t \text{ is even} \\ (\Lambda_s \Lambda_b)^{t-1} \Lambda_s \mathbf{1}\ , & t \text{ is odd} \end{cases}$$

and, similarly,

$$c_t^{\mathbf{b}} \;=\; K(t,N) \begin{cases} (\Lambda_b \Lambda_s)^{t-1} \mathbf{1}\ , & t \text{ is even} \\ (\Lambda_b \Lambda_s)^{t-1} \Lambda_s \mathbf{1}\ , & t \text{ is odd} \end{cases}$$

for all $t \geq 2$, where $K(t,N)$ is a universal combinatorial function. For a seller class $i \notin \mathbf{s}$, we have $c_{i,t} = K_i \sum_j \lambda_{i,b_j} c^{\mathbf{b}}_{j,t-1}$ and, similarly, $c_{i,t} = K_i \sum_j \lambda_{i,s_j} c^{\mathbf{s}}_{j,t-1}$ for a buyer of class $i \notin \mathbf{b}$, with come constants K_i, independent of the matching probabilities.

Now, using the Perron-Frobenius Theorem (see Meyer (2000), chapter 8, page 668), we show in the Appendix that the following is true.

Lemma C.3 *The matrix $\Lambda_s \Lambda_b$ has a unique largest positive eigenvalue e^r. Furthermore,*

$$\lim_{t \to \infty} \frac{1}{t} \log \frac{c_t^{\mathbf{b}}}{K(t,N)} \;=\; \frac{1}{t} \lim_{t \to \infty} \log \frac{c_t^{\mathbf{s}}}{K(t,N)} \;=\; r\,.$$

The eigenvalue r is strictly monotone increasing in λ_i, for any class i.

By analogy with DeMarzo, Vayanos and Zwiebel (2003), one could call r the rate of social influence. This rate captures the endogenous non-linear nature of cross-class learning externalities. In particular, more "central" classes have a larger impact on r, and therefore have a stronger impact on other classes' behaviour. The proof of Proposition 5.5 implies that for sufficiently large time horizon T the leading asymptotic term in the logarithm of the gains from information acquisition is proportional to r, and therefore monotonicity of r with respect to λ_i directly implies monotonicity of the gains from information acquisition.

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Supplementary Results for Information Percolation in Segmented Markets

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Abstract: This supplement to "Information Percolation in Segmented Markets" houses Appendices D through L of the main paper, Duffie, Malamud, and Manso (2013).

This supplement of Duffie, Malamud, and Manso (2013) houses the following appendices of our paper, “Information Percolation in Segmented Markets.” Appendices A through C are found in the main paper.

Supplementary Appendices

D. A proof of the existence and uniqueness of a strictly monotone equilibrium of the double auction.

E. An application of results from Appendix D to obtain approximations for the double-auction equilibrium for the case of large gains from trade.

F. An application of the results of Appendix E to derive approximations for the expected trading profits for the case of large gains from trade.

G. An application of the results of Appendix F to approximate the expected gains from information acquisition for the case of large gains from trade. From this, a derivation of general properties of equilibrium information acquisition. Appendices G.1 and G.2 apply these results to special cases of one and two classes of sellers, respectively.

H. The two-class model.

I. Results on endogenous investment in matching technology.

J. Proofs of results in Section I, on endogenous investment in matching technology.

K. Results for the case of dynamic information acquisition.

L. Proofs of results in Appendix K.

D Existence of Equilibrium for the Double Auction

Existence of equilibrium follows from Proposition 4.6 and the following general result.

Proposition D.1 *Fix a buyer class b and a seller class s such that*

$$\psi_b^H(x) \sim \mathrm{Exp}_{+\infty}(c, \gamma, -\alpha) \tag{1}$$

for some $c, \alpha > 0$ and some $\gamma \in \mathbb{R}$. If $\alpha < 1$, then there is no equilibrium associated with $V_0 = -\infty$. Suppose, however, that $\alpha > \alpha^*$ and that

$$\begin{aligned} -\gamma &< \frac{(\alpha+1)\log\alpha}{\log(\alpha+1) - \log\alpha}, \quad \text{if } \alpha \geq 2 \\ -\gamma &< \frac{\log(\alpha^2 - \alpha)\, 2^\alpha}{\log(\alpha+1) - \log\alpha}, \quad \text{if } \alpha < 2. \end{aligned}$$

Then, if the gain from trade $\overline{G}$ is sufficiently large, there exists a unique strictly monotone equilibrium with $V_0 = -\infty$. This equilibrium is in undominated strategies, and maximizes total welfare among all continuous nondecreasing equilibrium bidding policies.

In order to prove Proposition D.1, we apply the following auxiliary result.

Lemma D.2 *Suppose that $B, S : \mathbb{R} \to (v_b, v^H)$ are strictly increasing and that their inverses V_s and V_b satisfy*

$$v_b + \Delta_b P(V_s(z) + V_b(z)) = z.$$

Suppose further that $V_b'(z)$ solves (12) for all $z \in (v_b, v^H)$. Then $(B\,,\,S)$ is an equilibrium.

Proof. Recall that the seller maximizes

$$f_S(s) = \int_{V_b(s)}^{+\infty} (s - v_s - \Delta_s P(\theta + \phi))\ \Psi_b(P(\theta), \phi)\, d\phi. \tag{2}$$

To show that $S(\theta)$ is indeed optimal, it suffices to show that $f_S'(s) \geq 0$ for $s \leq S(\theta)$ and that $f_S'(s) \leq 0$ for $s \geq S(\theta)$. We prove only the first inequality. A proof of the second is analogous. So, let $s \leq S(\theta) \Leftrightarrow V_s(s) \leq \theta$. Then,

$$\begin{aligned} f_S'(s) &= V_b'(s)\ (-s + v_s + \Delta_s P(\theta + V_b(s)))\ \Psi_b(P(\theta), V_b(s)) + G_b(P(\theta), V_b(s)) \\ &= V_b'(s)\Psi_b(P(\theta), V_b(s)) \left(-s + v_s + \Delta_s P(\theta + V_b(s)) + \frac{1}{V_b'(s) h_b(P(\theta), V_b(s))} \right). \end{aligned}$$

By Lemma 4.1, $h_b(p, V_b(s))$ is monotone decreasing in p. Therefore, by (29),

$$\frac{1}{V_b'(s)\, h_b(P(\theta), V_b(s))} \;\geq\; \frac{1}{V_b'(s)\, h_b(P(V_s(S)), V_b(s))} \;=\; s \,-\, v_s \,-\, \Delta_s\, P(V_s(s) + V_b(s)).$$

Hence,

$$\begin{aligned} f_S'(s) \quad &\geq \quad V_b'(s)\, \Psi_b(P(\theta), V_b(s)) \\ &\qquad \times \left(-s + v_s \,+ \Delta_s P(\theta + V_b(s)) \;+\; s \,-\, v_s \,-\, \Delta_s\, P(V_s(s) + V_b(s))\right) \;\geq\; 0, \end{aligned}$$

because $\theta \,\geq\, V_s(s)$.

For the buyer, it suffices to show that

$$f_B(b) \;=\; \max_b \int_{-\infty}^{V_s(b)} \big(v_b \,+\, \Delta_b P(\theta + \phi) \,-\, S(\theta)\big)\, \Psi_s(P(\phi), \theta)\, d\theta \tag{3}$$

satisfies $f_B'(b) \;\geq\; 0$ for $b \;\leq\; B(\phi)$, and satisfies $f_B'(b) \;\leq\; 0$ for $b \,\geq\, B(\phi)$. That is,

$$v_b \,+\, \Delta_b\, P(\phi \,+\, V_s(b)) \,-\, S(V_s(b)) \;=\; v_b \,+\, \Delta_b\, P(\phi \,+\, V_s(b)) \,-\, b \;\geq\; 0$$

for $b \;\leq B(\phi)$, and the reverse inequality for $b \geq B(\phi)$. For $b \,\leq\, B(\phi)$, we have $\phi \,\geq\, V_b(b)$ and therefore

$$v_b \,+\, \Delta_b\, P(\phi \,+\, V_s(b)) \,-\, b \;\geq\; v_b \,+\, \Delta_b\, P(V_b(b) \,+\, V_s(b)) \,-\, b \;=\; 0,$$

as claimed. The case of $b \geq B(\phi)$ is analogous. ■

Proof of Proposition D.1. It follows from Proposition 4.3 and Lemma D.2 that a strictly monotone equilibrium in undominated strategies exists if and only if there exists a solution $V_b(z)$ to (12) such that $V_b(v_b) \;=\; -\infty$ and

$$V_s(z) \;=\; \log \frac{z - v_b}{v^H - z} \;-\; V_b(z) \,-\, \log R$$

is monotone increasing in z and satisfies $V_s(v_b) = -\infty$, $V_s(v^H) \,=\, +\infty$. Furthermore, such an equilibrium is unique if the solution to the ODE (12) with $V_b(v_b) \;=\; -\infty$ is unique.

Fix a $t \leq T$ and denote for brevity $\gamma \;=\; \gamma_{it}$, $c \;=\; c_{it}$. Let also

$$g(z) \;=\; e^{(\alpha+1)\, V_b(z)}\,.$$

Then, a direct calculation shows that $V_b(z)$ solves (12) with $V_b(v_b) \;=\; -\infty$ if and only if $g(z)$ solves

$$\begin{aligned} &g'(z) \\ &= g(z)\, \frac{\alpha + 1}{v_b - v_s} \left(\frac{z - v_b}{v^H - z}\, \frac{1}{h_b^H((\alpha+1)^{-1} \log g(z))} \,+\, \frac{1}{h_b^L((\alpha+1)^{-1} \log g(z))} \right), \end{aligned} \tag{4}$$

with $g(v_b) = 0$. By assumption and Lemma 4.1,

$$h_b^H(V) \sim c_i\,|V|^\gamma\, e^{(\alpha+1)V} \quad \text{and} \quad h_b^L(V) \sim c_i\,|V|^\gamma\, e^{\alpha V} \tag{5}$$

as $V \to -\infty$ because both $G_b^H(V)$ and $G_b^L(V)$ converge to 1. Hence, the right-hand side of (4) is continuous and the existence of a solution follows from the Euler theorem. Furthermore, when studying the asymptotic behavior of $g(z)$ as $z \downarrow v_b$, we can replace h_b^H and h_b^L by their respective asymptotics (5).

Indeed, let us consider

$$\begin{aligned} \tilde{g}'(z) \;=\; (\alpha+1)\,\tilde{g}(z)\,\frac{1}{v_b - v_s}\Bigg(\frac{z - v_b}{v^H - z}\,\frac{1}{c\,((\alpha+1)^{-1}\log 1/\tilde{g})^\gamma\,\tilde{g}} \\ +\; \frac{1}{c((\alpha+1)^{-1}\log 1/\tilde{g})^\gamma\,\tilde{g}^{\alpha/(\alpha+1)}}\Bigg), \end{aligned} \tag{6}$$

with the initial condition $\tilde{g}(v_b) = 0$. We consider only values of z sufficiently close to v_b, so that $\log \tilde{g}(z) < 0$.

It follows from standard ODE comparison arguments and the results below that for any $\varepsilon > 0$ there exists a $\bar{z} > v_b$ such that

$$\left|\frac{g(z)}{\tilde{g}(z)} - 1\right| + \left|\frac{g'(z)}{\tilde{g}'(z)} - 1\right| \;\le\; \varepsilon \tag{7}$$

for all $z \in (v_b, \bar{z})$. The assumptions of the Proposition guarantee that the same asymptotics hold for the derivatives of the hazard rates, which implies that the estimates obtained in this manner are uniform.

First, we will consider the case of general (not necessarily large) $v_b - v_s$ and show that, when $\alpha < 1$, $g(z)$ decays so fast as $z \downarrow v_b$ that $V_s(z)$ cannot remain monotone increasing. A similar argument then implies that $V_s(z)$ cannot remain monotone increasing when $\overline{G}\alpha < 1$.

At points in the proof, we will define suitable positive constants denoted C_1, C_2, $C_3, \ldots$ without further mention.

Denote

$$\zeta \;=\; \frac{(\alpha+1)^{\gamma+1}}{c\,(v_b - v_s)}\,. \tag{8}$$

Then, we can rewrite (6) in the form

$$\tilde{g}'(z) \;=\; \frac{\zeta}{(\log 1/\tilde{g})^\gamma}\left(\frac{z - v_b}{v^H - z} + \tilde{g}^{1/(\alpha+1)}\right). \tag{9}$$

From this point, throughout the proof, without loss of generality, we assume that $v_b = 0$. Furthermore, after rescaling if necessary, we may assume that $v^H - v_b = 1$. Then, the same asymptotic considerations as above imply that, when studying the behavior of $\tilde{g}$ as $z \downarrow v_b$, we may replace $v^H - z \sim v^H - v_b$ in (6) by 1.

Let $A(z)$ be the solution to

$$z \;=\; \int_0^{A(z)} \zeta^{-1} \, (-\log x)^{\gamma} \, x^{-1/(\alpha+1)} \, dx\,.$$

A direct calculation shows that

$$B(z) \;\stackrel{def}{=}\; \int_0^{z} \zeta^{-1} \, (-\log x)^{\gamma} \, x^{-1/(\alpha+1)} \, dx \;\sim\; \zeta^{-1} \frac{\alpha+1}{\alpha} \, (-\log z)^{\gamma} \, z^{\alpha/(\alpha+1)}\,.$$

Conjecturing the asymptotics

$$A(z) \;\sim\; K \, (-\log z)^{\gamma(\alpha+1)/\alpha} \, z^{(\alpha+1)/\alpha} \tag{10}$$

and substituting these into $B(A(z)) = z$, we get

$$K \;=\; \zeta^{\frac{\alpha+1}{\alpha}} \left(\frac{\alpha}{\alpha+1} \right)^{\frac{(\gamma+1)(\alpha+1)}{\alpha}}\,.$$

Standard considerations imply that this is indeed the asymptotic behavior of $A(z)$. It is then easy to see that

$$A'(z) \;\sim\; K \, \frac{\alpha+1}{\alpha} \, (-\log z)^{\gamma(\alpha+1)/\alpha} \, z^{1/\alpha}. \tag{11}$$

By (9),

$$\tilde{g}'(z) \;\geq\; \frac{\zeta}{(\log 1/\tilde{g})^{\gamma}} \, \tilde{g}^{1/(\alpha+1)}.$$

Integrating this inequality, we get $\tilde{g}(z) \;\geq\; A(z)$. Now, the factor $(\log 1/\tilde{g})^{\gamma}$ is asymptotically negligible as $z \downarrow v_b$. Namely, for any $\varepsilon > 0$ there exists a $C_1 > 0$ such that

$$C_1 \, \tilde{g}^{1/(\alpha+\varepsilon+1)} \;\geq\; \frac{\zeta}{(\log 1/\tilde{g})^{\gamma}} \, \tilde{g}^{1/(\alpha+1)} \;\geq\; C_1^{-1} \, \tilde{g}^{1/(\alpha-\varepsilon+1)}.$$

Thus,

$$\left((\tilde{g})^{\frac{\alpha-\varepsilon}{1+\alpha-\varepsilon}} \right)' \;\geq\; C_2\,.$$

Integrating this inequality, we get that

$$\tilde{g}(z) \;\geq\; C_3 \, (z - v_b)^{\frac{\alpha-\varepsilon+1}{\alpha-\varepsilon}}\,. \tag{12}$$

Let

$$l(z) \;=\; B(\tilde g(z)) \;-\; z\,.$$

Then, for small z, by (10),

$$\begin{aligned}
l'(z) \;&=\; \tilde g'(z)\,\zeta^{-1}\,(-\log \tilde g)^{\gamma}\,\tilde g^{-1/(\alpha+1)} \;-\; 1\\
&=\; \frac{\zeta}{(\log 1/\tilde g)^{\gamma}}\left(\frac{z}{v^H - z} \;+\; \tilde g^{1/(\alpha+1)}\right)\zeta^{-1}\,(-\log \tilde g)^{\gamma}\,\tilde g^{-1/(\alpha+1)} \;-\; 1\\
&=\; \frac{z}{1-z}\,\frac{1}{\tilde g^{1/(\alpha+1)}} \qquad (13)\\
&=\; \frac{z}{1-z}\,\frac{1}{(A(l(z)+z))^{1/(\alpha+1)}}\\
&\le\; \frac{z}{1-z}\,\frac{1}{(A(l(z)))^{1/(\alpha+1)}}\,,
\end{aligned}$$

where we have used the fact that $l(z) \ge 0$ because $h(0) = 0$ and $l'(z) \ge 0$. Integrating this inequality, we get that, for small z,

$$l(z) \;\le\; C_4\, z^{2(\alpha-\varepsilon)/(\alpha-\varepsilon+1)}.$$

Hence, for small z,

$$\tilde g(z) \;=\; A(l(z) \,+\, z) \;\le\; A((C_4+1)z^{2(\alpha-\varepsilon)/(\alpha-\varepsilon+1)}) \;\le\; C_5\, z^{2-\varepsilon}. \qquad (14)$$

Let $C(z)$ solve

$$\int_0^{C(z)} (-\log x)^{\gamma}\, dx \;=\; \zeta \int_0^z \frac{x}{1-x}\, dx\,.$$

A calculation similar to that for the function $A(z)$ implies that

$$C(z) \;\sim\; C_6\,(-\log z)^{\gamma}\, z^2 \qquad (15)$$

as $z \to 0$. Integrating the inequality

$$\tilde g'(z) \;\ge\; \frac{\zeta}{(-\log \tilde g)^{\gamma}}\,\frac{z}{1-z},$$

we get that

$$\tilde g(z) \;\ge\; C(z).$$

Let now $\alpha < 1$. Then, (14) immediately yields that the second term in the brackets in (6) is asymptotically negligible and, consequently,

$$\frac{\zeta}{(\log 1/\tilde g)^{\gamma}}\,\frac{z}{1-z} \;\le\; \tilde g'(z) \;\le\; \frac{(1+\varepsilon)\,\zeta}{(\log 1/\tilde g)^{\gamma}}\,\frac{z}{1-z} \qquad (16)$$

holds for sufficiently small z. Integrating this inequality implies that

$$C(z) \le \tilde{g}(z) \le (1+\varepsilon)\, C(z).$$

Now, (16) implies that

$$(1-\varepsilon)\, 2\, C(z) z^{-1} \le \tilde{g}'(z) \le 2\,(1+\varepsilon)\, C(z)\, z^{-1}$$

for sufficiently small[1] z.

Using the asymptotics (5) and repeating the same argument implies that $g(z)$ also satisfies these bounds. (The calculations for g are lengthier and omitted here.)

Now,

$$V_b'(z) = \frac{g'(z)}{(\alpha+1)\, g(z)} \ge (1-\varepsilon)\, \frac{2}{\alpha+1}\, z^{-1}.$$

Therefore,

$$V_s'(z) = \frac{1}{z\,(1-z)} - V_b'(z) < 0$$

for sufficiently small z. Thus, $V_s(z)$ cannot be monotone increasing and the equilibrium does not exist.

Let now $\alpha > 1$. We will now show that there exists a unique solution to (4) with $g(0) = 0$. Since the right-hand side loses Lipschitz continuity only at $z = 0$, it suffices to prove local uniqueness at $z = 0$. Hence, we need only consider the equation in a small neighborhood of $z = 0$. (It is recalled that we assume $v_b = 0$.)

As above, we prove the result directly for the ODE (6), and then explain how the argument extends directly to (4).

Suppose, to the contrary, that there exist two solutions $\tilde{g}_1$ and $\tilde{g}_2$ to (6). Define the corresponding functions l_1 and l_2 via $l_i = B(\tilde{g}_i) - z$. Both functions solve (13). Integrating over a small interval $[0, l]$, we get

$$|l_1(x) - l_2(x)| \le \int_0^x \frac{z}{1-z} \left| \frac{1}{(A(l_1(z)+z))^{1/(\alpha+1)}} - \frac{1}{(A(l_2(z)+z))^{1/(\alpha+1)}} \right| dz\,. \quad (17)$$

Now, we will use the following elementary inequality: There exists a constant $C_6 > 0$ such that

$$a^{1/\alpha} - b^{1/\alpha} \le \frac{C_6\,(a-b)}{a^{(\alpha-1)/\alpha} + b^{(\alpha-1)/\alpha}} \quad (18)$$

for $a > b > 0$. Indeed, let $x = b/a$ and $\beta = 1/\alpha$. Then, we need to show that

$$(1 + x^{1-\beta})\,(1 - x^{\beta}) \le C_6\,(1-x)$$

[1] We are using the same ε in all of these formulae. This can be achieved by shrinking if necessary the range of z under consideration.

for $x \in (0,1)$. That is, we must show that

$$x^{1-\beta} \;-\; x^{\beta} \;\leq\; (C_6 - 1)\,(1-x).$$

By continuity and compactness, it suffices to show that the limit

$$\lim_{x\to 1} \frac{x^{1-\beta} \;-\; x^{\beta}}{1-x}$$

is finite. This follows from L'Hôpital's rule.

By (10) and (11), we can replace the function $A(z)$ in (17) by its asymptotics (10) at the cost of getting a finite constant in front of the integral. Thus, for small z,

$$\begin{aligned} &|l_1(x) \;-\; l_2(x)| \\ &\leq\; C_7 \int_0^x z \left| \frac{((-\log(l_1+z))^{\gamma}\,(l_1+z))^{1/\alpha} - ((-\log(l_2+z))^{\gamma}\,(l_2+z))^{1/\alpha}}{((-\log(l_1+z))^{\gamma}\,(l_1+z))^{1/\alpha}((-\log(l_2+z))^{\gamma}\,(l_2+z))^{1/\alpha}} \right| dz\,. \end{aligned} \tag{19}$$

By (18),

$$\begin{aligned} &|((-\log(l_1+z))^{\gamma}\,(l_1+z))^{1/\alpha} - ((-\log(l_2+z))^{\gamma}\,(l_2+z))^{1/\alpha}| \\ &\leq\; C_6\, \frac{|(-\log(l_1+z))^{\gamma}\,(l_1+z) \;-\; (-\log(l_2+z))^{\gamma}\,(l_2+z)|}{((-\log(l_1+z))^{\gamma}\,(l_1+z))^{(\alpha-1)/\alpha} + ((-\log(l_2+z))^{\gamma}\,(l_2+z))^{(\alpha-1)/\alpha}}\,. \end{aligned} \tag{20}$$

Now, consider some $\gamma > 0$. Then, for any sufficiently small $a > b > 0$, a direct calculation shows that

$$0 \;<\; (\log(1/a))^{\gamma}\,a \;-\; (\log(1/b))^{\gamma}\,b \;\leq\; ((\log(1/a))^{\gamma} \;+\; (\log(1/b))^{\gamma})\,(a-b).$$

If, instead, $\gamma \leq 0$, then the function $a \mapsto (\log(1/a))^{\gamma}\,a$ is continuously differentiable at $a = 0$, and hence

$$0 \;<\; (\log(1/a))^{\gamma}\,a \;-\; (\log(1/b))^{\gamma}\,b \;\leq\; C_8\,(a-b)\,.$$

Since $\alpha > 1$, the same calculation as that preceding (16) implies that, for sufficiently small z,

$$A(z) \;\leq\; \tilde{g}_i(z) \;=\; A(z + l_i(z)) \;\leq\; (1+\varepsilon)\,A(z)\;,\;\; i \;=\; 1,\,2.$$

Thus, for $z \in [0, \bar{\varepsilon}]$,

$$\begin{aligned} &\left| \frac{((-\log(l_1+z))^{\gamma}\,(l_1+z))^{1/\alpha} - ((-\log(l_2+z))^{\gamma}\,(l_2+z))^{1/\alpha}}{((-\log(l_1+z))^{\gamma}\,(l_1+z))^{1/\alpha}((-\log(l_2+z))^{\gamma}\,(l_2+z))^{1/\alpha}} \right| \\ &\qquad\leq\; C_9\,|l_1(z) \;-\; l_2(z)|\,\frac{1}{z^{((\alpha+1)/\alpha)-\varepsilon}} \\ &\qquad\leq\; C_9 \left(\sup_{z\,\in\,[\,0,\,\bar{\varepsilon}\,]} |l_1(z) \;-\; l_2(z)| \right) \frac{1}{z^{((\alpha+1)/\alpha)-\varepsilon}}. \end{aligned} \tag{21}$$

Thus, (19) implies that

$$\begin{aligned} |l_1(x) \; - \; l_2(x)| \leq \; C_{10} \left(\sup_{z \in [0,\bar\varepsilon]} |l_1(z) \; - \; l_2(z)| \right) \int_0^x z \, \frac{1}{z^{((\alpha+1)/\alpha)+\varepsilon}} \, dz \\ = \; C_{11} \, (\bar\varepsilon)^{\frac{\alpha-1}{\alpha}-\varepsilon} \sup_{z \in [0,\bar\varepsilon]} |l_1(z) \; - \; l_2(z)| \end{aligned} \tag{22}$$

for all $l \leq \bar\varepsilon$. Taking the supremum over $l \in [0, \bar\varepsilon]$, we get

$$\sup_{z \in [0,\bar\varepsilon]} |l_1(z) \; - \; l_2(z)| \; \leq \; C_{11} \, (\bar\varepsilon)^{\frac{\alpha-1}{\alpha}-\varepsilon} \sup_{z \in [0,\bar\varepsilon]} |l_1(z) \; - \; l_2(z)| \, .$$

Picking $\bar\varepsilon$ so small that $C_{11} \, (\bar\varepsilon)^{\frac{\alpha-1}{\alpha}-\varepsilon} < 1$ immediately yields that $l_1 = l_2$ on $[0, \bar\varepsilon]$ and hence, since the right-hand side of (6) is Lipschitz continuous for $z \, l \neq 0$, we have $l_1 = l_2$ for all z by a standard uniqueness result for ODEs.

The fact that the same result holds for the original equation (4) follows by the same arguments as above.

It remains to prove the last claim, namely the existence of equilibrium for sufficiently large $v_b - v_s$. By Proposition 4.3, it suffices to show that

$$V_s'(z) \; = \; \frac{1}{z\,(1-z)} \; - \; V_b'(z) \; > \; 0 \tag{23}$$

for all $z \in (0,1)$ provided that $v_b - v_s$ is sufficiently large.

It follows from the proof of Lemma 4.1 that

$$G_L^{-1} \left((1-z)^{\frac{1}{(v_b - v_s)}} \right) \; \leq \; V_b(z) \; \leq G_H^{-1} \left((1-z)^{\frac{1}{(v_b - v_s)}} \right) \, .$$

Thus, as $v_b - v_s \uparrow +\infty$, $V_b(z)$ converges to $-\infty$ uniformly on compact subsets of $[0,1)$. By assumption,

$$\lim_{V \to +\infty} \frac{1}{h_b^H(V)} \; = \; \frac{1}{\alpha} \, , \qquad \lim_{V \to +\infty} \frac{1}{h_b^L(V)} \; = \; \frac{1}{\alpha+1}.$$

Thus, as $z \uparrow 1$,

$$V_b'(z) \; \sim \; \frac{1}{\alpha\,(v_b - v_s)} \, \frac{1}{1-z} \; < \; \frac{1}{z(1-z)}.$$

Fixing a sufficiently small $\varepsilon > 0$, we will show below that there exists a threshold W such that (23) holds for all $v_b - v_s > W$ and all z such that $V_b(z) \leq -\varepsilon^{-1}$. Since, by the assumptions made, $1/h_b^H(V)$ and $1/h_b^L(V)$ are uniformly bounded from above for $V \geq -\varepsilon^{-1}$, it will immediately follow from (12) that (23) holds for all z with $V_b(z) \geq -\varepsilon^{-1}$ as soon as $v_b - v_s$ is sufficiently large.

Thus, it remains to prove (23) when $V_b(z) \le -\varepsilon^{-1}$. We pick an ε so small that we can replace the ODE (4) by (6) when proving (23). That is, once we prove the claim for the "approximate" solution $\tilde{g}(z)$, the actual claim will follow from (7).

Let

$$\tilde{g}(z) \;=\; \frac{\zeta}{(-\log\zeta)^{\gamma}}\, f(z) \;\stackrel{def}{=}\; \varepsilon\, f(z)\;,\; \varepsilon \;=\; \frac{\zeta}{(-\log\zeta)^{\gamma}}\,.$$

Then, (4) is equivalent to the ODE

$$f'(z) \;=\; \left(\frac{\log(1/\zeta)}{\log(1/\zeta)\,+\,\log(1/f(z))}\right)^{\gamma}\left(\frac{z}{1-z}\,+\,\varepsilon^{\frac{1}{\alpha+1}}\,f(z)^{\frac{1}{\alpha+1}}\right)\,. \qquad (24)$$

As $v_b - v_s \to +\infty$, we get that $\zeta, \varepsilon \to 0$. Let

$$f_0(z) \;\stackrel{def}{=}\; \int_0^z \frac{x}{1-x}\,dx \;=\; -\log(1-z)\,-\,z\,.$$

Using bounds analogous to that preceding (16), it is easy to see that

$$\lim_{v_b - v_s \to +\infty} f(z) \;=\; f_0(z)\;, \qquad \lim_{v_b - v_s \to +\infty} f'(z) \;=\; f_0'(z),$$

and that the convergence is uniform on compact subsets of $(0,1)$. Fixing a small $\varepsilon_1 > 0$, we have, for $z > \varepsilon_1$,

$$\begin{aligned}
\lim_{v_b - v_s \to \infty} V_b'(z) \;&=\; \lim_{v_b - v_s \to \infty} \frac{\tilde{g}'(z)}{(\alpha+1)\tilde{g}(z)} \\
&=\; \lim_{v_b - v_s \to \infty} \frac{f'(z)}{(\alpha+1)f(z)} \\
&=\; \frac{f_0'(z)}{(\alpha+1)f_0(z)} \\
&=\; \frac{z}{(\alpha+1)\,(1-z)\,(-\log(1-z)-z)}.
\end{aligned}$$

We then have

$$\frac{d^2}{dz^2}(-\log(1-z)) \;=\; \frac{1}{(1-z)^2} \;\ge\; 1.$$

Therefore, by Taylor's formula,

$$-\log(1-z) - z \;\ge\; \frac{1}{2}\,z^2\,.$$

Hence,

$$\frac{z}{(\alpha+1)\,(1-z)\,(-\log(1-z)-z)} \;\le\; \frac{2}{\alpha+1}\,\frac{1}{z(1-z)}.$$

Therefore (23) holds for large $v_b - v_s$ because $\alpha > 1$. This argument does not work as $z \to 0$ because $f(0) = f_0(0) = 0$. So, we need to find a way to get uniform upper bounds for $f'(z)/f(z)$ when z is small. By the comparison argument used above, and picking ε_1 sufficiently small, since our goal is to prove inequality (23), we can replace $1-z$ by 1 in (24).

In this part of the proof, it will be more convenient to deal with $\tilde{g}$ instead of f. By the above, we may replace $\tilde{g}$ by the function g_1 solving

$$g_1'(z) \;=\; \frac{\zeta}{(-\log(g_1))^\gamma}\left(z \,+\, g_1^{\frac{1}{\alpha+1}}\right).$$

Let

$$d(z) \;=\; \int_0^z \left(\log\left(\frac{1}{x}\right)\right)^\gamma dx\;,$$

$D(z) \;=\; d^{-1}(z)$, and $k(z) \;=\; D(g_1(z))$. Then, we can rewrite the ODE for g_1 as

$$k'(z) \;=\; \zeta\,\left(z \,+\, (D(k(z)))^{1/(\alpha+1)}\right)\;, \qquad k(0)=0.$$

Define $L(z)$ via

$$\int_0^{L(z)} (D(x))^{-1/(\alpha+1)}\,dx \;=\; z,$$

and let

$$\phi(z) \;=\; L(\zeta\,z) \;+\; \frac{1}{2}\zeta\,z^2 \;\geq\; L(\zeta z).$$

Then, by the monotonicity of $D(z)$,

$$\phi'(z) \;=\; \zeta\,L'(\zeta z) \;+\; \zeta\,z \;=\; \zeta\,\left(z \;+\; (D(L(\zeta z)))^{1/(\alpha+1)}\right) \;\leq\; \zeta\,(z \;+\; (D(\phi(\zeta z)))^{1/(\alpha+1)}).$$

By a comparison theorem for ODEs (for example, Hartman (1982), Theorem 4.1, p. 26),[2] we have

$$k(z) \;\geq\; \phi(z) \;\Leftrightarrow\; g_1(z) \;=\; D(k(z)) \;\geq\; D(\phi(z))\,. \tag{25}$$

Therefore, since the functions $x(-\log x)^\gamma$ and $x^{\alpha/(\alpha+1)}\,(-\log x)^\gamma$ are monotone increasing for small x, we have

$$\begin{aligned}(1+\alpha)\,V_b'(z) \;&=\; \frac{g'(z)}{g(z)} \\ &\leq\; (1+\varepsilon)\,\frac{g_1'(z)}{(\alpha+1)\,g_1(z)} \\ &=\; \frac{(1+\varepsilon)\zeta\,z}{g_1\,(-\log g_1)^\gamma} \;+\; \frac{(1+\varepsilon)\zeta}{g_1^{\alpha/(\alpha+1)}\,(-\log g_1)^\gamma} \\ &\leq\; \frac{(1+\varepsilon)\zeta\,z}{D(\phi(z))\,(-\log D(\phi(z)))^\gamma} \;+\; \frac{(1+\varepsilon)\zeta}{D(\phi(z))^{\alpha/(\alpha+1)}\,(-\log D(\phi(z)))^\gamma}.\end{aligned} \tag{26}$$

[2]Even though the right-hand side of the ODE in question is not Lipschitz continuous, the proof of this comparison theorem easily extends to our case because of the uniqueness of the solution, due to (22).

Thus, it suffices to show that

$$\frac{\zeta\, z^2}{D(\phi(z))\,(-\log D(\phi(z)))^\gamma} + \frac{\zeta\, z}{D(\phi(z))^{\alpha/(\alpha+1)}\,(-\log D(\phi(z)))^\gamma} \;<\; (1-\varepsilon)(1+\alpha)$$

for some $\varepsilon > 0$, and for all sufficiently small z and ζ. Now, a direct calculation similar to that for the functions $A(z)$ and $C(z)$ implies that

$$d(z) \;\sim\; z\,(-\log z)^\gamma$$

and therefore that

$$D(z) \;\sim\; z\,(-\log z)^{-\gamma}\,.$$

Thus, it suffices to show that

$$\begin{aligned} &\frac{\zeta\, z^2}{\phi(z)\,(-\log\phi)^{-\gamma}\,(-\log(\phi(z)\,(-\log\phi)^{-\gamma}))^\gamma} \\ &\qquad + \frac{\zeta\, z}{(\phi(z)\,(-\log\phi)^{-\gamma})^{\alpha/(\alpha+1)}\,(-\log(\phi(z)\,(-\log\phi)^{-\gamma}))^\gamma} \\ &\quad < \; (1-\varepsilon)(1+\alpha). \end{aligned} \tag{27}$$

Leaving the leading asymptotic term, we need to show that

$$\frac{\zeta\, z^2}{\phi(z)} + \frac{\zeta\, z}{(\phi(z))^{\alpha/(\alpha+1)}\,(-\log(\phi(z)))^{\gamma/(\alpha+1)}} \;<\; (1-\varepsilon)(1+\alpha)\,.$$

We have

$$\int_0^z (D(x))^{-1/(\alpha+1)} dx \;\sim\; \frac{\alpha+1}{\alpha}\, z^{\alpha/(\alpha+1)}\,(-\log z)^{\gamma/(\alpha+1)}.$$

Therefore

$$L(z) \;\sim\; \left(\frac{\alpha}{\alpha+1}\, z\right)^{(\alpha+1)/\alpha}\,(-\log z)^{-\gamma/\alpha}\,.$$

Hence, we can replace $\phi(z)$ by

$$\tilde\phi(z) \;\stackrel{def}{=}\; \left(\frac{\alpha}{\alpha+1}\,\zeta z\right)^{(\alpha+1)/\alpha}\,(-\log(\zeta z))^{-\gamma/\alpha} \;+\; \frac{1}{2}\,\zeta\, z^2\,.$$

Let

$$x \;=\; \frac{\zeta\, z^2}{(\zeta z)^{(\alpha+1)/\alpha}\,(-\log(\zeta z))^{-\gamma/\alpha}}\,.$$

Then,

$$\begin{aligned} \frac{\zeta\, z^2}{\tilde\phi(z)} \quad &+ \quad \frac{\zeta\, z}{(\tilde\phi(z))^{\alpha/(\alpha+1)}\,(-\log(\tilde\phi(z)))^{\gamma/(\alpha+1)}} \\ &= \quad \frac{1}{\left(\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}} + 0.5x\right)^{\alpha/(\alpha+1)}} \left(\frac{-\log(\zeta z)}{-\log\tilde\phi}\right)^{\gamma/(\alpha+1)} + \frac{x}{\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}} + 0.5x}\,. \end{aligned}$$

We have

$$\begin{aligned}\log(\tilde{\phi}) \;&=\; \log(\zeta\, z) \,+\, \log\left(\left(\frac{\alpha}{\alpha+1}\right)^{(\alpha+1)/\alpha} (\zeta z)^{1/\alpha}\,(-\log(\zeta z))^{-\gamma/\alpha} \;+\; 0.5\, z\right)\\ &\leq\; \log(\zeta z)\end{aligned}$$

for small ζ, z. Furthermore, for any $\varepsilon > 0$ there exists a $\varepsilon > 0$ such that

$$\left(\frac{\alpha}{\alpha+1}\right)^{(\alpha+1)/\alpha} (\zeta z)^{1/\alpha}\,(-\log(\zeta z))^{-\gamma/\alpha} \;\geq\; (\zeta z)^{1/(\alpha-\varepsilon)}$$

for all $\zeta z \leq \varepsilon$. Hence,

$$\frac{\alpha-\varepsilon}{\alpha-\varepsilon+1} \;\leq\; \frac{-\log(\zeta z)}{-\log\tilde{\phi}} \;\leq\; 1$$

for all sufficiently small ζ, z. Consequently, to prove (26) it suffices to show that

$$\sup_{x>0}\; \chi(x) < \; 1+\alpha,$$

where

$$\chi(x) \;=\; \frac{1}{\left(\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}} \;+\; 0.5x\right)^{\alpha/(\alpha+1)}}\, A_\alpha \;+\; \frac{x}{\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}} \;+\; 0.5x},$$

with

$$A_\alpha \;=\; \max\left\{\left(\frac{\alpha}{\alpha+1}\right)^{\gamma/(\alpha+1)}\;,\, 1\right\}.$$

Let

$$K \;=\; \left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}}.$$

Then,

$$\chi'(x) \;=\; -\frac{0.5\,A_\alpha\,\alpha}{\alpha+1}\,\frac{1}{(K+0.5x)^{(2\alpha+1)/(\alpha+1)}} \;+\; \frac{K}{(K+0.5x)^2}\,.$$

Thus, $\chi'(x_*) = 0$ if and only if

$$K + 0.5x_* \;=\; \left(\frac{K}{\frac{0.5\,A_\alpha\,\alpha}{\alpha+1}}\right)^{\alpha+1},$$

which means that

$$x_* \;=\; 2\left(\left(\frac{2}{A_\alpha}\right)^{\alpha+1} - 1\right)\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}}.$$

Then,

$$
\begin{aligned}
&\chi(x_*)\\
&= \frac{1}{\left(\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}} + 0.5x_*\right)^{\alpha/(\alpha+1)}} A_\alpha \; + \; \frac{x_*}{\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}} + 0.5x_*}\\
&= \frac{1}{\left((2/A_\alpha)^{\alpha+1}\,(\alpha/(\alpha+1))^{(\alpha+1)/\alpha}\right)^{\alpha/(\alpha+1)}} A_\alpha \; + \; \frac{2\left(\left(\frac{2}{A_\alpha}\right)^{\alpha+1} - 1\right)\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}}}{(2/A_\alpha)^{\alpha+1}\,(\alpha/(\alpha+1))^{(\alpha+1)/\alpha}} \qquad (28)\\
&= \left(\frac{A_\alpha}{2}\right)^{\alpha} \frac{\alpha+1}{\alpha} A_\alpha \; + \; 2 - \; 2\left(\frac{A_\alpha}{2}\right)^{\alpha+1} \; = \; 2 \; + \; \frac{A_\alpha^{\alpha+1}}{2^\alpha\,\alpha}\,.
\end{aligned}
$$

There are three candidates for x that achieve a maximum of χ, namely $x = 0$, $x = +\infty$, and $x = x_*$, which is positive if and only if $A_\alpha < 2$.

If $\gamma \geq 0$, then $A_\alpha = 1$, so $x = 0$ and $x = +\infty$ satisfy the required inequality as soon as $\alpha > 1$, whereas $\chi(x_*) < \alpha + 1$ if and only if $\alpha > \alpha_*$, where

$$\alpha_* \; = \; 1 \; + \; \frac{1}{\alpha_*\,2^{\alpha_*}}.$$

A calculation shows that $\alpha^* \in (1.30, 1.31)$.

If $\gamma < 0$, then

$$\chi(0) \; = \; \frac{(\alpha+1)\,A_\alpha}{\alpha}\,, \qquad \chi(+\infty) \; = \; 2,$$

and this gives the condition $A_\alpha < \alpha$. If $A_\alpha > 2$, that is, if

$$-\gamma \; > \; (\alpha+1)\,\frac{\log 2}{\log((\alpha+1)/\alpha)},$$

then we are done. Otherwise, we need the property

$$2 \; + \; \frac{A_\alpha^{\alpha+1}}{2^\alpha\,\alpha} \; < \; \alpha + 1 \; \Leftrightarrow \; -\gamma \; < \; \frac{\log\left((\alpha^2-\alpha)\,2^\alpha\right)}{\log((\alpha+1)/\alpha)}\,.$$

■

E The Behavior of the Double Auction Equilibrium

Let

$$\zeta_{it} \; = \; \frac{(\alpha+1)}{c_{it}\,\overline{G}} \qquad (29)$$

and

$$\varepsilon_{it} \; = \; \frac{\zeta_{it}}{(|\log \zeta_{it}|/(\alpha+1))^{\gamma_{it}}}\,. \qquad (30)$$

Clearly, both ζ_{it} and ε_{it} are small when $\overline{G}$ is large.

Proposition E.1 *Let* $S_t = S_{i,j,t}$, $B_t = B_{i,j,t}$ *and* $\varepsilon_t = \varepsilon_{it}$. *We have, as* $\overline{G} \to \infty$,

$$S_t(\theta) \;\sim\; \mathcal{S}\left(\theta + \frac{1}{\alpha+1}\log\varepsilon_t\right),$$

where $\mathcal{S}(\theta)$ *is the inverse of the function in* z *defined by*

$$\log\frac{z-v_b}{v^H-z} - \frac{1}{\alpha+1}\log\left(\log\frac{1}{v^H-z} - (z-v_b)\right).$$

Similarly,

$$B_t(\theta) \;\sim\; \mathcal{B}\left(\theta - \frac{1}{\alpha+1}\log\varepsilon_t\right)$$

where $\mathcal{B}(z)$ *is the inverse of the function in* z *defined by*

$$\frac{1}{\alpha+1}\log\left(\log\frac{1}{v^H-z} - (z-v_b)\right).$$

Corollary E.2 *For any buyer-and-seller class pair* (i,j), $S_{i,j,t}(\theta)$ *is monotone decreasing in* t *and in any meeting probability* λ_i, *whereas* $B_{i,j,t}(\theta)$ *is monotone increasing in* t *and any* λ_i.

Proof. Without loss of generality, we assume for simplicity that $R = 1$. (This merely adds a constant to the inverse of the ask function, by Proposition 4.3.) We fix a time period $t \geq 0$ and omit the time index everywhere and write $V_b = V_{bt}$, $V_s = V_{st}$ for the inverses of the bid and ask functions. We also let $\gamma = \gamma_t$, $c = c_t$.

Let

$$\zeta \;=\; \zeta_t \;=\; \frac{(\alpha+1)^{\gamma+1}}{c\,\overline{G}}.$$

As in the proof of Proposition D.1, we define

$$g(z) \;=\; e^{(\alpha+1)V_b(z)} \;=\; \frac{\zeta}{(-\log\zeta)^{\gamma}}\,f(z) \;\stackrel{def}{=}\; \varepsilon\,f(z).$$

Then, as we have shown in the proof of Proposition D.1, we may assume that, for large $\overline{G}$,

$$f'(z) \;=\; \left(\frac{\log(1/\zeta)}{\log(1/\zeta) + \log(1/f(z))}\right)^{\gamma}\left(\frac{z-v_b}{v^H-z} + \varepsilon^{\frac{1}{\alpha+1}}\,f(z)^{\frac{1}{\alpha+1}}\right), \quad f(v_b) = 0. \quad (31)$$

See (24). Furthermore, as $\overline{G} \to \infty$, we have $\zeta, \varepsilon \to 0$,

$$\lim_{\overline{G}\to\infty} f(z) \;=\; f_0(z),$$

where

$$f_0(z) \;=\; (v^H - v_b)\,\log\frac{v^H - v_b}{v^H - z} \;-\; (z - v_b),$$

and the convergence is uniform on compact subsets of $[v_b, v^H)$.

From this point, for simplicity we take the case $\gamma = 0$. The general case follows by similar but lengthier arguments. Hence, we assume that f solves

$$f'(z) \;=\; \frac{z - v_b}{v^H - z} \;+\; \varepsilon^{1/\alpha+1}\, f^{1/(\alpha+1)}. \tag{32}$$

Since the solution $f(z)$ to (32) is uniformly bounded on compact subsets of $[v_b, v^H)$, by integrating (32) we find that

$$0 \;\le\; f(z) \;-\; f_0(z) \;=\; O(\varepsilon^{\frac{1}{\alpha+1}}\,(z \;-\; v_b)),$$

uniformly on compact subsets of $[v_b, v^H)$. Furthermore, $f_0(z) \;\le\; C_1\,(z - v_b)^2$, uniformly on compact subsets of $[v_b, v^H)$. Substituting these bounds into (32), we get

$$\begin{aligned} f(z) \;-\; f_0(z) \quad &\le\quad C_2\,\varepsilon^{\frac{1}{\alpha+1}} \int_{v_b}^{z} (\varepsilon^{1/\alpha+1}\,(z - v_b) \;+\; (z - v_b)^2)^{1/(\alpha+1)}\, dz \\ &\le\quad C_3\,\varepsilon^{\frac{1}{\alpha+1}}\,(z - v_b)\,(\varepsilon^{1/(\alpha+1)^2}\,(z - v_b)^{1/(\alpha+1)} \;+\; (z - v_b)^{2/(\alpha+1)}). \end{aligned}$$

Let now

$$l(z) \;=\; f(z)^{\alpha/(\alpha+1)} \;-\; \frac{\varepsilon^{1/\alpha+1}\alpha}{\alpha+1}\,(z - v_b)\,.$$

Then,

$$\begin{aligned} l'(z) \;&=\; \frac{\alpha}{\alpha+1}\, f'(z)\, f^{-1/(\alpha+1)} \;-\; \frac{\varepsilon^{1/\alpha+1}\alpha}{\alpha+1} \\ &=\; \frac{\alpha}{\alpha+1}\, \frac{z - v_b}{\left(\frac{\varepsilon^{1/\alpha+1}\alpha}{\alpha+1}\,(z - v_b) + l(z)\right)^{1/\alpha}} \\ &\le\; \frac{\alpha}{\alpha+1}\, \frac{z - v_b}{(l(z))^{1/\alpha}}\,. \end{aligned} \tag{33}$$

Integrating this inequality, we get

$$l(z) \;\le\; \frac{1}{2}(z - v_b)^2,$$

and therefore

$$f(z) \;\le\; C_4\,((z - v_b)^2 \;+\; \varepsilon^{1/\alpha}\,(z - v_b)^{(\alpha+1)/\alpha})\,. \tag{34}$$

Consequently,

$$e^{V_b(z)} \;=\; \varepsilon^{\frac{1}{\alpha+1}} \left(f_0(z) \;+\; o(\varepsilon^{\frac{1}{\alpha+1}}\,(z - v_b))\right)^{1/(\alpha+1)} \tag{35}$$

uniformly on compact subsets of $[v_b, v^H)$. Therefore,

$$\lim_{\varepsilon\to 0}\left(V_b(z) \;-\; \frac{1}{\alpha+1}\log\varepsilon\right) \;=\; \frac{1}{\alpha+1}\,\log f_0(z),$$

uniformly on compact subsets of (v_b, v^H).

Now, since $V_b \to -\infty$ uniformly on compact subsets of $[v_b, v^H)$,

$$V_s(z) \;=\; \log\frac{z-v_b}{v^H-z} \;-\; V_b(z)$$

converges to $+\infty$, uniformly on compact subsets of (v_b, v^H). Since $S(-\infty) = v_b$, standard arguments imply that $S(\theta)$ converges to v_b uniformly on compact subsets of $[-\infty, +\infty)$ (with $-\infty$ included). Furthermore,

$$\lim_{\varepsilon\to 0}\left(V_s(z) \;+\; \frac{1}{\alpha+1}\log\varepsilon\right) \;=\; \log\frac{z-v_b}{v^H-z} \;-\; \frac{1}{\alpha+1}\,\log f_0(z) \;\stackrel{def}{=}\; M(z),$$

uniformly on compact subsets of (v_b, v^H). Let $\mathcal{S}(z) \;=\; M^{-1}(z)$. We claim that

$$\lim_{\varepsilon\to 0} S\left(\theta \;-\; \frac{1}{\alpha+1}\log\varepsilon\right) \;=\; \mathcal{S}(\theta)\,, \tag{36}$$

uniformly on compact subsets of $\mathbb{R}$. Indeed, $S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)$ is the unique solution to the equation in y given by

$$\theta \;=\; V_s(y) \;+\; \frac{1}{\alpha+1}\log\varepsilon\,.$$

Since the right-hand side converges uniformly to the strictly monotone function $M(\,\cdot\,)$, this unique solution also converges uniformly to $\mathcal{S}(\theta)$. Furthermore, the equality

$$v_b \;+\; \Delta_b\,P(V_s(z) \;+\; V_b(z)) \;=\; z \;\Leftrightarrow v_b \;+\; \Delta_b\,P(\theta \;+\; V_b(S(\theta))) \;=\; S(\theta)$$

implies that

$$V_b\left(S\left(\theta \;-\; \frac{1}{\alpha+1}\log\varepsilon\right)\right) \;=\; \log\left(\frac{S-v_b}{v^H-S}\right) \;-\; \theta \;+\; \frac{1}{\alpha+1}\log\varepsilon$$

and therefore

$$V_b\left(S\left(\theta \;-\; \frac{1}{\alpha+1}\log\varepsilon\right)\right) \;-\; \frac{1}{\alpha+1}\log\varepsilon \;\to\; \log\left(\frac{\mathcal{S}(\theta)-v_b}{v^H-\mathcal{S}(\theta)}\right) \;-\; \theta.$$

We have

$$M(z) \;=\; \log\left(\frac{z-v_b}{(v^H-z)\left((v^H-v_b)\,\log\left(\frac{v^H-v_b}{v^H-z}\right) \;-\; (z-v_b)\right)^{1/(\alpha+1)}}\right).$$

Now, for $z \sim v_b$,

$$\log\left(\frac{v^H - v_b}{v^H - z}\right) \;=\; -\log\left(1 \;-\; \frac{z - v_b}{v^H - v_b}\right) \;\sim\; \frac{z - v_b}{v^H - v_b} \;+\; \frac{1}{2}\left(\frac{z - v_b}{v^H - v_b}\right)^2, \tag{37}$$

and therefore

$$M(z) \;\sim\; (1+\alpha)^{-1}\log(2(v^H - v_b)) \;+\; \frac{\alpha-1}{\alpha+1}\log\left(\frac{z - v_b}{v^H - v_b}\right) \tag{38}$$

as $z \to v_b$. Consequently, as $\theta \to -\infty$, we have

$$\mathcal{S}(\theta) \;\sim\; v_b \;+\; K\, e^{\frac{\alpha+1}{\alpha-1}\theta}$$

for some constant $K = K(\alpha)$. ■

F The Behavior of Some Important Integrals

For simplicity, many results in this section will be established under technical conditions on α. The general case can be handled similarly, but is significantly more messy. As above, we fix a pair $(i,j) = (b,s)$ and use S_t and B_t to denote the corresponding double auction equilibrium. Recall that ψ^H_{st} is the cross-sectional density of the information type of sellers at time t.

As previously, we consider the case of large $\overline{G}$ and use the notation $A \sim B$ to denote that $A/B \to 1$ when $\overline{G} \to \infty$.

Lemma F.1 *Let*

$$\frac{\alpha+1}{\alpha-1} > \alpha\,.$$

Then

$$\int_{\mathbb{R}} (v_b \;-\; S_\tau(y))\, \psi^H_{s\tau}(y)\, dy \;\sim\; c_{s\tau}\, \varepsilon^{\frac{\alpha}{\alpha+1}} \left|\frac{\log\varepsilon}{1+\alpha}\right|^{\gamma_{s\tau}} \int_{\mathbb{R}} (v_b - \mathcal{S}(y))\, e^{-\alpha y}\, dy$$

and

$$\int_{\mathbb{R}} (v_b \;-\; S_\tau(y))\, \psi^L_{s\tau}(y)\, dy \;=\; o(\varepsilon^{\frac{\alpha}{\alpha+1}})$$

as $\overline{G} \to \infty$.

Proof. In the following, we handle the cases of $\psi^L_{s\tau}$ and $\psi^H_{s\tau}$ simultaneously by using the notation "$\psi^{H,L}_{s\tau}$." Changing variables, we get

$$\begin{aligned} &\int_{\mathbb{R}} (v_b \;-\; S_\tau(y))\, \psi^{H,L}_{s\tau}(y)\, dy \\ &= \int_{\mathbb{R}} \psi^{H,L}_{s\tau}\left(y - \frac{1}{\alpha+1}\log\varepsilon\right)\left(v_b \;-\; S_\tau\left(y - \frac{1}{\alpha+1}\log\varepsilon\right)\right) dy\,. \end{aligned} \tag{39}$$

Furthermore, by Lemma G.6,

$$\lim_{\varepsilon\to 0} c_{s\tau}^{-1}\,\varepsilon^{-\{\alpha,\alpha+1\}/(\alpha+1)}\,\left|\frac{\log\varepsilon}{1+\alpha}\right|^{-\gamma_{s\tau}}\,\psi_{s\tau}^{H,L}\left(y-\frac{1}{\alpha+1}\log\varepsilon\right)\;=\;e^{-\{\alpha,\alpha+1\}\,y}.$$

By (36),

$$v_b\;-\;S_\tau\left(y-\frac{1}{\alpha+1}\log\varepsilon\right)\;\to\;v_b\;-\;\mathcal{S}(y)\,.$$

In order to conclude that

$$\begin{aligned}\lim_{\varepsilon\to 0}\varepsilon^{-\alpha/(\alpha+1)}\int_{\mathbb{R}}\psi_{s\tau}^{H}\left(y-\frac{1}{\alpha+1}\log\varepsilon\right)\left(v_b\;-\;S_\tau\left(y-\frac{1}{\alpha+1}\log\varepsilon\right)\right)dy\\ =\;c_{s\tau}\int_{\mathbb{R}}e^{-\alpha y}\,(v_b\;-\;\mathcal{S}(y))\,dy,\end{aligned}\tag{40}$$

and that

$$\int_{\mathbb{R}}\psi_{s\tau}^{L}\left(y-\frac{1}{\alpha+1}\log\varepsilon\right)\left(v_b\;-\;S_\tau\left(y-\frac{1}{\alpha+1}\log\varepsilon\right)\right)dy\;=\;o(\varepsilon^{\alpha/(\alpha+1)}),$$

we will show that the integrands

$$I(y)=\varepsilon^{-\alpha/(\alpha+1)}\,\psi_{s\tau}^{H}\left(y-\frac{1}{\alpha+1}\log\varepsilon\right)\left(v_b\;-\;S_\tau\left(y-\frac{1}{\alpha+1}\log\varepsilon\right)\right)$$

and

$$\varepsilon^{-\epsilon}\varepsilon^{-\alpha/(\alpha+1)}\,\psi_{s\tau}^{L}\left(y-\frac{1}{\alpha+1}\log\varepsilon\right)\left(v_b\;-\;S_\tau\left(y-\frac{1}{\alpha+1}\log\varepsilon\right)\right)$$

have an integrable majorant for some $\epsilon>0$. Then, (40) will follow from the Lebesgue dominated convergence theorem.

We decompose the integral in question into three parts, as

$$\int_{-\infty}^{\frac{1}{1+\alpha}\log\varepsilon}I_1(y)\,dy\;+\;\int_{\frac{1}{1+\alpha^L}\log\varepsilon}^{A}I_2(y)\,dy\;+\;\int_{A}^{+\infty}I_3(y)\,dy,$$

and prove the required limit behavior for each integral separately. To this end, we will need to establish sharp bounds for $S(\theta)$ and $V_b(\theta)$.

Lemma F.2 *Let $\Omega\subset\mathbb{R}_+^2$ be a bounded open set and $\mathcal{L}(\theta,\varepsilon)\in C^b(\Omega)$ be a bounded, continuous function. Then we have*

$$S\left(\theta-\frac{1}{\alpha+1}\log\varepsilon\right)\;\le\;v_b\;+\;C_1\,\mathcal{L}(\theta,\varepsilon)\tag{41}$$

for all $(\varepsilon, \theta) \in \Omega$ *if and only if*

$$\frac{1}{\alpha+1}\, \log f(v_b \,+\, \mathcal{L}(\theta,\varepsilon)) \;-\; \log(\mathcal{L}(\theta,\varepsilon)) \le\; C_2 \,-\, \theta\,. \tag{42}$$

If (41) *holds, we have*

$$V_b\left(S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\right) \;\le\; \frac{\log\varepsilon}{1+\alpha} \,+\, C_3 \,+\, \log\mathcal{L}(\theta,\varepsilon) \,-\, \theta. \tag{43}$$

Proof. Applying V_s to both sides of (41) and using the fact that V_s is strictly increasing, we see that the desired inequality is equivalent to

$$\theta - \frac{1}{\alpha+1}\log\varepsilon \;\le\; V_s(v_b \,+\, C_1\,\mathcal{L})\,.$$

Now,

$$V_s(z) + \frac{1}{\alpha+1}\log\varepsilon \;=\; \log\frac{z-v_b}{v^H-z} - V_b(z) + \frac{1}{\alpha+1}\log\varepsilon \;=\; \log\frac{z-v_b}{v^H-z} - \frac{1}{\alpha+1}\,\log f(z)\,.$$

The claim follows because we are in the regime when $v^H - z$ is uniformly bounded away from zero.

Furthermore,

$$-\,\frac{\log\varepsilon}{1+\alpha} \,+\, V_b(S) \;=\; \log\left(\frac{S-v_b}{v^H-S}\right) \,-\, \theta - \log R\,. \tag{44}$$

If θ is bounded from above, then S is uniformly bounded away from v^H, and hence

$$\log\left(\frac{S-v_b}{v^H-S}\right) \,-\, \theta \;\le\; C_4 \,+\, \log(S-v_b) \,-\, \theta.$$

The claim follows. ■

Lemma F.3 *Suppose that* $\varepsilon > 0$ *is sufficiently small. Fix an* $A > 0$. *Then, for*

$$\theta \in \left(\frac{1}{\alpha+1}\log\varepsilon, A\right) \tag{45}$$

we have

$$S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right) \;\le\; v_b \,+\, C_5\, e^{\frac{\alpha+1}{\alpha-1}\theta}\,, \tag{46}$$

and for

$$\theta \;<\; \frac{1}{\alpha+1}\log\varepsilon, \tag{47}$$

we have that

$$S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right) \;\le\; v_b \,+\, C_6\,\varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\, e^{\frac{\alpha}{\alpha-1}\theta}\,. \tag{48}$$

Proof. By Lemma F.2, inequality (48) is equivalent to

$$\frac{1}{\alpha+1}\,\log f(v_b\,+\,C_6\,\varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\,e^{\frac{\alpha}{\alpha-1}\theta})\;-\;\log(C_6\,\varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\,e^{\frac{\alpha}{\alpha-1}\theta})\;\le\;-\theta\,+\,C_7\,. \tag{49}$$

Under the condition (47),

$$\max\left\{(z-v_b)^2\,,\,\varepsilon^{1/\alpha}\,(z-v_b)^{(\alpha+1)/\alpha}\right\}\;=\;\varepsilon^{1/\alpha}\,(z-v_b)^{(\alpha+1)/\alpha} \tag{50}$$

for

$$z\;=\;C_8\,\varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\,e^{\frac{\alpha}{\alpha-1}\theta}\,.$$

Hence, by (34),

$$f(z)\;\le\;C_9\,\varepsilon^{1/\alpha}\,(z-v_b)^{(\alpha+1)/\alpha}\,.$$

Consequently,

$$\begin{aligned}
&\frac{1}{\alpha+1}\,\log f(v_b\,+\,C_6\,\varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\,e^{\frac{\alpha}{\alpha-1}\theta})\;-\;\log\left(C_6\,\varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\,e^{\frac{\alpha}{\alpha-1}\theta}\right)\\
&\quad\le\;C_{10}\;+\;\frac{1}{(\alpha+1)\,\alpha}\log\varepsilon\;+\;\frac{1}{\alpha}\left(\frac{\alpha}{\alpha-1}\theta\;+\;\frac{1}{(\alpha+1)(\alpha-1)}\log\varepsilon\right)\\
&\qquad-\;\left(\frac{\alpha}{\alpha-1}\theta\;+\;\frac{1}{(\alpha+1)(\alpha-1)}\log\varepsilon\right)\\
&\quad=\;-\theta\;+\;C_{10}\,,
\end{aligned} \tag{51}$$

and (48) follows.

Similarly, when θ satisfies (45), a direct calculation shows that

$$\max\left\{(z-v_b)^2\,,\,\varepsilon^{1/\alpha}\,(z-v_b)^{(\alpha+1)/\alpha}\right\}\;=\;(z-v_b)^2 \tag{52}$$

for

$$z\;=\;v_b\;+\;C_5\,e^{\frac{\alpha+1}{\alpha-1}\theta}\,.$$

Since Therefore, by (34),

$$\begin{aligned}
&\frac{1}{\alpha+1}\,\log f(v_b\,+\,C_5\,e^{\frac{\alpha+1}{\alpha-1}\theta})\;-\;\log(C_5\,e^{\frac{\alpha+1}{\alpha-1}\theta}\,)\\
&\qquad\qquad\le\;C_{11}\;+\;\frac{2}{\alpha-1}\theta-\frac{\alpha+1}{\alpha-1}\theta\;=\;-\theta\;+\;C_{11},
\end{aligned} \tag{53}$$

and (46) follows. ■

As above, we recall that $\psi^H_{s\tau}$ is the cross-sectional density of the information type of sellers at time τ. As above, we handle the cases of $\psi^L_{s\tau}$ and $\psi^H_{s\tau}$ simultaneously by using the notation "$\psi^{H,L}_{s\tau}$."

Lemma F.4 *If*

$$\frac{\alpha+1}{\alpha-1} \;>\; \alpha,$$

then

$$\int_{-\infty}^{\frac{1}{\alpha+1}\,\log\varepsilon} \psi^{H,L}_{s\tau}\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\,\left(v_b \;-\; S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\right)\,d\theta \;=\; o\big(\varepsilon^{\alpha/(\alpha+1)}\big)\,.$$

Proof. By (47), since $\psi^{H,L}_{s\tau}$ is bounded, we get

$$\begin{aligned}
&\int_{-\infty}^{\frac{1}{\alpha+1}\,\log\varepsilon} \psi^{H,L}_{s\tau}\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\,\left(v_b \;-\; S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\right)\,d\theta \\
&\leq\; C_{12}\int_{-\infty}^{\frac{1}{\alpha+1}\,\log\varepsilon} \varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\, e^{\frac{\alpha}{\alpha-1}\theta}\,d\theta \\
&=\; \varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\,\frac{\alpha-1}{\alpha}\,\varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}+\frac{\alpha}{(\alpha+1)(\alpha-1)}} \\
&=\; o\big(\varepsilon^{\alpha/(\alpha+1)}\big)\,.
\end{aligned} \tag{54}$$

■

Lemma F.5 *If*

$$\frac{\alpha+1}{\alpha-1} \;>\; \alpha,$$

then

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\,\varepsilon^{-\frac{\alpha}{\alpha+1}}\int_{\frac{1}{\alpha+1}\,\log\varepsilon}^{A} \psi^{H}_{s\tau}\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\,\left(v_b \;-\; S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\right)\,d\theta \\
&\qquad =\; c_{s\tau}\int_{-\infty}^{A} (v_b - \mathcal{S}(\theta))\,e^{-\alpha\,\theta}\,d\theta
\end{aligned} \tag{55}$$

and

$$\int_{\frac{1}{\alpha+1}\log\varepsilon}^{A} \psi^{L}_{s\tau}\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\,\left(v_b \;-\; S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\right)\,d\theta \;=\; o\big(\varepsilon^{\alpha/(\alpha+1)}\big).$$

Proof. By assumption, as $x \to \infty$,

$$\psi^{H}_{s\tau}(x) \;\sim\; c_{s\tau}\,e^{-\alpha x}.$$

The claim follows from (36) and (45), which provides an integrable majorant. ■

The same argument implies the following result.

Lemma F.6 *We have*

$$\lim_{\varepsilon\to 0}\varepsilon^{-\frac{\alpha}{\alpha+1}}\int_A^{+\infty}\psi^H_{s\tau}\left(\theta-\frac{1}{\alpha+1}\log\varepsilon\right)\left(v_b\;-\;S\left(\theta-\frac{1}{\alpha+1}\log\varepsilon\right)\right)d\theta$$
$$=\;c_{s\tau}\int_A^{+\infty}(v_b-\mathcal{S}(\theta))\,e^{-\alpha\,\theta}\,d\theta \tag{56}$$

and

$$\int_A^{+\infty}\psi^L_{s\tau}\left(\theta-\frac{1}{\alpha+1}\log\varepsilon\right)\left(v_b\;-\;S\left(\theta-\frac{1}{\alpha+1}\log\varepsilon\right)\right)d\theta\;=\;o(\varepsilon^{\alpha/(\alpha+1)})\,.$$

■

We define, for $K\in\{H,L\}$,

$$G^K_{\eta,q_{0,\tau-1}}(x)=\int_x^{+\infty}(\eta^K\,*\,q^K_{0,\tau-1})(y)\,dy$$
$$F^K_{\eta,q_{0,\tau-1}}(x)=1\;-\;G^K_{\eta,q_{0,\tau-1}}(x), \tag{57}$$

where $q_{0,\tau}=q_{i,0,\tau}$ is the density of increment to information type that an agent of class i will get during the time interval $[0,\tau]$ from trading with counterparties of class j. That is,

$$q_{i,0,0}\;=\;(1-\lambda_i)\,\delta_0\;+\;\lambda_i\,\psi_{j\,0}.$$

and

$$q_{i,0,\tau+1}\;=\;(1-\lambda_i)\,q_{i,0,\tau}\;+\;\lambda_i\sum_j\kappa_{ij}\,q_{i,0,\tau}*\psi_{j\,\tau+1}.$$

Furthermore, everywhere in the sequel we assume that the density η of the type of an acquired signal packet satisfies $\eta^H\sim\mathrm{Exp}_{+\infty}(c_\eta,\gamma_\eta,-\alpha)$ for some $c_\eta,\;\gamma_\eta>0$. This is without loss of generality by Condition 2 and Lemma 4.5 on p. 29, which together imply that any number of acquired signal packets satisfies this condition. That is, a convolution of densities satisfying the specified tail condition also satisfies the same condition. The same argument also implies that

$$q^H_{i,0,\tau}\sim\mathrm{Exp}_{+\infty}(c_{i,0,\tau},\gamma_{i,0,\tau},-\alpha)$$

for some $c_{i,0,\tau},\gamma_{i,0,\tau}>0$ and

$$\eta^H\,*\,q^H_{i,0,\tau}\;\sim\;\mathrm{Exp}_{+\infty}(C_{i,\eta,0,\tau},\gamma_{i,0,\tau}+\gamma_\eta+1,-\alpha)$$

for some $C_{i,\eta,0,\tau}>0$.

Lemma F.7 *Suppose that*

$$\frac{(\alpha+1)^2}{\alpha-1} \;>\; 2\alpha+1\,.$$

Then,

$$\begin{aligned}&\int_{\mathbb{R}} \psi^H_{s\tau}(y)\,(v^H \,-\, S_\tau(y))\,F^H_{\eta,q_{0,\tau-1}}(V_{b\tau}(S_\tau(y)))\,dy \;\sim\; R^{-(\alpha+1)}\,c_{s\tau}\,\frac{c_{s,0,\tau-1}}{\alpha+1}\,C_{s,\eta,0,\tau-1}\\ &\times\varepsilon^{\frac{2\alpha+1}{\alpha+1}}\left|\frac{\log\varepsilon}{1+\alpha}\right|^{\gamma_{s\tau}+\gamma_{s,0,\tau-1}+\gamma_\eta+1}\int_{\mathbb{R}}(v^H \,-\, \mathcal{S}(y))\left(\frac{\mathcal{S}(y)-v_b}{v^H-\mathcal{S}(y)}\right)^{\alpha+1}e^{-y(2\alpha+1)}\,dy\end{aligned} \tag{58}$$

and

$$\begin{aligned}&\int_{\mathbb{R}} \psi^L_{s\tau}(y)\,(S_\tau(y)-v_b)\,F^L_{\eta,q_{0,\tau-1}}(V_{b\tau}(S_\tau(y)))\,dy \;\sim\; R^{-\alpha}\,c_{s\tau}\,\frac{c_{s,0,\tau-1}}{\alpha}\,C_{s,\eta,0,\tau-1}\\ &\times\varepsilon^{\frac{2\alpha+1}{\alpha+1}}\left|\frac{\log\varepsilon}{1+\alpha}\right|^{\gamma_{s\tau}+\gamma_{s,0,\tau-1}+\gamma_\eta+1}\int_{\mathbb{R}}(\mathcal{S}(y)-v_b)\left(\frac{\mathcal{S}(y)-v_b}{v^H-\mathcal{S}(y)}\right)^{\alpha}e^{-y(2\alpha+1)}\,dy\,.\end{aligned} \tag{59}$$

as $\overline{G}\to\infty$.

Proof. As $x\to-\infty$, we have

$$F^{H,L}_{\eta,q_{0,\tau-1}}(x)\;\sim\;\frac{c_{s,0,\tau-1}\,C_{s,\eta,0,\tau}}{\{\alpha+1,\alpha\}}\,e^{x\,\{\alpha+1,\alpha\}}\,|x|^{\gamma_{s,0,\tau-1}+\gamma_\eta+1}\,.$$

The claim follows by the arguments used in the proof of Lemma F.1. Special care is needed only because $(v^H-S)^{-1}$ blows up as $\theta\uparrow+\infty$.

By (44),

$$\begin{aligned}&F^H_{\eta,q_{0,\tau-1}}\left(V_b\left(S\left(\theta-\frac{1}{\alpha+1}\log\varepsilon\right)\right)\right)\\ &\le\; C_{13}\,\varepsilon\left(\frac{S-v_b}{v^H-S}\,e^{-\theta}\right)^{\alpha+1}\left|\log\left(\frac{S-v_b}{v^H-S}\,e^{-\theta}\varepsilon^{\frac{1}{\alpha+1}}\right)\right|^{\gamma_{s,0,\tau-1}+\gamma_\eta+1}.\end{aligned} \tag{60}$$

Thus, to get an integrable majorant in a neighborhood of $+\infty$, it would suffice to have a bound

$$v^H-S\;\ge\;C_{14}\,e^{-\beta\theta}$$

with some $\beta>0$ such that $\beta\alpha<2\alpha+1$, because this would guarantee that

$$\left(\frac{S-v_b}{v^H-S}\,e^{-\theta}\right)^{\alpha}\left|\log\left(\frac{S-v_b}{v^H-S}\,e^{-\theta}\varepsilon^{\frac{1}{\alpha+1}}\right)\right|^{\gamma_{s,0,\tau-1}+\gamma_\eta+1}e^{-\alpha\theta}\;\le\;\tilde{C}_{14}e^{-\bar{\varepsilon}\theta}$$

for some $\bar{\varepsilon}>0$. By the argument used in the proof of Lemma F.2, it suffices to show that for sufficiently large θ,

$$\frac{1}{\alpha+1}\,\log f(v_H\,-\,C_{14}\,e^{-\beta\theta})\;\le\;C_{15}\,+\,(\beta-1)\,\theta\,.$$

Now, it follows from (32) that

$$f'(z) \ \le\ f(z)^{1/(\alpha+1)} \ +\ \frac{v^H - v_b}{v^H - z}.$$

Since, for sufficiently small ε, $f(z)$ is uniformly bounded away from zero on compact subsets of $(v_b, v^H]$, we get

$$\frac{d}{dz}(f(z)^{\alpha/(\alpha+1)}) \ \le\ C_{16}\,(1 \,+\, (v^H - z)^{-1}),$$

for some $K > 0$ when z is close to v^H. Integrating this inequality, we get

$$f(z)^{\alpha/(\alpha+1)} \ \le\ C_{17}\,(1 \,-\, \log(v^H - z)).$$

Consequently,

$$\frac{1}{\alpha+1}\log f(v_H \,-\, C_{14}\,e^{-\beta\theta}) \ \le\ C_{18}\,\log\theta$$

if θ is sufficiently large. Hence, the required inequality holds for any $\beta > 1$ with a sufficiently large C_{14}, and the claim follows. ■

Lemma F.8 *Let*

$$\frac{\alpha+1}{\alpha-1} \ >\ \alpha\,.$$

Then

$$\begin{aligned}&\int_{\mathbb{R}} (S_\tau(y) - v_b) \,\times\, (\eta^H * q^H_{t,\tau-1})(y-\theta)\,dy \\ &\sim \frac{c_{b,t,\tau-1}}{\alpha+1}\,C_{b,\eta,0,\tau-1}\left|\frac{\log\varepsilon}{1+\alpha}\right|^{\gamma_{b,t,\tau-1}+\gamma_\eta+1}\ \varepsilon^{\frac{\alpha}{\alpha+1}}\ \int_{\mathbb{R}}(\mathcal{S}(y) - v_b)\,e^{-\alpha(y-\theta)}\,dy\end{aligned} \tag{61}$$

and

$$\int_{\mathbb{R}} (S_\tau(y) - v_b) \,\times\, (\eta^L * q^L_{t,\tau-1})(y-\theta)\,dy \ =\ o\left(\left|\frac{\log\varepsilon}{1+\alpha}\right|^{\gamma_{b,t,\tau-1}+\gamma_\eta+1}\ \varepsilon^{\frac{\alpha}{\alpha+1}}\right)\,. \tag{62}$$

as $\overline{G} \to \infty$.

Lemma F.9 *Let*

$$\frac{(\alpha+1)\alpha}{\alpha-1} \ >\ \alpha\,.$$

Then we have, as $\overline{G} \to \infty$,

$$\begin{aligned}\int_{\mathbb{R}} (S_\tau(y) - v_s)\, F^L_{b\tau}(V_{b\tau}(S_\tau(y)))\, (\eta^L * q^L_{t,\tau})(y-\theta)dy \sim\ & c_{b,t,\tau-1}\, R^{-\alpha} C_{b,\eta,0,\tau-1} e^{(\alpha+1)\theta} \\ \times \left|\frac{\log\varepsilon}{1+\alpha}\right|^{\gamma_{b,t,\tau-1}+\gamma_\eta+1} \varepsilon^{\frac{\alpha}{\alpha+1}}\, \frac{\alpha+1}{\alpha} & \int_{\mathbb{R}} e^{-(2\alpha+1)y} \left(\frac{\mathcal{S}(y)-v_b}{v^H - \mathcal{S}(y)}\right)^{\alpha} dy\end{aligned} \tag{63}$$

and

$$\begin{aligned}&\int_{\mathbb{R}} (v^H - S_\tau(y))\, F^H_{b\tau}(V_{b\tau}(S_\tau(y)))\, (\eta^H * h^H_{t,\tau})(y-\theta)dy \\ &= o\left(\left|\frac{\log\varepsilon}{1+\alpha}\right|^{\gamma_{b,t,\tau-1}+\gamma_\eta+1} \varepsilon^{\frac{\alpha}{\alpha+1}}\right).\end{aligned} \tag{64}$$

G Proofs for Case of Initial Information Acquisition

For any given agent i, the expected utility $U_{i,t,\tau}$ from trading during the time interval $[t,\tau]$ is

$$U_{i,t,\tau}(\theta) \;=\; \sum_{r=t}^{\tau} u_{i,t,r}(\theta)\,,$$

where $u_{i,t,r}$ is the expected utility from trading at time r conditional on the agent's information at time t, evaluated at the information type outcome θ.

We denote further by $u_{i,t,r}(\theta;\eta)$ the expected utility from trading at time r conditional on the agent's information at time t after the agent has made the decision to acquire a signal packet with type density $\eta^{H,L}$, before the type of the acquired signal is observed. With this notation, $u_{i,t,r}(\theta) = u_{i,t,r}(\theta;\delta_0)$. The following lemma provides expressions for $u_{i,t,r}(\theta;\eta)$. These expressions follows directly from the definition of the double-auction trading mechanism.

Lemma G.1 *For a given buyer with posterior information type* θ *at time* 0,

$$\begin{aligned}u_{b,0,\tau}(\theta;\eta) \;=\;& P(\theta)\lambda \int_{\mathbb{R}} (v^H - S_\tau(y))\, G^H_{\eta,q_{0,\tau-1}}(V_{b\tau}(S_\tau(y)) - \theta)\, \psi^H_{s\tau}(y)\, dy \\ &+\; (1-P(\theta))\lambda \int_{\mathbb{R}} (v_b - S_\tau(y))\, G^L_{\eta,q_{0,\tau-1}}(V_{b\tau}(S_\tau(y)) - \theta)\, \psi^L_{s\tau}(y)\, dy,\end{aligned} \tag{65}$$

whereas a seller's utility is

$$\begin{aligned}u_{s,0,\tau}(\theta;\eta) \;=\;& P(\theta)\,\lambda \int_{\mathbb{R}} (S_\tau(y) - v^H)\, G^H_{b\tau}(V_{b\tau}(S_\tau(y)))\, (\eta^H * q^H_{0,\tau-1})(y-\theta)\, dy \\ &+\; (1-P(\theta))\,\lambda \int_{\mathbb{R}} (S_\tau(y) - v_s)\, G^L_{b\tau}(V_{b\tau}(S_\tau(y)))\, (\eta^L * q^L_{0,\tau-1})(y-\theta)\, dy.\end{aligned} \tag{66}$$

Here, by convention, we set $q^K_{t,t-1} \;=\; \delta_0$.

The next result provides approximate expressions for the gains from information acquisition when $\overline{G}$ is sufficiently large. Recall that the asymptotic behaviour for large $\overline{G}$ in the double auction between a class i of buyers and a class j of seller is determined by

$$\zeta_{it} \;=\; \frac{(\alpha+1)}{c_{it}\,\overline{G}}\,.$$

Lemma G.2 *Let b be a buyer of class i. Denote by $\mathbf{s}$ the set of seller classes with which buyers of class i trade and let*

$$\gamma_{s\tau} \;\equiv\; \max_{j\in\mathbf{s}} \gamma_{j\tau}.$$

Further, let

$$\mathbf{s}_m \;=\; \{j\in\mathbf{s}\;:\;\gamma_{j\tau}=\gamma_{s\tau}\}\,.$$

Let also $\gamma_\tau \;\equiv\; \gamma_{b\tau}$. Then

$$u_{b,0,\tau}(\theta;\eta)-u_{b,0,\tau}(\theta) \;\sim\; \frac{e^{-\alpha\theta}R^{-\alpha}}{1+Re^{\theta}}\, I_b^{\text{gain}}\,\lambda \int_{\mathbb{R}} (v^H\;-\;\mathcal{S}(y))\,\left(\frac{\mathcal{S}(y)-v_b}{v^H-\mathcal{S}(y)}\right)^{\alpha+1} e^{-y(2\alpha+1)}\,dy, \tag{67}$$

as $\overline{G}\to\infty$, where

$$\begin{aligned} I_b^{\text{gain}} \;&=\; \sum_{j\in\mathbf{s}_m} c_{j\tau}\,\frac{1}{\alpha(\alpha+1)}\left(\frac{\zeta_\tau}{(|\log\zeta_\tau|/(\alpha+1))^{\gamma_\tau}}\right)^{\frac{2\alpha+1}{\alpha+1}} \left|\frac{\log\zeta_\tau}{1+\alpha}\right|^{\gamma_{j\tau}} c_{s,0,\tau-1}\left|\frac{\log\zeta_\tau}{1+\alpha}\right|^{\gamma_{b,0,\tau-1}} \\ &\quad\times\left(C_{b,\eta,0,\tau}(\bar N_b,\bar N_s)\left|\frac{\log\zeta_\tau}{1+\alpha}\right|^{\gamma_\eta+1}-1\right) \\ &=\; \sum_{j\in\mathbf{s}_m} c_{j\tau}\,\frac{1}{\alpha(\alpha+1)}\,\zeta_\tau^{\frac{2\alpha+1}{\alpha+1}}\,c_{s,0,\tau-1}\left|\frac{\log\zeta_\tau}{1+\alpha}\right|^{\gamma_{b,0,\tau-1}-\frac{\alpha}{\alpha+1}\gamma_\tau+(\gamma_{j\tau}-\gamma_\tau)} \\ &\quad\times\left(C_{b,\eta,0,\tau}(\bar N_b,\bar N_s)\left|\frac{\log\zeta_\tau}{1+\alpha}\right|^{\gamma_\eta+1}-1\right)\,. \end{aligned} \tag{68}$$

Lemma G.3 *Let s be a seller of class i. Denote by $\mathbf{b}$ the set buyer classes with which seller of class i trade and let*

$$\gamma_\tau \;\equiv\; \max_{j\in\mathbf{b}} \gamma_{j\tau}.$$

Further, let

$$\mathbf{b}_m \;=\; \{j\in\mathbf{b}\;:\;\gamma_{j\tau}=\gamma_\tau\}\,.$$

Then

$$u_{s,0,\tau}(\theta;\eta) - u_{s,0,\tau}(\theta) \ \sim \ \frac{e^{(\alpha+1)\theta}\, R^{-\alpha}}{1+R\,e^{\theta}}\, I_s^{\text{gain}} \times G_s$$

as $\overline{G}\to\infty$, *where*

$$G_s = \lambda \int_{\mathbb{R}} \left((\mathcal{S}(y)-v_b) - \frac{\alpha+1}{\alpha}\, e^{-(\alpha+1)y} \left(\frac{\mathcal{S}(y)-v_b}{v^H-\mathcal{S}(y)}\right)^{\alpha}\right) e^{-\alpha y}\, dy$$

and

$$I_s^{\text{gain}} \ = \ \sum_{j\in \mathbf{b}_m} \zeta_{j\tau}^{\frac{\alpha}{\alpha+1}} \left(C_{b,\eta,0,\tau}(\bar N_b,\bar N_s) \left|\frac{\log\zeta_{j\tau}}{1+\alpha}\right|^{\gamma_\eta+1} - 1\right) c_{b,0,\tau-1} \left|\frac{\log\zeta_{j\tau}}{1+\alpha}\right|^{\gamma_{s,0,\tau-1}-\frac{\alpha}{\alpha+1}\gamma_\tau}.$$

Lemmas G.2 and G.3 follow directly from Lemmas F.1-F.9 above. The following result is then an immediate consequence.

Corollary G.4 *For buyers and sellers, the utility gain from acquiring information is convex in the number of signal packets acquired. Consequently, any optimal pure strategy is either to acquire the maximum number* $\bar n$ *of signal packets, or to acquire none.*

The next lemma is a direct consequence of Lemma 4.5.

Lemma G.5 *Suppose that* $\lambda_{ij}\equiv\lambda_i\kappa_{ij}\neq 0$ *for all* i,j.[3] *Let* $\bar N_s$ *be the maximal number of signal packets for sellers, and* $\bar N_b$ *the maximal number of signals for buyers. Then, for any class* i,

$$\gamma_{i1} \ = \ \bar N_i + \mathbf{1}_{i\in s}\bar N_b + \mathbf{1}_{i\in b}\bar N_s,$$

and thus, for all $t\geq 2$,

$$\gamma_{it} \ = \ 2^{t-1}(\bar N_b+\bar N_s) \ - \ 1 \ + \ \bar N_i - \mathbf{1}_{i\in s}\bar N_s - \mathbf{1}_{i\in b}\bar N_b,$$

where we write $i\in b$ *if class* i *is a buyer class, and similarly for the sellers' classes. Furthermore,*

$$\gamma_{i,0,\tau-1} \ = \ \gamma_{it}-\bar N_i\,.$$

Proof of Theorem 5.2. It follows from Lemmas G.2-G.3 that it suffices to show that the exponents for $|\log\zeta|$ are monotone increasing in N if T is sufficiently large. For buyers, we have

$$\gamma_{b,0,\tau-1} - \frac{2\alpha+1}{\alpha+1}\gamma_\tau + \gamma_{j\tau} \ = \ -N_b - \frac{\alpha}{\alpha+1}\gamma_{b\tau} + \ \gamma_{j\tau}$$

[3]The case when some of the matching probabilities are zero can be studied by a limiting procedure.

whereas, for sellers, we need to show that

$$\gamma_{s,0,\tau-1} - \frac{\alpha}{\alpha+1}\gamma_\tau$$

is monotone increasing in the number of acquired signals. This follows directly from Lemma G.5. ■

We will now study examples illustrating our general model. We will first treat the case of one class of sellers, and then consider the case of two classes of sellers.

G.1 One Class of Sellers

In order to calculate the equilibria, we will first need to determine the dependence of the cross-sectional type distributions on the model parameters. Suppose that buyers and sellers acquire N_b and N_s signal packets respectively. Then, let $\bar{N}_i = N_{\min} + N_i$ be the total number of signals packets that class i possesses. The maximum feasible number of signal packets is $N_{\max} = N_{\min} + \bar{n}$. Using Lemma 4.5, we immediately get the following two technical lemmas.

Lemma G.6 *Suppose that at time* 0 *buyers and seller acquire* $\bar{N}_b$ *and* $\bar{N}_s$ *signals respectively. Then,* $c_{bt} = c_{st} = c_t$ *and* $\gamma_{bt} = \gamma_{st} = \gamma_t$ *so that* $\psi_{st}\ ,\ \psi_{bt}\ \sim\ \mathrm{Exp}_{-\infty}(c_t, \gamma_t, \alpha+1)$ *for all* $t \geq 1$*, where* $\gamma_1 = \bar{N}_b + \bar{N}_s - 1$*. It follows that* $\gamma_t = 2\gamma_{t-1} + 1$ *for* $t \geq 2$*,*

$$c_1 \;=\; \lambda\, c_{s0} c_{b0} \frac{(\bar{N}_s - 1)!\,(\bar{N}_b - 1)!}{(\bar{N}_s + \bar{N}_b - 1)!}$$

and

$$c_{t+1} \;=\; \lambda\, c_t^2\, \frac{(\gamma_t!)^2}{\gamma_{t+1}!}\,.$$

In particular,

$$\gamma_t \;=\; 2^{t-1}(\bar{N}_b + \bar{N}_s) \;-\; 1$$

and

$$c_t \;=\; D_{\bar{N}_b, \bar{N}_s}(t)\, c_0^{2^{t-1}(\bar{N}_s + \bar{N}_b)}\, \lambda^{2^t - 1},$$

for a model-independent combinatorial function $D_{\bar{N}_b, \bar{N}_s}(t)$*.*

Lemma G.7 *For* $i = b$ *or* $i = s$*, we have* $q^H_{i,0,\tau}\ \sim\ \mathrm{Exp}_{-\infty}(c_{i,0,\tau}, \gamma_{i,0,\tau}, \alpha+1)$*, where*

$$\gamma_{i,0,\tau} \;=\; (2^\tau \;-\; 1)\,(\bar{N}_b + \bar{N}_s) \;-\; 1 \;+\; \bar{N}_j$$

and

$$c_{i,0,\tau} \;=\; D_{i,\bar{N}_b,\bar{N}_s}(0, \tau, c_0)\, \lambda^{2^{\tau+1}-1},$$

for model-independent combinatorial functions $D_{\bar{N}_b,\bar{N}_s}(t,\tau)$ *and* $D_{i,\bar{N}_b,\bar{N}_s}(0,\tau)$*.*

Proof of Lemma C.3. By the Perron-Frobenius Theorem (see Meyer (2000), chapter 8, page 668), we have

$$(\Lambda_s \Lambda_b)^{t-1} \sim r_s^{t-1} p_s q_s^T$$

where p_s and q_s are right and left Perron eigenvectors of $\Lambda_s \Lambda_b$ respectively, and r_s is the corresponding Perron eigenvalue. Similarly,

$$(\Lambda_b \Lambda_s)^{t-1} \sim r_b^{t-1} p_b q_b^T$$

where p_b and q_b are the right and left Perron eigenvectors of $\Lambda_b \Lambda_s$ respectively, and r_b is the corresponding Perron eigenvalue. Now, applying Λ_b to the identity $\Lambda_s \Lambda_b p_s = r_s p_s$, we get that $\Lambda_b p_s$ is a positive right eigenvector of $\Lambda_b \Lambda_s$ corresponding to a positive eigenvalue r_s. Uniqueness part of the Perron-Frobenius Theorem (see Meyer (2000), chapter 8, page 667) implies that $r_s = r_b$. To prove the last statement, we note that, by the Collatz-Wielandt formula (see Meyer (2000), chapter 8, page 667),

$$r_s = \max_x \min_i \frac{(\Lambda_s \Lambda_b x)_i}{x_i} = \min_i \frac{(\Lambda_s \Lambda_b p_s)_i}{p_{si}}$$

If we increase one of the elements of Λ_s or Λ_b, all coordinates of $\Lambda_s \Lambda_b p_s$ become strictly larger since $p_s > 0$, and hence the Collatz-Wielandt formula implies that r_s also strictly increases. ■

Proof of Proposition 5.5. The claim of monotonicity in $N_{\min}$ and $\bar{n}$ follows directly from Lemmas F.1-F.9 and the proof of Theorem 5.2. Furthermore, for large t, $c_{s_i,t}/c_{b_j,t}^{\alpha/(\alpha+1)}$ is monotone increasing in λ_{s_k,b_l} if and only if so does the principal eigenvalue r_s, and hence the claim follows from Lemma C.3.

This completes the proof of the claim for seller and buyer classes from $\mathbf{s}$ and $\mathbf{b}$.

For a seller class $i \notin \mathbf{s}$, we have $c_{i,t} = \sum_j \lambda_{i,b_j} c_{j,t-1}^b$ by Lemma C.2, and the claim follows. A similar argument applies for a buyer of class $i \notin \mathbf{b}$, with the only exception that λ_{i,s_j} appear in the denominator leading to within-class strategic substitutability of matching probabilities. The latter however is offset by the factor λ_i entering the expected gains from trade. ■

We will also need the following auxiliary lemma, whose proof is straightforward.

Lemma G.8 *For $i \in \{b, s\}$, let $\mathrm{Gain}_i(\bar{N}_s, \bar{N}_b)$ denote the utility gain from acquiring the maximum number $\bar{n} = N_{\max} - N_{\min}$ of signal packets, for a market in which all other* *buyers and sellers have $\bar{N}_b$ and $\bar{N}_s$ signal packets, respectively. Let*

$$\begin{aligned} &\pi_1 \equiv \mathrm{Gain}_s(N_{\max}, N_{\min}), \quad \pi_2 \equiv \mathrm{Gain}_s(N_{\min}, N_{\min}), \\ &\pi_3 \equiv \mathrm{Gain}_b(N_{\max}, N_{\max}), \quad \pi_4 \equiv \mathrm{Gain}(N_{\max}, N_{\min}). \end{aligned} \tag{69}$$

Then:

- $(N_{\max}, N_{\min})$ *is an equilibrium if and only if* $\pi \in [\pi_4, \pi_1]$.
- $(N_{\max}, N_{\max})$ *is an equilibrium if and only if* $\pi \le \pi_3$.
- $(N_{\min}, N_{\min})$ *is an equilibrium if and only if* $\pi \ge \pi_2$.

Lemma G.9 *Let* $\widetilde{T} \equiv \log_2(\alpha+1)+1$. *Then, the following are true:*

- *If* $T = 0$ *then* $\pi_1 = \pi_2 > \pi_4 > \pi_3$. *Thus, an equilibrium exists if and only if* $\pi \notin (\pi_3, \pi_4)$.
- *If* $0 < T < \widetilde{T}$ *then* $\pi_1 > \pi_2 > \pi_4 > \pi_3$, *and an equilibrium exists if and only if* $\pi \notin (\pi_3, \pi_4)$.
- *If* $t > \widetilde{T}$ *then* $\pi_1 > \pi_2 > \pi_3 > \pi_4$, *and an equilibrium always exists.*
- *For all* i, π_i *is increasing in* $N_{\min}$ *and in* $\bar{n}$.

Proof. For small values of ε, the constants π_1, π_2, π_3, and π_4 satisfy

$$\pi_k \ \sim\ \mathfrak{A}_i(0, \bar{N}_s, \bar{N}_b)\, Z_i(0, \bar{N}_s, \bar{N}_b),$$

for corresponding pairs of $\bar{N}_b, \bar{N}_s$. Here,

$$\mathfrak{A}_i(0, \bar{N}_s, \bar{N}_b) \ = \ (N_{\max} - N_{\min})^{-1} \left(C_{j,\eta_{N_{\max}},0,T} \left| \frac{\log \zeta}{1+\alpha} \right|^{N_{\max}} - C_{j,\eta_{N_{\min}},0,T} \left| \frac{\log \zeta}{1+\alpha} \right|^{N_{\min}} \right), \tag{70}$$

where $j = s$ when $i = b$, and where $j = b$ when $i = s$. Furthermore,

$$\begin{aligned} Z_b(0, \bar{N}_s, \bar{N}_b) \ &\sim\ \lambda \frac{R^{-\alpha}}{1+R} \mathfrak{D}_b(0, c_0, \bar{N}_b, \bar{N}_s, \alpha) \lambda^{2^T-1} \left(\frac{1}{\lambda^{2^T-1}\,\overline{G}} \right)^{\frac{2\alpha+1}{\alpha+1}} \\ &\qquad \times\ \lambda^{2^T-1} \left|\log(\overline{G})\right|^{(2^{T-1}\ -\ 1)\,(\bar{N}_b+\bar{N}_s)\ -\ 1+\bar{N}_s - \frac{\alpha}{\alpha+1}(2^{T-1}(\bar{N}_b+\bar{N}_s)-1)} \\ &= \ \frac{R^{-\alpha}}{1+R} \mathfrak{D}_b(0, c_0, \bar{N}_b, \bar{N}_s, \alpha)\, \lambda^{\frac{2^T-(\alpha+1)+2\alpha+1}{\alpha+1}}\, (\overline{G})^{-\frac{2\alpha+1}{\alpha+1}} \\ &\qquad \times\ |\log \overline{G}|^{\frac{2^{T-1}}{\alpha+1}(\bar{N}_b+\bar{N}_s)-\bar{N}_b-\frac{1}{\alpha+1}}, \end{aligned} \tag{71}$$

for some function $\mathfrak{D}_b(0, c_0, \bar{N}_b, \bar{N}_s, \alpha)$. Similarly,

$$Z_s(0, \bar{N}_s, \bar{N}_b) \sim \frac{R^{-\alpha}}{1+R} \mathfrak{D}_s(0, c_0, \bar{N}_b, \bar{N}_s, \alpha) \lambda^{\frac{2^T - (\alpha+1) + 2\alpha+1}{\alpha+1}} (\overline{G})^{-\frac{\alpha}{\alpha+1}} \times |\log \overline{G}|^{\frac{2^T-1}{\alpha+1}(\bar{N}_b+\bar{N}_s) - \bar{N}_b - \frac{1}{\alpha+1}}, \tag{72}$$

for some function $\mathfrak{D}_s(c_0, \bar{N}_b, \bar{N}_s, \alpha)$. For $T = 0$, there is only one trading round and therefore

$$Z_s(0, \bar{N}_s, \bar{N}_b) = \frac{R^{-\alpha}}{1+R} \mathfrak{D}_b(c_0, \bar{N}_b, \bar{N}_s, \alpha) \lambda (\overline{G})^{-\frac{\alpha}{\alpha+1}} |\log(\overline{G})|^{-(\bar{N}_b-1)\alpha/(\alpha+1)}$$

and

$$Z_b(0, \bar{N}_s, \bar{N}_b) = \frac{R^{-\alpha}}{1+R} \mathfrak{D}_s(c_0, \bar{N}_b, \bar{N}_s, \alpha) \lambda (\overline{G})^{-\frac{2\alpha+1}{\alpha+1}} |\log \overline{G}|^{-(\bar{N}_b-1)\alpha/(\alpha+1) + (\bar{N}_s - \bar{N}_b)}.$$

When $\overline{G}$ is sufficiently large, $Z_s > Z_b$ and the impact of $\mathfrak{D}_i$ and $C_{i,\eta,0,\tau}(\bar{N}_b, \bar{N}_s)$ is small and does not affect the monotonicity results. The claim follows by direct calculation. ■

G.2 Two Classes of Sellers

As above, we denote by $\bar{N}_i = N_{\min} + N_i$ the total number of signal packets held by agents of class i. We have the following results.

Let $\bar{N}_s = \max\{\bar{N}_1, \bar{N}_2\}$ and let $m \in \{1, 2\}$ be the corresponding seller class that acquired more information and $-m$ be the other seller class. Then,

$$\lambda = \begin{cases} 0.5\lambda_m, & \bar{N}_1 \neq \bar{N}_2 \\ 0.5(\lambda_1 + \lambda_2), & \bar{N}_1 = \bar{N}_2 . \end{cases}$$

Lemma G.10 *We have $\psi_{l,t} \sim \operatorname{Exp}_{-\infty}(c_{lt}, \gamma_{lt}, \alpha+1)$ for $l \in \{s_1, s_2, b\}$ for all $t \geq 1$, where $\gamma_{s_k,1} = \bar{N}_k + \bar{N}_b - 1$ and $\gamma_{b1} = \bar{N}_s + \bar{N}_b - 1$, and where, for $t \geq 2$,*

$$\gamma_{s_k,t} = \gamma_{s_k,t-1} + \gamma_{b,t-1} + 1 \tag{73}$$

$$\gamma_{b,t} = \gamma_{b,t-1} + \gamma_{s_m,t-1} + 1 \tag{74}$$

and where further

$$c_{b1} = \lambda c_{s0} c_{b0} \frac{(\bar{N}_s - 1)!\,(\bar{N}_b - 1)!}{(\bar{N}_s + \bar{N}_b - 1)!}, \quad c_{s_k,1} = \lambda_k c_{s_k,0} c_{b0} \frac{(\bar{N}_k - 1)!\,(\bar{N}_b - 1)!}{(\bar{N}_k + \bar{N}_b - 1)!},$$

$$c_{b,t+1} = c_{bt} \frac{\gamma_{bt}!\,\gamma_{s_m,t}!}{\gamma_{b,t+1}!} \begin{cases} \lambda c_{s_m,t}, & \bar{N}_1 \neq \bar{N}_2 \\ 0.5(\lambda_1 c_{s_1,t} + \lambda_2 c_{s_2,t}), & \bar{N}_1 = \bar{N}_2, \end{cases}$$

and

$$c_{s_k,t+1} \;=\; c_{bt}\,\frac{\gamma_{bt}!\,\gamma_{s_m,t}!}{\gamma_{b,t+1}!}\,\lambda_k\,c_{s_k,t}.$$

Consequently,

$$\gamma_{bt} \;=\; \gamma_{s_m,t} \;=\; 2^{t-1}(\bar N_b+\bar N_s)\,-\,1$$

and, for $t\geq 2$,

$$\gamma_{s_{-m},t} \;=\; 2^{t-1}(\bar N_b+\bar N_s)\,-\,1\;+\;\bar N_{-m}-\bar N_s.$$

Thus, for $\bar N_1\neq\bar N_2$,

$$c_{bt} \;=\; D_{\bar N_b,\bar N_s}(t)\,c_0^{2^{t-1}(\bar N_s+\bar N_b)}\,(0.5\lambda_m)^{2^t-1}\;,\;\; c_{s_k,t} \;=\; D_{\bar N_b,\bar N_1,\bar N_2}(t)\,\lambda_k^t\,(0.5\lambda_m)^{2^t-t-1},$$

for some combinatorial functions $D_{\bar N_b,\bar N_s}(t), D_{k,\bar N_b,\bar N_1,\bar N_2}(t)$.

However, when $\bar N_1=\bar N_2$, we get

$$c_{b,t} \;=\; d_{b,\bar N_b,\bar N_s}(t)\,(\lambda_1^t+\lambda_2^t)\prod_{r=1}^{t-1}(\lambda_1^r+\lambda_2^r)^{2^{t-r-1}}$$

and

$$c_{s_k,t} \;=\; d_{s,\bar N_b,\bar N_s}(t)\,\lambda_k^t\prod_{r=1}^{t-1}(\lambda_1^r+\lambda_2^r)^{2^{t-r-1}},$$

for some combinatorial functions $d_{b,\bar N_b,\bar N_s}(t)$ and $d_{s,\bar N_b,\bar N_s}(t)$.

Now, we need to calculate $\gamma_{t,\tau}$.

Lemma G.11 *We have $h^H_{l,t,\tau}\;\sim\;\mathrm{Exp}_{-\infty}(c_{l,t,\tau},\gamma_{t,\tau},\alpha+1)$, where*

$$c_{s_k,t,t} \;=\; \lambda_k\,c_{bt},\qquad \gamma_{t,t} \;=\; \gamma_{bt},$$

and

$$c_{b,t,t} \;=\; \begin{cases}0.5\lambda_m\,c_{s_m,t}\;, & \bar N_1\neq\bar N_2\\ 0.5(\lambda_1 c_{s_1,t}+\lambda_2\,c_{s_2,t})\;, & \bar N_1=\bar N_2\,.\end{cases}$$

Then we define inductively

$$c_{s_k,t,\tau+1} \;=\; \lambda_k\,c_{s_k,t,\tau}\,c_{b,\tau+1}\,\frac{\gamma_{s_k,t,\tau}!\,\gamma_{b,\tau+1}!}{\gamma_{s,0,\tau+1}!},\;\; \gamma_{s,0,\tau+1} \;=\; \gamma_{s,0,\tau}+\gamma_{b,\tau+1}+1$$

and

$$c_{b,t,\tau+1} \;=\; c_{b,t,\tau}\,\frac{\gamma_{b,t,\tau}!\,\gamma_{s_m,\tau+1}!}{\gamma_{b,t,\tau+1}!}\begin{cases}0.5\lambda_m c_{s_m,\tau+1}\;, & \bar N_1\neq\bar N_2\\ 0.5\,(\lambda_1 c_{s_1,\tau+1}\;+\;\lambda_2\,c_{s_2,\tau+1})\;, & \bar N_1=\bar N_2\,,\end{cases}$$

and

$$\gamma_{b,t,\tau+1} \;=\; \gamma_{b,t,\tau} + \gamma_{s_m,\tau+1} + 1.$$

In particular, for $t > 0$,

$$\gamma_{l,t,\tau} \;=\; (2^{\tau} \;-\; 2^{t-1})\,(\bar{N}_b + \bar{N}_s) \;-\; 1\;,\; l \in \{s_1, s_2, b\},$$

For $t = 0$,

$$\gamma_{s,0,\tau} \;=\; (2^{\tau} \;-\; 1)\,(\bar{N}_b + \bar{N}_s) \;-\; 1 \;+\; \bar{N}_b\;,\;\; \gamma_{b,0,\tau} \;=\; (2^{\tau} \;-\; 1)\,(\bar{N}_b + \bar{N}_s) \;-\; 1 \;+\; \bar{N}_s\,.$$

If $\bar{N}_1 \neq \bar{N}_2$ *then*

$$c_{b,t,\tau} \;=\; D_{b,\bar{N}_b,\bar{N}_s}(t,\tau,c_0)\,\lambda_m^{2^{\tau+1}-2^t}, \qquad c_{s_m,t,\tau} \;=\; D_{s,\bar{N}_b,\bar{N}_s}(t,\tau,c_0)\,\lambda_m^{2^{\tau+1}-2^t},$$

and

$$c_{s_{-m},t,\tau} \;=\; \left(\frac{\lambda_{-m}}{\lambda_m}\right)^{\tau-t+1} c_{s_m,t,\tau}$$

for all $t \geq 0$, *for some combinatorial functions* $D_{\bar{N}_b,\bar{N}_s}(t,\tau)$ *and* $D_{l,\bar{N}_b,\bar{N}_s}(0,\tau)$.

When $\bar{N}_1 = \bar{N}_2$,

$$c_{s_k,t,\tau} \;=\; d_{s,\bar{N}_b,\bar{N}_s}(t,\tau)\lambda_k^{\tau-t+1}\left(\prod_{r=1}^{t-1}(\lambda_1^r + \lambda_2^r)^{2^{\tau-r}-2^{t-r-1}}\right)\prod_{r=t}^{\tau}(\lambda_1^r + \lambda_2^r)^{2^{\tau-r}}$$

and

$$c_{b,t,\tau} \;=\; d_{b,\bar{N}_b,\bar{N}_s}(t,\tau)\frac{\lambda_1^{\tau+1} + \lambda_2^{\tau+1}}{\lambda_1^t + \lambda_2^t}\left(\prod_{r=1}^{t-1}(\lambda_1^r + \lambda_2^r)^{2^{\tau-r}-2^{t-r-1}}\right)\prod_{r=t}^{\tau}(\lambda_1^r + \lambda_2^r)^{2^{\tau-r}}$$

for all $t \geq 0$, *for some combinatorial functions* $d_{k,\bar{N}_b,\bar{N}_s}(t,\tau)$ *and* $d_{b,\bar{N}_b,\bar{N}_s}(t,\tau)$.

Proposition G.12 *Suppose that* $T > \widetilde{T}$. *Let* $\lambda_1 \leq \lambda_2$. *In equilibrium, we always have* $\bar{N}_b \leq \bar{N}_1 \leq \bar{N}_2$. *Furthermore, there exist constants* $\pi_1 > \pi_2 > \pi_3 > \pi_4 > \pi_5 > \pi_6$ *such that the following are true:*

1. *If* $\pi > \pi_1$ *then the unique equilibrium is* $(\bar{N}_b, \bar{N}_1, \bar{N}_2) = (N_{\min}, N_{\min}, N_{\min})$.

2. *If* $\pi_1 > \pi > \pi_2$ *then there are two equilibria:*

    - $(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\min}, N_{\min}, N_{\min})$
    - $(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\min}, N_{\max}, N_{\max})$.

3. If $\pi_2 > \pi > \pi_3$ *then there are three equilibria:*

- $(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\min}, N_{\min}, N_{\min})$
- $(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\min}, N_{\max}, N_{\max})$
- $(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\min}, N_{\min}, N_{\max})$.

4. If $\pi_3 > \pi > \pi_4$ *then there are two equilibria:*

- $(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\min}, N_{\min}, N_{\min})$
- $(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\min}, N_{\max}, N_{\max})$.

5. If $\pi_4 > \pi > \pi_5$ *then there is a unique equilibrium*

$$(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\min}, N_{\max}, N_{\max}).$$

6. If $\pi_5 > \pi > \pi_6$ *there are two equilibria:*

- $(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\max}, N_{\max}, N_{\max})$
- $(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\min}, N_{\max}, N_{\max})$.

7. If $\pi_6 > \pi$ *then there is a unique equilibrium*

$$(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\max}, N_{\max}, N_{\max}).$$

Proof. Denote by $\text{Gain}_i(\bar{N}_b, \bar{N}_1, \bar{N}_2)$ the gains from acquiring the maximal number of signals for an agent of class i, conditional on the numbers of signals packets acquired by all other agents. As in Lemma G.8, we define

$$\begin{aligned}
\pi_1 \;&\equiv\; \text{Gain}_1(N_{\min}, N_{\max}, N_{\max}), \quad \pi_2 \;\equiv\; \text{Gain}_2(N_{\min}, N_{\min}, N_{\max}) \\
\pi_3 \;&\equiv\; \text{Gain}_1(N_{\min}, N_{\min}, N_{\max}), \quad \pi_4 \;\equiv\; \text{Gain}_2(N_{\min}, N_{\min}, N_{\min}) \\
\pi_5 \;&\equiv\; \text{Gain}_b(N_{\max}, N_{\max}, N_{\max}), \quad \pi_6 \;\equiv\; \text{Gain}_b(N_{\min}, N_{\max}, N_{\max})\,.
\end{aligned} \tag{75}$$

Then, it suffices to prove that π_i are monotone decreasing in i. As in the proof of Lemma G.9, we have

$$\pi_i \;\sim\; \mathfrak{A}_i\, Z_i,$$

and it remains to study the asymptotic behavior of Z_i. We have

$$Z_b(0, \bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; Z_b^{s_1}(0, \bar{N}_b, \bar{N}_1, \bar{N}_2) \;+\; Z_b^{s_2}(0, \bar{N}_b, \bar{N}_1, \bar{N}_2),$$

where

$$Z_b^{s_k}(0,\bar N_b,\bar N_1,\bar N_2)\ \sim\ 0.5\lambda_k\,\frac{R^{-\alpha}}{1+R}\,\mathfrak{D}_b(0,c_0,\bar N_b,\bar N_s,\alpha)\,c_{bT}\,\left(\frac{1}{c_{bT}\,\overline{G}}\right)^{\frac{2\alpha+1}{\alpha+1}} \times\ c_{b,0,\tau-1}\,|\log\overline{G}|^{\gamma_{b,0,T-1}-\frac{\alpha}{\alpha+1}\gamma_T} \tag{76}$$

for some function $\mathfrak{D}_b(0,c_0,\bar N_b,\bar N_s,\alpha)$. Similarly,

$$Z_{s_k}(0,\bar N_b,\bar N_1,\bar N_2)\ \sim\ \frac{R^{-\alpha}}{1+R}\,\mathfrak{D}_s(0,c_0,\bar N_b,\bar N_s,\alpha)\,\lambda_k\,c_{s_k,0,T-1}\,(c_{bT}\,\overline{G})^{-\frac{\alpha}{\alpha+1}} \times|\log\overline{G}|^{\gamma_{s,0,T-1}-\frac{\alpha}{\alpha+1}\gamma_T}\,. \tag{77}$$

We first study equilibria with $\bar N_1 = N_{\min} < \bar N_2 = N_{\max}$. Since, for both seller classes, the surpluses from acquiring information are of comparable magnitude and are much larger than those of the buyers, we ought to have $\bar N_b = N_{\min}$. This will be an equilibrium if

$$\pi > (Z_b^{s_1}(0,N_{\min},N_{\min},N_{\max})\ +\ Z_b^{s_2}(0,N_{\min},N_{\min},N_{\max}))\,\mathfrak{A}_b,$$

but this automatically follows from

$$\pi_3\ \sim\ \mathfrak{A}Z_{s_1}(0,N_{\min},N_{\min},N_{\max})\ <\ \pi\ <\ \mathfrak{A}Z_{s_2}(0,N_{\min},N_{\min},N_{\max})\ \sim\ \pi_2.$$

Since $Z_{s_1}/Z_{s_2}=(\lambda_1/\lambda_2)^{\tau+1}$, this is only possible if $\lambda_1<\lambda_2$. Furthermore,

$$Z_{s_k}(0,N_{\min},N_{\min},N_{\max})\ \sim\ \frac{R^{-\alpha}}{1+R}\,\tilde{\mathfrak{D}}_s\,\lambda_k\,\left(\frac{\lambda_k}{\lambda_2}\right)^T\ \lambda_2^{\frac{1}{\alpha+1}(2^T-1)}\overline{G}^{-\frac{\alpha}{\alpha+1}} \times|\log\overline{G}|^{(2^{T-1}\ -\ 1)\,(N_{\min}+N_{\max})\ -\ 1+N_{\min}-\frac{\alpha}{\alpha+1}(2^{T-1}(N_{\min}+N_{\max})-1)}. \tag{78}$$

Now, $\bar N_1=\bar N_2=N_{\max},\ \ \bar N_b=N_{\min}$ forms an equilibrium if and only if

$$\pi>\pi_6\ \sim\ (Z_b^{s_1}(0,N_{\min},N_{\max},N_{\max})\ +\ Z_b^{s_2}(0,N_{\min},N_{\max},N_{\max}))\,\mathfrak{A}_b$$

and

$$\begin{aligned}\pi<\pi_1\ &\sim\ Z_{s_1}(0,N_{\min},N_{\max},N_{\max})\\ &\sim\ \frac{R^{-\alpha}}{1+R}\,\tilde{\mathfrak{D}}_s\,\lambda_1\,\frac{\lambda_1^T}{(\lambda_1^T+\lambda_2^T)^{\alpha/(\alpha+1)}}\prod_{r=1}^{T-1}(\lambda_1^r+\lambda_2^r)^{\frac{1}{\alpha+1}2^{T-r}}\\ &\times\overline{G}^{-\frac{\alpha}{\alpha+1}}|\log\overline{G}|^{(2^{T-1}\ -\ 1)\,(N_{\min}+N_{\max})\ -\ 1+N_{\min}-\frac{\alpha}{\alpha+1}(2^{T-1}(N_{\min}+N_{\max})-1)}.\end{aligned} \tag{79}$$

Next, $\bar{N}_b = \bar{N}_1 = \bar{N}_2 = N_{\min}$ is an equilibrium if and only if

$$\begin{aligned} \pi > \pi_4 \ &\sim\ Z_{s_2}(0, N_{\min}, N_{\min}, N_{\min}) \\ &\sim\ \frac{R^{-\alpha}}{1+R}\,\tilde{\mathfrak{D}}_s\,\lambda_2\,\frac{\lambda_2^T}{(\lambda_1^T+\lambda_2^T)^{\alpha/(\alpha+1)}} \\ &\times \prod_{r=1}^{T-1}(\lambda_1^r+\lambda_2^r)^{\frac{1}{\alpha+1}2^{T-r}}\,\overline{G}^{-\frac{\alpha}{\alpha+1}}\,|\log\overline{G}|^{(2^{T-1}\ -\ 1)\,(2N_{\min})\ -\ 1+N_{\min}-\frac{\alpha}{\alpha+1}(2^{T-1}(2N_{\min})-1)}\,. \end{aligned} \tag{80}$$

Finally, $\bar{N}_b = \bar{N}_1 = \bar{N}_2 = N_{\max}$ is an equilibrium if and only if

$$\pi < \pi_5 \ \sim\ (Z_b^{s_1}(0, N_{\max}, N_{\max}, N_{\max})\ +\ Z_b^{s_2}(0, N_{\max}, N_{\max}, N_{\max}))\,\mathfrak{A}_b\,. \tag{81}$$

The fact that π_i decreases with i follows directly from their asymptotic expressions. ■

Lemma G.13 *There exists a unique solution $\hat{T} > \max\{2, \widetilde{T}\}$ to the equation $(\alpha+1)\hat{T} = 2^{\hat{T}} - 1$, and a unique solution $\bar{T}$ to the equation $(2\alpha+1)\bar{T} = 2^{\bar{T}} - 1$. Furthermore,*

$$\frac{\prod_{r=0}^{T-1}(\lambda_1^r+\lambda_2^r)^{2^{T-1-r}}}{(\lambda_1^T+\lambda_2^T)^{\alpha}}$$

- *is monotone decreasing in λ_2 for all $\lambda_2 \ge \lambda_1$ if $T \le \hat{T}$.*
- *is monotone increasing in λ_2 for all $\lambda_2 \ge \lambda_1$ if $T \ge \bar{T}$.*

Proof. The fact that $\hat{T}$ exists and is unique follows directly from the convexity of the function 2^T. To prove that $\widetilde{T} < \hat{T}$, we need to show that $(\alpha+1)\widetilde{T} > 2^{\widetilde{T}} - 1$. Substituting $\widetilde{T} = \log_2(\alpha+1)+1$, we get

$$2^{\widetilde{T}} - 1 - (\alpha+1)\widetilde{T} = 2(\alpha+1) - 1 - (\alpha+1)(\log_2(\alpha+1)+1) = \alpha - (\alpha+1)\log_2(\alpha+1) < 0,$$

because $\alpha+1 > 2$ implies that $\log_2(\alpha+1) > 1$.

Let now $x = \lambda_2/\lambda_1 \ge 1$. Then, by homogeneity, it suffices to show that

$$\frac{\prod_{r=0}^{T-1}(1+x^r)^{2^{T-1-r}}}{(1+x^T)^{\alpha}}$$

is monotone decreasing in x. Differentiating, we see that we need to show that

$$\sum_{r=1}^{T-1} 2^{T-1-r} r\,\frac{x^r}{1+x^r}\ \le\ \alpha\,T\,\frac{x^T}{1+x^T}\,.$$

Since $x \geq 1$, we have

$$\frac{x^r}{1+x^r} \leq \frac{x^T}{1+x^T}.$$

Therefore, using the simple identity

$$\sum_{r=1}^{T-1} 2^{T-1-r} r \;=\; 2^T - 1 - T,$$

we get

$$\sum_{r=1}^{T-1} 2^{T-1-r} r \, \frac{x^r}{1+x^r} \;\leq\; (2^T - 1 - T)\, \frac{x^T}{1+x^T} \leq \alpha\, T\, \frac{x^T}{1+x^T}$$

for all $T \leq \hat{T}$. Similarly, since

$$\frac{x^r}{1+x^r} \geq \frac{1}{2} \;\geq\; \frac{1}{2} \frac{x^T}{1+x^T},$$

we get that

$$\sum_{r=1}^{T-1} 2^{T-1-r} r \, \frac{x^r}{1+x^r} \;\geq\; (2^T - 1 - T)\, \frac{1}{2} \frac{x^T}{1+x^T} \geq \alpha\, T\, \frac{x^T}{1+x^T}$$

for all $T \geq \bar{T}$. ■

The next proposition gives the partial-equilibrium impact on the information gathering incentives of class-1 sellers of increasing the contact probability λ_2 of the more active sellers.

Proposition G.14 *Suppose Condition 2 holds and $\lambda_1 \leq \lambda_2$. Fixing the numbers N_1, N_2, and N_b of signal packets gathered by all agents, consider the utility $u_{1n} - u_{1N_1}$ of a particular class-1 seller for gathering n signal packets. There exist integers $\bar{T}$ and $\hat{T}$, larger than the time $\widetilde{T}$ of Proposition 5.6 such that, for any $n > N_1$, the utility gain $u_{1n} - u_{1N_1}$ of acquiring additional signal packets is decreasing in λ_2 for $0 < T < \hat{T}$ and is increasing in λ_2 for $T > \bar{T}$.*

Proofs of Propositions G.14 and 5.7. Monotonicity of the gains Gain_1 follows from Lemma G.13 and the expressions for this gain, provided in the proof of Proposition G.12. Proposition 5.7 follows from Lemma G.13 if we set $\mathcal{K} = \pi_1$. ■

H Two-Class Case

This appendix focuses more closely on information acquisition externalities by specializing to the case in which all investors have the same contact probability λ. In this case, there are only two classes of investors, buyers b and sellers s. For a small time horizon T, the lack of complementarity suggested by Proposition 5.7 implies that symmetric equilibria may fail to exist. For larger T, symmetric equilibria always exist and are generally non-unique.

Definition H.1 *An asymmetric rational expectations equilibrium is: for each class $i \in \{b, s\}$, the masses $p_{in}, n = 0, \ldots, \bar{n}$, where p_{in} is the mass of the sub-group of group i that acquires exactly n packets; for each time t and seller-buyer pair (i, j), a pair (S_{ijt}, B_{ijt}) of bid and ask functions; and for each class i and time t, a cross-sectional type distribution ψ_{it} such that:*

(1) The cross-sectional type distribution ψ_{it} is initially $\psi_{i0} = \sum_{n=0}^{\bar{n}} p_{in} \, \overline{\psi}^{*(N_{\min}+n)}$ and satisfies the evolution equation (6).

(2) The bid and ask functions (S_{ijt}, B_{ijt}) form the equilibrium uniquely defined by Theorem 4.7.

(3) Each $n \in \{0, \ldots, \bar{n}\}$ with $p_{in} > 0$ solves $\max_{n \in \{0,\ldots,\bar{n}\}} u_{in}$, for each class i.

It turns out that, in all asymmetric equilibria, agents in each sub-group either do not acquire information at all or acquire the maximal number $\bar{n}$ of signal packets. We will denote the corresponding strategy $(\bar{n}, p_i)$, meaning that a group of mass p_i of agents of class i acquires the maximal number $\bar{n}$ of packets and the other sub-group (of mass $1 - p_i$) does not acquire any information.

Proposition H.2 *There exist thresholds $\overline{\pi} > \hat{\pi} > \underline{\pi}$ such that the following are true.*

1. *If $T < \widetilde{T}$ then:*
   - *A symmetric equilibrium exists if and only if $\pi \notin (\underline{\pi}, \hat{\pi})$.*
   - *An asymmetric equilibrium exists if and only if $\pi \geq \underline{\pi}$.*
2. *If $T > \widetilde{T}$, then:*
   - *A symmetric equilibrium always exists.*

- *Asymmetric equilibria exist if and only if $\pi \leq \overline{\pi}$.*

Furthermore, there is always at most one equilibrium in which different sub-groups of sellers acquire different amounts of information, and at most one equilibrium in which different sub-groups of buyers acquire different amounts of information.

In order to determine how the equilibrium mass of those agents who acquire information depends on the model parameters, we need to study the behavior of the gain from acquiring information. The next proposition studies externalities from information acquisition by other agents on the information acquisition incentives of any given agent in an out-of-equilibrium setting.

Proposition H.3 *For all i, the gain*

$$\mathrm{Gain}_i \;=\; \max_{n>0}\{(u_{in}-u_{i0})/n\}$$

from information acquisition is increasing in $N_{\min}, \bar{n}$, and λ. Fix an i and suppose that only a subgroup of mass p_i of class-i agents acquire information. Let us also fix the information acquisition policy of the other class.

1. *If $T > \widetilde{T}$, then Gain_i/p_i is monotone increasing in p_i.*
2. *If $T < \widetilde{T}$, then Gain_i/p_i is monotone decreasing in p_i.*

Proof of Proposition H.3. Let $\overline{\pi} \equiv \pi_1 > \pi_2 \equiv \underline{\pi}$. Suppose that a mass p of buyers acquire $\bar{n}$ packets and the rest (mass $1-p$) do acquire no packets. For our asymptotic formulae, this is equivalent to simply multiplying c_0 by $p^{1/N_{\max}}$ for the initial density of the buyers' type distribution. Furthermore, the same recursive calculation as above implies that $c_{i,\tau}$ is proportional to $p^{2^{\tau-1}}$ for $\tau > 0$ whereas $c_{i,0,\tau}$ is proportional to $p^{2^{\tau}-1}$. By the same argument as above, sellers always acquire more information and therefore we ought to have that $\bar{N}_s = N_{\max}$. The equilibrium condition is just the indifference condition for a buyer,

$$p\pi \;=\; \mathrm{Gain}_b,$$

because then a seller will always acquire information since the gain from doing so is always higher for him. Substituting the asymptotic expressions for the gains of information acquisition, we get the asymptotic relation

$$p\pi \;\sim\; p^{\min\{1,2^{T-1}+1\}}\, p^{\max\{2^{T-1}-1,0\}}\, p^{-\frac{2\alpha+1}{\alpha+1}\min\{1,2^{T-1}\}}\,\pi_3.$$

For $T < \widetilde{T}$, this gives a unique equilibrium value of p for any $\pi \geq \pi_3$. For $T > \widetilde{T}$, this gives a unique value of p for all $\pi \leq \pi_3$.

Similarly, for the case when different groups of sellers acquire different amounts of information, the equilibrium condition is

$$p\pi \;\sim\; p^{\max\{2^{T-1},1\}}\, p^{-\frac{\alpha}{\alpha+1}\min\{1,2^{T-1}\}}\, \pi_1$$

For $T < \widetilde{T}$, this gives a unique equilibrium value for p for any $\pi \geq \pi_1$. For $T > \widetilde{T}$, this gives a unique value for p for all $\pi \leq \pi_1$.

The fact that there are no equilibria in which both buyers and sellers acquire information asymmetrically follows from the expressions for the asymptotic size of the gains of information acquisition. ∎

The intuition behind Proposition H.3 is similar to that behind Proposition G.14. An increase in the mass p gives rise to both a learning effect and a pricing effect. The learning effect dominates the pricing effect if and only if there are sufficiently many trading rounds, that is, when $T > \widetilde{T}$.

Now, the equilibrium indifference condition, determining the mass p_i is given by

$$\pi \;=\; p_i^{-1}\,\mathrm{Gain}_i(p_i, \lambda, N_{\min}, \bar{n}). \tag{82}$$

Proposition H.3 immediately yields the following result.

Proposition H.4 *The following are true:*[4]

- *If $T > \widetilde{T}$ then equilibrium masses p_b and p_s are decreasing in $\lambda, N_{\min}, \bar{n}$;*
- *If $T < \widetilde{T}$ then equilibrium masses p_b and p_s are increasing in $\lambda, N_{\min}, \bar{n}$.*

We note that a stark difference between the monotonicity results of Propositions G.14 and H.3. By Proposition H.3, in the two-class model, gains from information acquisition *are always increasing in the "market liquidity" parameter* λ. By contrast, Proposition G.14 shows that, with more than two classes, this is not true anymore. *Gains may decrease with liquidity.* The effect of this monotonicity of gains differs, however, between symmetric and asymmetric equilibria. In symmetric equilibiria, the effect goes in the intuitive direction: Since gains increase with λ, so does the equilibrium amount

[4]Recall that, by Proposition H.2, equilibrium masses p_b and p_s are always unique (if they exist).

of information acquisition. By contrast, equation (82) shows that, for asymmetric equilibria, the effect goes in the opposite direction: Since the gains increase in both λ and the mass p of agents that acquire information (when $T > \widetilde{T}$), this mass must go down in equilibrium in order to make the agents indifferent between acquiring and not acquiring information.

From this result, we can also consider the effect of "education policies" such as the following.

- Educating agents before they enter the market by increasing the number $N_{\min}$ of endowed signal packets.
- Increasing the number $\overline{n}$ of signal packets that can be acquired.

Proposition H.4 implies that, in a dynamic model with sufficiently many trading rounds, both policies improve market efficiency. By contrast, a static model that does not account for the effects of information percolation would lead to the opposite policy implications.

I Endogenous Investment in Matching Technology

In this section, we take initial information endowments as given and instead focus on endogenous investment in matching technologies. In particular, the initial type densities are characterized by a fixed vector $N = (N_1, \ldots, N_M)$ of initially acquired signal packets. Before the initial signals are revealed to each agent, agents in class i individually choose an amount $\chi_{ij} \in K \equiv \{\underline{\chi}, \overline{\chi}\}$ to invest in a technology for meeting investors of class j, for some minimum investment $\underline{\chi} > 0$ and maximum investment $\overline{\chi} > \underline{\chi}$. We examine the case of symmetric choices within classes, so that agents of class i commonly choose the investment χ_{ij}. Given these choices, in each period the probability with which an agent of class i meets some agent in class j is $f_{ij}(\chi_{ij}, \chi_{ji})$, for a given function $f_{ij} : K \times K \to (0, 1)$. By the exact law of large numbers, this technology satisfies

$$m_i f_{ij}(\chi_{ij}, \chi_{ji}) \;=\; m_j f_{ji}(\chi_{ji}, \chi_{ij}).$$

We always make the non-satiation assumption that

$$\sum_{j \neq i} f_{ij}(\overline{\chi}, \overline{\chi}) < 1.$$

Given the $M \times (M-1)$ matching-technology investments $\chi = (\chi_{ij})$, the cross-sectional type density ψ_{it} of the class-i agents satisfies the evolution equation

$$\psi_{i,t+1} \;=\; \left(1 - \sum_{j \neq i} f_{ij}(\chi_{ij}, \chi_{ji})\right) \psi_{i,t} \;+\; \sum_{j \neq i} f_{ij}(\chi_{ij}, \chi_{ji})\, \psi_{i,t} * \psi_{j,t}. \tag{83}$$

Similarly, given χ, a particular agent of class i who makes the technology choice $c \in K^{M-1}$ has a Markov type process whose probability density $\psi_t^{c,\chi}$ at time t satisfies the Kolmogorov forward equation

$$\psi_{t+1}^{c,\chi} \;=\; \left(1 - \sum_{j \neq i} f_{ij}(c_j, \chi_{ji})\right) \psi_t^{c,\chi} \;+\; \sum_{j \neq i} f_{ij}(c_j, \chi_{ji})\, \psi_t^{c,\chi} * \psi_{j,t}. \tag{84}$$

We will be applying the following technical assumption.

Condition 3 . *For any integer $T > 1$ and any pair (i,j) of agent classes, the function $c \mapsto (f_{ij}(\overline{\chi}, c))^T - (f_{ij}(\underline{\chi}, c))^T$ is nonnegative and monotone increasing in c.*

This assumption guarantees that the increase in matching probabilities associated with investing in a more effective matching technology is increasing in the investments in matching technology by other agents. The complementarity property holds, for example, for the constant-returns-to-scale technology of Duffie, Malamud and Manso (2009), by which $f_{ij}(\chi_{ij}, \chi_{ji}) = k_{ij}\chi_{ij}\chi_{ji}$ for some constant k_{ij}. The idea is natural: the greater the efforts of other agents at being matched, the more easily are they found by improving one's own search technology.

Definition I.1 *A (symmetric) rational expectations equilibrium consists of matching technology investments $\chi = (\chi_{ij})$; for each time t and each seller-buyer pair (i,j), a pair (S_{ijt}, B_{ijt}) of bid and ask functions; and for each class i and time t, a cross-sectional type density ψ_{it} such that:*

1. *The cross-sectional type density ψ_{it} satisfies the evolution equation (83).*

2. *The bid and ask functions (S_{ijt}, B_{ijt}) are the equilibrium bidding strategies uniquely defined by Theorem 4.7.*

3. *The matching-technology investments* $\chi_i = (\chi_{i1}, \ldots, \chi_{iM})$ *of class* i *maximize, for any agent of class* i*, the expected total trading gains net of matching-technology costs. That is,* χ_i *solves*

$$\sup_{c \in K^{M-1}} \quad U_i(c, \chi),$$

where

$$U_i(c,\chi) = E\left(\sum_{t=1}^{T}\sum_{j} f_{ij}(c_j, \chi_{ji})\, v_{ijt}(\Theta_t^{c,\chi}; B_{ijt}, S_{ijt})\right) - (\chi_{i1} + \cdots + \chi_{iM}), \quad (85)$$

where the agent's type process $\Theta_t^{c,\chi}$ *has probability density* $\psi_t^{c,\chi}$ *satisfying (84) and the expected gain* v_{ijt} *associated with a given sort of trading encounter is as defined by (9) or (11), depending on whether class-*i *agents are sellers or buyers, respectively.*

We say that *search is a strategic complement* if, for any agent class i and any matching technology investments $\chi = (\chi_{ij})$, the utility gain $U_i(c', \chi) - U_i(c, \chi)$ associated with increasing the matching technology investments from c to $c' \geq c$ is increasing in χ_{-i}, the matching-technology investments of classes $j \neq i$. The main result of this section is the following theorem.

Theorem I.2 *Suppose Conditions 2 and 3 hold. Let* $\bar{T}$ *and* $\bar{g}$ *be as in Proposition 5.5. Then, for any proportional gain from trade* $\overline{G} > \overline{g}$ *and market duration* $T > \bar{T}$*, search is a strategic complement.*

The intuition for this result is analogous to that of Proposition 5.5. If other agents are assumed to have increased their ability to find counterparties, and thereby collect more information from trading encounters, then under the stated conditions a given agent is encouraged to do the same in order to mitigate adverse selection in trade with better informed counterparties.

This complementarity can be responsible for the existence or non-existence of equilibria, depending on the duration T of markets, just as in the previous section. The Tarski (1955) fixed point theorem implies the following analogue of Corollary 5.3.

Corollary I.3 *Suppose Conditions 2 and 3 hold. For any proportional gain from trade* $\overline{G} > \overline{g}$ *and market duration* $T > \bar{T}$*, there exists a symmetric equilibrium. Furthermore, the set of equilibria investments in matching technology is a lattice with respect to the natural partial order on* $K^{M\times(M-1)}$*.*

Just as with the discussion following Corollary 5.3, a maximal and a minimal element of the set of equilibria can be selected by the same standard iteration procedure. We also have the following comparative-statics variant of Proposition 5.5.

Proposition I.4 *Under Conditions 2 and 3, there exist some $\overline{g}$ and $\bar{T}$ such that for any proportional gain from trade $\overline{G} > \overline{g}$ and market duration $T > \bar{T}$, equilibrium investment in matching technology is increasing in the initial vector N of acquired signal packets.*

The intuition behind this result is analogous to that behind Theorem 5.2. If traders are initially better informed and T is large, then the learning effect dominates, giving agents an incentive to invest more in search technologies.

This result also illustrates the role of cross-class externalities. Even if agents in class j do not trade with those in class i, an increase in the initial information endowment of class i increases the search incentives of class j. This is a "pure" learning externality in that, if class i is better informed, this information will eventually percolate to the trading counterparties of class j. This encourages class j to have a better search technology.[5]

We also consider the incentive effects for the formation of "trading networks." (Because our model is based on a continuum agents, the network effect is with respect to agent classes, not individual agents.) With respect to information gathering incentives, agents prefer to trade with better informed agents. This incentive can even overcome the associated direct impact of adverse selection.

Condition 4 (Symmetry). Classes are symmetric in the sense that they have equal masses $m_i = m_j$ and f_{ij} does not depend on (i, j).

Theorem I.5 *Suppose that class-i agents are initially better informed than those of class j, that is, $N_i > N_j$. Under Conditions 2, 3, and 4, there exists some $\overline{g}$ and some $\bar{T}$ such that for any proportional gain from trade $\overline{G} > \overline{g}$ and market duration $T > \bar{T}$, in any equilibrium:*

1. *There is more investment in matching with class i than with class j. That is, $\chi_{ki} \geq \chi_{kj}$ for all k.*

2. *Class-i agents invest more in matching technology than do class-j agents. That is, $\chi_i \geq \chi_j$.*

[5] As before, this is based on the assumption that there is an ordered path of classes connecting class i with class j.

The incentive effects associated with this result naturally support the existence of "hub-spoke trading networks," with better informed agents situated in the "center," and with other agents trading more with central agents by virtue of establishing trading relationships, meaning investment in the associated matching technologies. As a result, one expects a positive correlation between the frequency of trade of a class of agents and its information quality. While this effect accounts for learning opportunities, pricing effects, and adverse selection, we do not capture some other important effects, such as those associated with size variation in trades and risk aversion.

Finally, we note that if the market duration is moderate, meaning that $T \in (\widetilde{T}, \bar{T})$, the learning effect may not be strong enough to create the complementarity effects that we have described. Indeed, there are counterexamples for the 3-class model of the previous section.

J Proofs: Endogenous Matching Technology

Proof of Theorem I.2. To prove the theorem, we need to show that, for any $k \neq i$, the utility gain for an agent of class i from searching more for agents of class k is monotone increasing in the search efforts of all other agents. This utility gain is given by

$$\begin{aligned}
&\sum_{t=0}^{T} \sum_{j \neq i,k} f_{ij}(\chi_{ij}, \chi_{ji}) \int_{\mathbb{R}} 0.5(\pi^H_{i,j,t}(\theta) h^H_{i,t+1}(\chi_i^{k,+}, \theta) + \pi^L_{i,j,t}(\theta) h^L_{i,t+1}(\chi_i^{+,k}, \theta))\, d\theta \\
&+ \sum_{t=0}^{T} f_{ik}(\overline{\chi}, \chi_{ki}) \int_{\mathbb{R}} 0.5(\pi^H_{i,k,t}(\theta) h^H_{i,t+1}(\chi_i^{k,+}, \theta) + \pi^L_{i,j,t}(\theta) h^L_{i,t+1}(\chi_i^{k,+}, \theta))\, d\theta \\
&- \sum_{t=0}^{T} \sum_{j \neq i,k} f_{ij}(\chi_{ij}, \chi_{ji}) \int_{\mathbb{R}} 0.5(\pi^H_{i,j,t}(\theta) h^H_{i,t+1}(\chi_i^{k,-}, \theta) + \pi^L_{i,j,t}(\theta) h^L_{i,t+1}(\chi_i^{k,-}, \theta))\, d\theta \\
&- \sum_{t=0}^{T} f_{ik}(\underline{\chi}, \chi_{ki}) \int_{\mathbb{R}} 0.5(\pi^H_{i,k,t}(\theta) h^H_{i,t+1}(\chi_i^{k,-}, \theta) + \pi^L_{i,j,t}(\theta) h^L_{i,t+1}(\chi_i^{k,-}\theta))\, d\theta,
\end{aligned} \tag{86}$$

where $\chi_i^{k,\pm}$ coincides with χ_i, but with χ_{ik} replaced by $\underline{\chi}$ and $\overline{\chi}$, respectively. Lemmas G.2-G.3 imply that the leading asymptotic term of this gains from search can be written as

$$\sum_{t=0}^{T} (f_{ik}(\overline{\chi}, \chi))^t - (f_{ik}(\underline{\chi}, \chi))^t) K_{ijt},$$

for some nonnegative coefficients K_{ijt} that do not depend on χ_{ik}. Furthermore, a slight modification of the proof of Proposition 5.5 implies that these coefficients K_{ijt} are monotone increasing in the matching-technology investments of other classes whenever T is sufficiently large. The claim follows. ∎

Proof of Proposition I.4. The proof follows directly from the arguments in the proof of Theorem I.2 because the coefficients K_{ijt} are monotone in the initial amount of information. ∎

Proof of Theorem I.5. It follows from Lemmas G.2-G.3 that the expected profit from trading with better informed classes is always larger when T is sufficiently large, and that these profits are larger for initially better informed agents. The claim follows. ∎

K Dynamic Information Acquisition

For settings in which side investment in information gathering can be done dynamically, based on learning over time, we are able to get analytical results only with a sufficiently low cost of information acquisition, corresponding to a per-packet cost of signals of $\pi_4 > \pi$, the case considered in the previous appendix.

In this case, we can show that there is a "threshold equilibrium," characterized by thresholds $\underline{X}_{it} < \overline{X}_{it}$, such that agents of type i acquire additional information only when their log-likelihood is in the interval $(\underline{X}_{it}, \overline{X}_{it})$.

The timing of the game is as follows. At the beginning of each period t, an agent may acquire information . Trading then takes place after an agent meets a counterparty with probability λ. Without loss of generality, when they acquire information, they choose between $N_{\min}$ and $N_{\max}$ packets. Otherwise, there will be multiple thresholds for each intermediate level number of signals. This is feasible to model, but much more complicated.

We let $\psi_{i,t}$ denote the cross-sectional density of types after information acquisition, and before trading takes place, and let $\chi_{i,t}$ denote the cross-sectional density of types after the auctions take place. Thus,

$$\psi_{i,t+1} \;=\; (\chi_{i,t}\, I_{(\underline{X}_{i,t+1}, \overline{X}_{i,t+1})}) * \eta_{N_{\max}} \;+\; (\chi_{i,t}\, I_{\mathbb{R}\setminus(\underline{X}_{i,t+1}, \overline{X}_{i,t+1})}) * \eta_{N_{\min}}$$

is the density that determines the bid and ask functions, and

$$\chi_{i,t+1} \;=\; (1-\lambda)\, \psi_{i,t+1} \;+\; \lambda\, \psi^{K}_{b,t+1} * \psi_{s,0+1}$$

is the cross-sectional density of types after the auctions took place.

We now denote by $Q_{i,t,\tau}(\theta, x)$ the cross-sectional type density at time τ right before the auctions take place of an agent of class i conditional on his type being θ at time t *after the information has been acquired.* Then, conditional on his type being θ at time t before information has been acquired, depending on whether the agent acquires $N_i \in \{N_{\max}, N_{\min}\}$ signals, his type density at time τ is

$$R^{N_i}_{i,t,\tau}(\theta, x) \;=\; \int_{\mathbb{R}} \eta_{N_i}(z-\theta)\, Q_{i,t,\tau}(z, x)\, dz\,.$$

Furthermore, $Q_{i,t,\tau}(z, x)$ satisfies the recursion

$$Q_{i,t,\tau+1}(\theta, x) \;=\; \Big(q_{i,t,\tau}(\theta,\cdot) I_{[\underline{X}_{i,\tau+1},\overline{X}_{i,\tau+1}]}\Big) * \eta_{N_{\max}} \;+\; \Big(q_{i,t,\tau}(\theta,\cdot) I_{[\underline{X}_{i,\tau+1},\overline{X}_{i,\tau+1}]}\Big) * \eta_{N_{\min}}$$

where

$$q_{i,t,\tau}(\theta,\cdot) \;=\; \lambda\, Q_{i,t,\tau}(\theta,\cdot) * \psi_{j,\tau} \;+\; (1-\lambda)\, Q_{i,t,\tau}\,.$$

We will also need the following additional technical condition.

Condition 3. Suppose there exist $K, \epsilon > 0$ such that

$$|\overline{\psi}^H(-x) e^{(\alpha+1)x} - c_0| + |\overline{\psi}^H(x) e^{\alpha x} - c_0| \;\le\; K\, e^{-\epsilon\, x} \tag{87}$$

for all $x > 0$.

Theorem K.1 *There exist $A, g > 0$ such that, for all $\overline{G} > g$ and all $\pi < e^{-A\overline{G}}$, there exists a threshold equilibrium.*

We let $M^{H,L}_{it}$ note the mass of agents of class i who acquire information at time t, indicating with a superscript the corresponding outcome of Y, H or L.

Theorem K.2 *There exists a critical time t^* such that the following hold in any threshold equilibrium under the conditions of Theorem K.1.*

- *Sellers:*
  - *For both H and L, the mass $M^{H,L}_{st}$ is monotone decreasing with t, and increasing in $\overline{G}^{-1}, T, N_{\max}$.*
  - *$M^{H,L}_{st}$ is monotone increasing in λ for $t < t^*$ and is monotone decreasing in λ for $t \ge t^*$.*

- *Buyers:*
  - *The mass* $M_{bt}^{H,L}$ *is monotone decreasing in* t *and is increasing in* $T, N_{\max}$*.*
  - M_{bt}^{H} *is monotone increasing in* $\overline{G}^{-1}$*.*
  - M_{bt}^{H} *is monotone increasing in* λ *for* $t < t^*$ *and is monotone decreasing in* λ *for* $t \ge t^*$*.*
  - M_{bt}^{L} *is monotone decreasing in* $\overline{G}^{-1}$*.*

L Proofs: Dynamic Information Acquisition

In this section we study asymptotic equilibrium behavior when $\overline{G}$ and π^{-1} become large. Furthermore, we will assume that π^{-1} is significantly larger than $\overline{G}$, so that $\pi^{-1}/\overline{G}^A$ is large for a sufficiently large $A > 0$. Throughout the proof, we will constantly use the notation $X >> Y$ if, asymptotically, $X - Y \to +\infty$.

L.1 Exponential Tails

Note that, by Lemma 4.5,

$$\chi_{i0}^H(x) \;=\; (1-\lambda)\,\psi_{i,0}^K \;+\; \lambda\,\psi_{b,0}^H * \psi_{s,0}^H \;\sim\; c_0\, e^{-\alpha x}\,|x|^{2\,N_{\max}-1} \;=\; c_0\, e^{-\alpha x}\,|x|^{\gamma_0}\,.$$

Furthermore, as we show below, in any equilibrium we always have

$$\underline{X}_{bt} << \underline{X}_{b,t+1} << \underline{X}_{st} << \underline{X}_{s,t+1} \tag{88}$$

and

$$\overline{X}_{b,t+1} << \overline{X}_{bt} << \overline{X}_{s,t+1} << \overline{X}_{st}. \tag{89}$$

Lemma L.1 *Suppose that* $x \to +\infty$ *and* $\overline{X}_{it} \;\to\; +\infty$ *in such a way that*

$$\overline{X}_{b,t+1} << \overline{X}_{bt} << \overline{X}_{s,t+1} << \overline{X}_{st}$$

for all t *and such that, for any fixed* i, t*, the difference* $x - \overline{X}_{i,t}$ *either stays bounded or converges to* $+\infty$ *or converges to* $-\infty$*. Then,*

$$\psi_{it}(x) \;\sim\; C_{it}^{\psi}\, e^{-(\alpha+I_L)x}\, x^{\gamma_t^{\psi}}$$

and

$$\chi_{it}(x) \;\sim\; C_{it}^{\chi}\, e^{-(\alpha+I_L)x}\, x^{\gamma_t^{\chi}},$$

where

$$\gamma_t^{\psi} \;=\; N_{\max} + \gamma_{t-1}^{\chi}$$

and

$$\gamma_t^{\chi} \;=\; 2\gamma_t^{\psi} + 1\,.$$

The powers $m_t^{\psi},\ m_t^{\chi}$ *with which* λ *enters* $C_{it}^{\psi,\chi}$ *satisfy*

$$m_t^{\chi} = 2m_t^{\psi} + 1\ ,\ m_t^{\psi} = m_{t-1}^{\chi}\,.$$

Furthermore, there exists a constant $\mathfrak{K}_1$ *such that*

$$|\psi_{it}(x)| \;\le\; \mathfrak{K}_1\, e^{-(\alpha+I_L)x}\, x^{\gamma_t^{\psi}}$$

and

$$|\chi_{it}(x)| \;\le\; \mathfrak{K}_1\, e^{-(\alpha+I_L)x}\, x^{\gamma_t^{\chi}}\,.$$

In addition, there exists a $\delta_{it} > 0$ *such that*

$$|\psi_{it}(x)e^{\alpha x}\, x^{-\gamma_t^{\psi}} - C_{it}^{\psi}| \;<\; \mathfrak{K}_1 e^{-\delta_{it}x}$$

for all $x > A$.

Proof. The proof is by induction. Recall that

$$\eta_N \;=\; \overline{\psi}^{*N}$$

Fix a sufficiently large $A > 0$. Then,

$$\begin{aligned}
\chi_{i,t+1} \;&=\; \int_{\mathbb{R}} \psi_{bt}(x-y)\,\psi_{st}(y)\,dy \;=\; \int_{-\infty}^{x} \psi_{bt}(x-y)\psi_{st}(y)\,dy \\
&\quad + \int_{x}^{+\infty} \psi_{bt}(x-y)\,\psi_{st}(y)\,dy \\
&=\; \int_0^{+\infty} \psi_{bt}(y)\psi_{st}(x-y)dy \;+\; \int_{-\infty}^{0} \psi_{bt}(y)\psi_{st}(x-y)\,dy \\
&=\; \int_0^{A} \psi_{bt}(y)\psi_{st}(x-y)dy \;+\; \int_{A}^{+\infty} \psi_{bt}(y)\psi_{st}(x-y)\,dy \\
&\quad + \int_{-\infty}^{0} \psi_{bt}(y)\psi_{st}(x-y)\,dy \\
&\equiv\; I_1 \;+\; I_2 \;+\; I_3\,.
\end{aligned}$$

Pick an A so large that ψ_{bt} can be replaced by its asymptotic from the induction hypothesis. Note that we can only take the "relevant" asymptotic coming from the values of y satisfying $y < \overline{X}_{b,T}$ because the tail behavior coming from "further away" regimes are asymptotically negligible. Then,

$$
\begin{aligned}
I_2 \;=\; & \int_A^{+\infty} \psi_{bt}(y)\psi_{st}(x-y)\,dy \\
& \sim \int_A^{+\infty} C\,e^{-(\alpha+I_L)y}\,y^{\gamma_t^\psi}\,\psi_{st}(x-y)\,dy \\
& = C\,e^{-(\alpha+I_L)x}\,x^{\gamma_t^\psi}\int_{-\infty}^{x-A} e^{(\alpha+I_L)y}\,|1-y/x|^{\gamma_t^\psi}\,\psi_{st}(y)\,dy\,. \quad (90)
\end{aligned}
$$

Now, applying l'Hôpital's rule and using the induction hypothesis, we get that

$$
\frac{\int_{-\infty}^{x-A} e^{(\alpha+I_L)y}\,\psi_{st}(y)\,dy}{(\gamma_t^\psi+1)^{-1}x^{\gamma_t^\psi+1}} \;\sim\; C_{st}.
$$

Thus, we have proved the required asymptotic for the term I_2.

To bound the term I_1, we again use the induction hypothesis and get

$$
\int_0^A \psi_{bt}(y)\psi_{st}(x-y)\,dy \;\le\; \mathfrak{K}_1\int_0^A \psi_{bt}(y)e^{-(\alpha+I_L)(x-y)}\,|x-y|^{\gamma_t^\psi}\,dy \;\sim\; e^{-(\alpha+I_L)x}|x|^{\gamma_t^\psi}\tilde{C}_2,
$$

for some constant $\tilde{C}_2$, so the term I_1 is asymptotically negligible relative to I_2.

Finally, for the term I_3, we have

$$
\int_x^{+\infty} \psi_{bt}(x-y)\psi_{st}(y)\,dy \;=\; \int_{-\infty}^0 \psi_{bt}(y)\psi_{st}(x-y)\,dy\,. \quad (91)
$$

Now, picking a sufficiently large $A > 0$ and using the same argument as above, we can replace the integral by $\int_{-A}^0$ and then use the induction hypothesis to replace $\psi_{st}(x-y)$ by $C_{st}e^{-(\alpha+I_L)(x-y)}|x-y|^{\gamma_t^\psi}$. Therefore,

$$
\begin{aligned}
\int_{-\infty}^0 \psi_{bt}(y)\psi_{st}(x-y)\,dy \;&\sim\; \int_{-\infty}^0 \psi_{bt}(y)C_{st}e^{-(\alpha+I_L)(x-y)}\,|x-y|^{\gamma_t^\psi}\,dy \\
&\sim\; C_{st}e^{-(\alpha+I_L)x}\,x^{\gamma_t^\psi}\int_{-\infty}^0 \psi_{bt}(y)e^{(\alpha+I_L)y}\,dy\,,
\end{aligned} \quad (92)
$$

which is negligible relative to I_2. Thus, we have completed the proof of the induction step for χ_{it}. It remains to prove it for ψ_{it}. We have

$$\begin{aligned}\psi_{it}(x) &= \int_{\underline{X}_{it}}^{\overline{X}_{it}} \chi_{i,t-1}(y)\eta_{N_{\max}}(x-y)\,dy + \int_{-\infty}^{\underline{X}_{it}} \chi_{i,t-1}(y)\eta_{N_{\min}}(x-y)\,dy \\ &\quad + \int_{\overline{X}_{it}}^{+\infty} \chi^H_{i,t-1}(y)\eta^H_{N_{\min}}(x-y)\,dy \\ &= \int_{-\infty}^{\overline{X}_{it}} \chi_{i,t-1}(y)\eta_{N_{\max}}(x-y)\,dy - \int_{-\infty}^{\underline{X}_{it}} \chi_{i,t-1}(y)\eta^H_{N_{\max}}(x-y)\,dy \\ &\quad + \int_{-\infty}^{\underline{X}_{i1}} \chi^H_{i0}(y)\eta_{N_{\min}}(x-y)\,dy + \int_{\overline{X}_{i1}}^{+\infty} \chi_{i0}(y)\eta_{N_{\min}}(x-y)\,dy\,. \end{aligned} \tag{93}$$

Since $\underline{X}_{i1} \to -\infty$, the induction hypothesis implies that

$$\begin{aligned}&\int_{-\infty}^{\underline{X}_{it}} \chi_{i,t-1}(y)\eta_N(x-y)\,dy \sim \int_{-\infty}^{\underline{X}_{it}} C_{i,t-1}e^{(\alpha+I_H)y}|y|^{\gamma^\chi_{i,t-1}}\,c_0^N e^{-(\alpha+I_L)(x-y)}\,|x-y|^{N-1}\,dy \\ &= C_{i,t-1}\,c_0^N e^{-(\alpha+I_L)x}\,x^{N-1}\int_{-\infty}^{0} e^{(2\alpha+1)(y+\underline{X}_{it})}|y+\underline{X}_{it}|^{\gamma^\chi_{i,t-1}}\,|1-(y+\underline{X}_{it})/x|^{N-1}\,dy \\ &= o\left(e^{-(\alpha+I_L)x}\,x^{N-1}\,e^{(2\alpha+1)\underline{X}_{it}}|\underline{X}_{it}|^{\gamma^\chi_{i,t-1}}\right)\,. \end{aligned} \tag{94}$$

The same argument as above (the induction step for χ_{it}) implies that

$$\int_{\mathbb{R}} \chi_{i,t-1}(y)\eta_N(x-y)\,dy \sim C\,e^{-(\alpha+I_L)x}\,x^{\gamma^\chi_{i,t-1}+N}\,.$$

Now, we will have to consider two different cases. If $x-\overline{X}_{it} \to +\infty$, we can replace $\eta^H_N(x-y)$ in the integral below by $c_0^N\,|x-y|^{N-1}\,e^{-\alpha(x-y)}$ and get

$$\begin{aligned}&\int_{-\infty}^{\overline{X}_{i1}} \chi_{i,t-1}(y)\eta_N(x-y)\,dy \\ &\sim c_0^N e^{-(\alpha+I_L)x}\,|x|^{N-1}\int_{-\infty}^{\overline{X}_{i1}} \chi_{i,t-1}(y)\,e^{(\alpha+I_L)y}\,|1-y/x|^{N-1}\,dy \end{aligned} \tag{95}$$

Using l'Hopital's rule and the induction hypothesis, we get

$$\int_{-\infty}^{\overline{X}_{it}} \chi_{i,t-1}(y)\,e^{(\alpha+I_L)y}\,dy \sim C\,(\overline{X}_{it})^{\gamma^\chi_{i,t-1}+1}.$$

It remains to consider the case when $x-\overline{X}_{it}$ stays bounded from above. In this case,

$$\int_{-\infty}^{\overline{X}_{it}} \chi_{i,t-1}(y)\eta_N(x-y)\,dy = \int_{x-\overline{X}_{it}}^{+\infty} \chi_{i,t-1}(x-z)\eta_N(z)\,dy\,. \tag{96}$$

Now, the same argument as in (94) implies that $\int_{-\infty}^{x-\overline{X}_{i1}} \chi_{i0}(x-z)\eta_N^H(z)\,dy$ is asymptotically negligible relative to $\int_{x-\overline{X}_{i1}}^{+\infty} \chi_{i0}(x-z)\eta_N^H(z)\,dy$ because $x-\overline{X}_{i1}$ is bounded from above. Therefore,

$$\int_{x-\overline{X}_{it}}^{+\infty} \chi_{i,t-1}(x-z)\eta_N(z)\,dy \;\sim\; \int_{-\infty}^{+\infty} \chi_{i0}^H(x-z)\eta_N^H(z)\,dy \;\sim\; C\,e^{-\alpha x}\,x^{\gamma_{i,t-1}^{\chi}+N}\,.$$

The induction step follows now from (93). The proof of the upper bounds for the densities is analogous. ■

The arguments in the proof of Lemma L.1 also imply the following result.

Lemma L.2 *Under the hypothesis of Lemma L.1, we have that, when $\theta \to +\infty$ so that $\theta - x \to +\infty$,*

$$\begin{aligned} q_{i,t,\tau}(\theta,x) &\sim C_{i,t,\tau}^{q}\,e^{(\alpha+I_H)(x-\theta)}\,|x-\theta|^{\gamma_{t,\tau}^{q}} \\ Q_{i,t,\tau}(\theta,x) &\sim C_{i,t,\tau}^{Q}\,e^{(\alpha+I_H)(x-\theta)}\,|x-\theta|^{\gamma_{t,\tau}^{Q}} \\ R_{i,t,\tau}^{N_i}(\theta,x) &\sim C_{i,t,\tau}^{R,N_i}\,e^{(\alpha+I_H)(x-\theta)}\,|x-\theta|^{\gamma_{t,\tau}^{Q}+N_i}, \end{aligned} \tag{97}$$

where

$$\begin{aligned} \gamma_{t,\tau}^{q} &= \gamma_{t,\tau}^{Q} + \gamma_{\tau}^{\psi} + 1 \\ \gamma_{t,\tau}^{Q} &= \gamma_{t,\tau-1}^{q} + N_{\max}\,. \end{aligned} \tag{98}$$

Lemmas L.1 and L.2 immediately yield the next result.

Lemma L.3 *We have:*

$$\begin{aligned} \gamma_t^{\psi} &= (2^{t+1}-1)N_{\max} - 1 \\ \gamma_t^{\chi} &= (2^{t+2}-2)N_{\max} - 1 \\ \gamma_{t,\tau}^{q} &= (2^{\tau+2}-2^{t+1}-1)N_{\max} - 1 \\ \gamma_{t,\tau}^{Q} &= (2^{\tau+1}-2^{t+1})N_{\max} - 1\,. \end{aligned} \tag{99}$$

Furthermore, the powers $m_{t,\tau}$ of λ with which λ enters the corresponding coefficients c_t and $C_{t,\tau}$ are given by:

$$\begin{aligned} m_t^{\psi} &= 2^{t-1} - 1 \\ m_t^{\chi} &= 2^{t} - 1 \\ m_{t,\tau}^{q} &= 2^{\tau+1} - 2^{t} \\ m_{t,\tau}^{Q} &= 2^{\tau} - 2^{t}\,. \end{aligned} \tag{100}$$

L.2 Gains from Information Acquisition

For any given agent i, the expected utility $U_{i,t,\tau}$ from trading during the time interval from t to τ immediately before information is acquired can be represented as

$$U_{i,t,\tau}(\theta) \;=\; \sum_{r=t}^{\tau} \; u_{i,t,\tau}(\theta)\,.$$

Suppose that, at time t, an agent of type i with posterior θ makes a decision to acquire information with type density η. Then, the agent knows that his type at time τ, at the moment when the next auction takes place, his posterior will be distributed as $\delta_\theta * \eta * g^K_{i,t,\tau-1}$.

We will use the following notation:

$$G^{K,R,N}_{t,\tau}(\theta,x) \;=\; \int_x^{+\infty} R^{K,N}_{t,\tau}(\theta,y)\,dy\;,\;\; F^{K,R,N}_{t,\tau}(\theta,x) \;=\; 1 \;-\; G^{K,R,N}_{t,\tau}(\theta,x),$$

for $K\in\{H,L\}$.

The following analog of Lemma G.1 holds.

Proposition L.4 *For a given buyer with posterior θ at time t, before the time-t auction takes place,*

$$\begin{aligned} u_{b,t,\tau}(N,\theta) \;&=\; P(\theta)\lambda\int_{\mathbb{R}}(v^H - S_\tau(y))\,G^{H,R,N}_{t,\tau}(\theta,V_{b\tau}(S_\tau(y)))\,\psi^H_{s\tau}(y)\,dy \\ &\quad+\;(1-P(\theta))\lambda\int_{\mathbb{R}}(v_b - S_\tau(y))\,G^{L,R,N}_{t,\tau}(\theta,V_{b\tau}(S_\tau(y)))\,\psi^L_{s\tau}(y)\,dy, \end{aligned} \tag{101}$$

whereas for a seller,

$$\begin{aligned} u_{s,t,\tau}(N,\theta) \;&=\; P(\theta)\,\lambda\int_{\mathbb{R}}(S_\tau(y)-v^H)\,G^H_{b\tau}(V_{b\tau}(S_\tau(y)))\,R^{H,N}_{t,\tau}(\theta,y)\,dy \\ &\quad+\;(1-P(\theta))\,\lambda\int_{\mathbb{R}}(S_\tau(y)-v_s)\,G^L_{b\tau}(V_{b\tau}(S_\tau(y)))\,R^{L,N}_{t,\tau}(\theta,y)\,dy. \end{aligned} \tag{102}$$

Thus, the gain from acquiring additional information is given by

$$\sum_{\tau>t}(u_{i,t,\tau}(N_{\max},\theta) \;-\; u_{i,t,\tau}(N_{\min},\theta))\,.$$

The following lemma provides asymptotic behavior of these gains for extreme type values.

Lemma L.5 *We have*

- *For a buyer:*
  - *As* $\theta \to +\infty$,

$$u_{b,t,\tau}(N_{\max},\theta) - u_{b,t,\tau}(N_{\min},\theta) \;\sim\; C_{b,t,\tau}^{R,N_{\max}} e^{-(\alpha+1)\theta} |\theta|^{\gamma^Q_{t,\tau}+N_{\max}} \times \int_{\mathbb{R}} (v^H \;-\; S_\tau(y)) \left(\frac{S_\tau(y) - v_b}{v^H - S_\tau(y)} \right)^{\alpha+1} e^{-(\alpha+1)y}\, \psi^H_{s\tau}(y)\, dy. \tag{103}$$

  - *As* $\theta \to -\infty$,

$$u_{b,t,\tau}(N_{\max},\theta) - u_{b,t,\tau}(N_{\min},\theta) \;\sim\; C_{b,t,\tau}^{R,N_{\max}}\, Re^{(\alpha+1)\theta} |\theta|^{\gamma^Q_{t,\tau}+N_{\max}} \times \int_{\mathbb{R}} (v^H \;-\; S_\tau(y)) \left(\frac{S_\tau(y) - v_b}{v^H - S_\tau(y)} \right)^{-\alpha} e^{\alpha y}\, \psi^H_{s\tau}(y)\, dy. \tag{104}$$

- *For a seller, as* $\theta \to -\infty$,[6]

$$\begin{aligned} u_{s,0,\tau}(N_{\max},\theta) - u_{s,0,\tau}(N_{\min},\theta) \;&\sim\; C_{s,0,\tau}^{R,N_{\max}}\, Re^{(\alpha+1)\theta} |\theta|^{\gamma^Q_{t,\tau}+N_{\max}} \\ &\times \int_{\mathbb{R}} \Bigg(\Big((S_\tau(y) - v_b) + (v^H - S_\tau(y))\, F^H_{b\tau}(V_{b\tau}(S_\tau(y))) \Big) e^y \\ &+ \Big((S_\tau(y) - v_b) + (v_s - S_\tau(y))\, F^L_{b\tau}(V_{b\tau}(S_\tau(y))) \Big) \Bigg) e^{-(\alpha+1)y}\, dy. \end{aligned} \tag{105}$$

Furthermore, the derivatives of $u_{i,t,\tau}(N_{\max},\theta) \,-\, u_{i,t,\tau}(N_{\min},\theta)$ *with respect to* θ *have the same asymptotic behavior, but with all constants on the right-hand sides multiplied by* $\alpha + 1$ *when* $\theta \to -\infty$ *and by* $-(\alpha+1)$ *when* $\theta \to +\infty$.

Proof. Throughout the proof, we will often interchange limit and integration without showing the formal justification, which is based the same arguments as in the case of initial information acquisition considered above. However, the calculations are lengthy and omitted for the reader's convenience.

We have

$$\int_{\mathbb{R}} (v^H - S_\tau(y))\, G^{H,R,N}_{t,\tau}(\theta, V_{b\tau}(S_\tau(y)))\, \psi^H_{s\tau}(y)\, dy \;=\; (v^H - v_b) + \int_{\mathbb{R}} (v_b - S_\tau(y))\, \psi^H_{s\tau}(y)\, dy \;-\; \int_{\mathbb{R}} (v^H - S_\tau(y))\, F^{H,R,N}_{t,\tau}(\theta, V_{b\tau}(S_\tau(y)))\, \psi^H_{s\tau}(y)\, dy \tag{106}$$

[6]The case $\theta \to +\infty$ will be considered separately below.

and

$$\int_{\mathbb{R}} (v^b - S_\tau(y))\, G^{L,R,N}_{t,\tau}(\theta, V_{b\tau}(S_\tau(y)))\, \psi^L_{s\tau}(y)\, dy$$
$$= \int_{\mathbb{R}} (v_b - S_\tau(y))\, \psi^L_{s\tau}(y)\, dy \;-\; \int_{\mathbb{R}} (v^b - S_\tau(y))\, F^{L,R,N}_{t,\tau}(\theta, V_{b\tau}(S_\tau(y)))\, \psi^L_{s\tau}(y)\, dy. \tag{107}$$

By Lemma L.2, for a fixed x, we have

$$F^{R,N}_{t,\tau}(\theta, V_{b\tau}(S_\tau(y))) \;\sim\; C^{F,R,N}_{b,t,\tau}\, e^{(\alpha+I_H)(V_{b\tau}(S_\tau(y))-\theta)}\, |\theta|^{\gamma^Q_{t,\tau}+N},$$

and the first claim follows from the identity

$$V_{b\tau}(S_\tau(y)) \;=\; \log \frac{S_\tau(y) - v_b}{v^H - S_\tau(y)} - y.$$

The case of the limit $\theta \to -\infty$ is completely analogous.

It remains to consider the case of a seller. The term corresponding to state H gives

$$\int_{\mathbb{R}} (S_\tau(y) - v^H)\, G^H_{b\tau}(V_{b\tau}(S_\tau(y)))\, R^{L,N}_{t,\tau}(\theta, y)\, dy$$
$$= (v_b - v^H) \;+\; \int_{\mathbb{R}} \Big((S_\tau(y) - v_b) + (v^H - S_\tau(y))\, F^H_{b\tau}(V_{b\tau}(S_\tau(y))) \Big)\, R^{L,N}_{t,\tau}(\theta, y)\, dy. \tag{108}$$

In the limit as $\theta \to -\infty$,

$$\int_{\mathbb{R}} \Big((S_\tau(y) - v_b) + (v^H - S_\tau(y))\, F^H_{b\tau}(V_{b\tau}(S_\tau(y))) \Big)\, R^{L,N}_{t,\tau}(\theta, y)\, dy$$
$$\sim\; C^{R,N}_{s,0,\tau} e^{\alpha\theta}\, |\theta|^{\gamma^Q_{t,\tau}+N} \int_{\mathbb{R}} \Big((S_\tau(y) - v_b) + (v^H - S_\tau(y))\, F^H_{b\tau}(V_{b\tau}(S_\tau(y))) \Big) e^{-\alpha y}\, dy. \tag{109}$$

The term corresponding to state L gives

$$\int_{\mathbb{R}} (S_\tau(y) - v_s)\, G^L_{b\tau}(V_{b\tau}(S_\tau(y)))\, R^{L,N}_{t,\tau}(\theta, y)\, dy$$
$$= (v_b - v_s) \;+\; \int_{\mathbb{R}} \Big((S_\tau(y) - v_b) + (v_s - S_\tau(y))\, F^L_{b\tau}(V_{b\tau}(S_\tau(y))) \Big)\, R^{L,N}_{t,\tau}(\theta, y)\, dy. \tag{110}$$

In the limit as $\theta \to -\infty$,

$$\int_{\mathbb{R}} \Big((S_\tau(y) - v_b) + (v_s - S_\tau(y))\, F^L_{b\tau}(V_{b\tau}(S_\tau(y))) \Big)\, R^{L,N}_{t,\tau}(\theta, y)\, dy$$
$$\sim\; C^{R,N}_{s,0,\tau} e^{(\alpha+1)\theta}\, |\theta|^{\gamma^Q_{t,\tau}+N} \int_{\mathbb{R}} \Big((S_\tau(y) - v_b) + (v_s - S_\tau(y))\, F^L_{b\tau}(V_{b\tau}(S_\tau(y))) \Big) e^{-(\alpha+1)y}\, dy. \tag{111}$$

This completes the proof.

The claim concerning the derivatives with respect to θ is proved analogously. ■

The arguments of the proof of Lemmas F.1-F.9 imply the following result.

Lemma L.6 *Let*

$$\frac{(\alpha+1)^2}{\alpha-1} > \alpha\,.$$

Then

$$\begin{aligned}&\int_{\mathbb{R}} (v^H \,-\, S_\tau(y))\left(\frac{S_\tau(y)-v_b}{v^H - S_\tau(y)}\right)^{\alpha+1} e^{-(\alpha+1)y}\,\psi^H_{s\tau}(y)\,dy\\ &\sim\; c_{s\tau}\,\varepsilon^{\frac{2\alpha+1}{\alpha+1}}\left|\frac{\log\varepsilon}{\alpha+1}\right|^{\gamma_\tau}\int_{\mathbb{R}}(v^H \,-\, \mathcal{S}(y))\left(\frac{\mathcal{S}(y)-v_b}{v^H-\mathcal{S}(y)}\right)^{\alpha+1} e^{-(2\alpha+1)y}\,dy.\end{aligned} \tag{112}$$

Similarly, we have the following result.

Lemma L.7 *Let*

$$\frac{\alpha+1}{\alpha-1} \;>\; \alpha\,.$$

Then

$$\begin{aligned}&\int_{\mathbb{R}}\Bigg(\Big((S_\tau(y)-v_b)+(v^H-S_\tau(y))\,F^H_{b\tau}(V_{b\tau}(S_\tau(y)))\Big)e^{y}\\ &+\Big((S_\tau(y)-v_b)+(v_s-S_\tau(y))\,F^L_{b\tau}(V_{b\tau}(S_\tau(y)))\Big)\Bigg)\,e^{-(\alpha+1)y}\,dy\\ &\sim\;\varepsilon^{\frac{\alpha}{\alpha+1}}\int_{\mathbb{R}}\left((\mathcal{S}(y)-v_b)e^{-\alpha y}-\frac{\alpha+1}{\alpha}\,e^{-(2\alpha+1)y}\left(\frac{\mathcal{S}(y)-v_b}{v^H-\mathcal{S}(y)}\right)^{\alpha}\right)\,dy.\end{aligned} \tag{113}$$

In order to prove the next asymptotic result, we will need the following auxiliary lemma.

Lemma L.8 *Let $f(z)$ solve*

$$f'(z)\;=\;\left(\frac{\log(1/\zeta)}{\log(1/\zeta)\;+\;\log(1/f(z))}\right)^{\gamma}\left(z\;+\;\varepsilon^{\frac{1}{\alpha+1}}\,f(z)^{\frac{1}{\alpha+1}}\right), \tag{114}$$

with $f(0)=0$. Then, $r_\varepsilon(y)\;=\;f(\varepsilon^{\frac{1}{\alpha-1}}y)\varepsilon^{-2/(\alpha-1)}$ converges to the function $r(y)$ that is the unique solution to

$$r'(y)\;=\;y\;+\;(r(y))^{1/(\alpha+1)}\,,\qquad r(0)=0\,.$$

Proof. We have

$$
\begin{aligned}
r'_\varepsilon(y) &= \varepsilon^{-\frac{1}{\alpha-1}} f'(\varepsilon^{\frac{1}{\alpha-1}} y) \\
&= \varepsilon^{-\frac{1}{\alpha-1}} \left(\frac{\log(1/\zeta)}{\log(1/\zeta) + \log(1/f(\varepsilon^{\frac{1}{\alpha-1}} y))} \right)^\gamma \left(\varepsilon^{\frac{1}{\alpha-1}} y + \varepsilon^{\frac{1}{\alpha+1}} f(\varepsilon^{\frac{1}{\alpha-1}} y)^{\frac{1}{\alpha+1}} \right) \quad (115) \\
&= \left(\frac{\log(1/\zeta)}{\log(1/\zeta) + \log(\varepsilon^{-2/(\alpha-1)}/r_\varepsilon(y))} \right)^\gamma \left(y + (r_\varepsilon(y))^{\frac{1}{\alpha+1}} \right).
\end{aligned}
$$

The right-hand side of this equation converges to $y + (r_\varepsilon(y))^{1/(\alpha+1)}$. The fact that $r_\varepsilon(y)$ converges to $r(y)$ follows from the uniqueness part of the proof of Proposition D.1 and standard continuity arguments. ■

Lemma L.9 *We have*

$$
\begin{aligned}
&\int_{\mathbb{R}} (v^H - S_\tau(y)) \left(\frac{S_\tau(y) - v_b}{v^H - S_\tau(y)} \right)^{-\alpha} e^{\alpha y} \psi^H_{s\tau}(y)\, dy \\
&\qquad \sim \varepsilon^{-\alpha/(\alpha-1)} \int_0^\infty y^{-\alpha-1} \phi_{s\tau} \left(y^{-1} r(y)^{1/(\alpha+1)} \right) dy,
\end{aligned} \quad (116)
$$

where

$$
\phi_{s\tau}(y) = y^{-\alpha} \psi^H_{s\tau}(-\log y).
$$

Proof. For simplicity, we make the normalization $v^H = 1,\ v_b = 0$.

We make the change of variable $S_\tau(y) = z,\ y = V_{s\tau}(z)$, $dy = V'_{s\tau}(z)\, dz$. Using the identity $V_{s\tau}(z) = \log \frac{z}{1-z} - V_{b\tau}(z)$, we get

$$
V'_{s\tau}(z) = \frac{1}{z(1-z)} - V_{b\tau}(z).
$$

We will also use the notation $g(z) = e^{(\alpha+1)V_{b\tau}(z)}$ from the proof of Proposition D.1. Then, we have

$$
\begin{aligned}
&\int_{\mathbb{R}} (v^H - S_\tau(y)) \left(\frac{S_\tau(y) - v_b}{v^H - S_\tau(y)} \right)^{-\alpha} e^{\alpha y} \psi^H_{s\tau}(y)\, dy \\
&= \int_0^1 (1-z) \left(\frac{1-z}{z} \right)^\alpha e^{\alpha V_{s\tau}(z)} \psi^H_{s\tau}(V_{s\tau}(z))\, V'_{s\tau}(z)\, dz \\
&= \int_0^1 (1-z)\, e^{-\alpha V_{b\tau}(z)} \psi^H_{s\tau} \left(\log \frac{z}{1-z} - V_{b\tau}(z) \right) \left(\frac{1}{z(1-z)} - V'_{b\tau}(z) \right) dz \\
&= \int_0^1 (1-z)\, g(z)^{-\frac{\alpha}{\alpha+1}} \psi^H_{s\tau} \left(\log \frac{z}{1-z} - \frac{\log g(z)}{\alpha+1} \right) \left(\frac{1}{z(1-z)} - \frac{g'(z)}{(\alpha+1) g(z)} \right) dz.
\end{aligned} \quad (117)
$$

As we have shown in the proof of Proposition D.1, $g(z)/\varepsilon$ converges to a limit $f_0(z)$ when $\varepsilon \to 0$. A direct calculation based on the dominated convergence theorem and the bounds for $f(z)$ established in the proof of Proposition D.1 imply that the limit

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{\alpha}{\alpha+1}} \int_r^1 (1-z)\, g(z)^{-\frac{\alpha}{\alpha+1}}\, \psi^H_{s\tau}\left(\log \frac{z}{1-z} - \frac{\log g(z)}{\alpha+1}\right) \left(\frac{1}{z(1-z)} - \frac{g'(z)}{(\alpha+1) g(z)}\right) dz$$

exists and is finite for any $r > 0$. By contrast, as we will show below,

$$\varepsilon^{\frac{\alpha}{\alpha+1}} \int_0^r (1-z)\, g(z)^{-\frac{\alpha}{\alpha+1}}\, \psi^H_{s\tau}\left(\log \frac{z}{1-z} - \frac{\log g(z)}{\alpha+1}\right) \left(\frac{1}{z(1-z)} - \frac{g'(z)}{(\alpha+1) g(z)}\right) dz$$

blows up to $+\infty$ as $\varepsilon \to 0$. Therefore, the part $\int_r^1$ of the integral is asymptotically negligible and we will in the sequel only consider the integral $\int_0^r$ with a sufficiently small $r > 0$. Then, it follows from the proof of Proposition D.1 that we may assume that $g(z) = \varepsilon f(z)$ where $f(z)$ solves the ODE (114). For the same reason, we may replace $1-z$ by 1. It also follows from the proof of Proposition D.1 that

$$K_2\, D(q(z)) \;\le\; g(z) \;\le\; K_1\, D(q(z)) \tag{118}$$

for some $K_1 > K_2 > 0$, where

$$D(x) \;=\; x\,(-\log x)^{-\gamma},$$

with $\gamma = \gamma_\tau$, and

$$q(z) \;=\; \zeta^{1+1/\alpha}\, C\, z^{(\alpha+1)/\alpha}\, (-\log(\zeta z))^{-\gamma/\alpha} \;+\; \frac{1}{2}\,\zeta\, z^2$$

for some constant $C > 0$.

Denote

$$\phi(e^{-x}) \;=\; e^{\alpha x}\, \psi^H_{s\tau}(x)\,.$$

We have $\psi^H_{s\tau}(x) \sim\; c_{s\tau} e^{-\alpha\, x} |x|^{\gamma_\tau}$ when $x \to +\infty$ and $\psi^H_{s\tau}(x) \sim\; c_{s\tau} e^{(\alpha+1)x} |x|^{\gamma_\tau}$ when $x \to -\infty$. Therefore,

$$\phi(y) \;=\; y^{-\alpha}\, \psi^H_{s\tau}(-\log y) \;\sim\; c_{s\tau} y^{-\alpha}\, e^{-\alpha(-\log y)} |\log y|^{\gamma_\tau} \;=\; c_{s\tau} |\log y|^{\gamma_\tau}$$

when $y \to 0$ and, similarly,

$$\phi(y) \;\sim\; c_{s\tau} y^{-2\alpha-1} |\log y|^{\gamma_\tau}$$

as $y \to +\infty$.

With this notation, we have

$$\begin{aligned}
&\int_0^r (1-z)\, g(z)^{-\frac{\alpha}{\alpha+1}}\, \psi^H_{s\tau}\left(\log z - \frac{\log g(z)}{\alpha+1}\right)\left(\frac{1}{z(1-z)} - \frac{g'(z)}{(\alpha+1)g(z)}\right) dz \\
&\quad = \int_0^r z^{-\alpha}\, \phi\left(z^{-1}\, g^{1/(\alpha+1)}\right)\left(\frac{1}{z(1-z)} - \frac{g'(z)}{(\alpha+1)g(z)}\right) dz\,.
\end{aligned} \tag{119}$$

By (118), for some $K_3 > 0$,

$$\begin{aligned}
\frac{g'(z)}{g(z)} \;\le\; K_3\, &\frac{\zeta^{1/\alpha} C z^{1/\alpha}\, (-\log(\zeta z))^{-\gamma/\alpha}\, \left(\frac{\alpha+1}{\alpha} \;+\; \frac{\gamma}{\alpha}\,(-\log(\zeta z))^{-1}\right) +\; z}{\zeta^{1/\alpha}\, C\, z^{(\alpha+1)/\alpha}\, (-\log(\zeta z))^{-\gamma/\alpha} \;+\; \frac{1}{2}\, z^2} \\
&\times (-\log q(z))^{-\gamma}\, (1 \;+\; \gamma\, (-\log q(z))^{-1}).
\end{aligned} \tag{120}$$

Since we are in the regime when both z and ζ are small, $1 \;+\; \gamma\,(-\log q(z))^{-1} \;\sim\; 1$, so we can ignore this factor when we determine the asymptotic behavior. Furthermore, for the same reason,

$$\frac{\alpha+1}{\alpha} \;\le\; z\, \frac{\zeta^{1/\alpha} C z^{1/\alpha}\, (-\log(\zeta z))^{-\gamma/\alpha}\, \left(\frac{\alpha+1}{\alpha} \;+\; \frac{\gamma}{\alpha}\,(-\log(\zeta z))^{-1}\right) +\; z}{\zeta^{1/\alpha}\, C\, z^{(\alpha+1)/\alpha}\, (-\log(\zeta z))^{-\gamma/\alpha} \;+\; \frac{1}{2}\, z^2} \;\le\; 2$$

for small ζ, z. Therefore, since for small ζ, z $(-\log q(z))^{-\gamma}$ is sufficiently small, we have

$$\frac{1}{z} \;-\; \frac{g'(z)}{g(z)} \;\sim\; \frac{1}{z} \tag{121}$$

for small z, ζ.

Making the transformation $z \;=\; \zeta^{1/(\alpha-1)}\, (-\log \zeta)^{-\gamma/(\alpha-1)} y$, standard dominated convergence arguments together with Lemma L.8 imply that

$$\begin{aligned}
&\int_0^r z^{-\alpha-1}\, \phi\left(z^{-1}\, g^{1/(\alpha+1)}\right) dz \\
&= \left(\frac{\zeta}{(-\log \zeta)^{\gamma}}\right)^{-\alpha/(\alpha-1)} \int_0^{r\left(\frac{\zeta}{(-\log \zeta)^{\gamma}}\right)^{-1/(\alpha-1)}} y^{-\alpha-1}\, \phi\left(y^{-1}\, (r_\varepsilon(y))^{1/(\alpha+1)}\right) dy \\
&\sim \left(\frac{\zeta}{(-\log \zeta)^{\gamma}}\right)^{-\alpha/(\alpha-1)} \int_0^{\infty} y^{-\alpha-1}\, \phi\left(y^{-1}\, r(y)^{1/(\alpha+1)}\right) dy,
\end{aligned} \tag{122}$$

completing the proof. ■

Lemma L.10 *We have*

$$V_{b\tau}(z) \;\approx\; \frac{1}{\overline{G}\alpha} \log \frac{1}{1-z} \;+\; K(\varepsilon)$$

as $z \uparrow 1$, for some constant $K(\varepsilon)$.

Proof. As above, we will everywhere use the normalization $v^H = 1,\ v_b = 0$. For brevity, let $h^{H,L} = h^{H,L}_{b\tau}$. We have

$$V'_{b\tau}(z) \;=\; (\overline{G})^{-1}\left(\frac{z}{1-z}\,\frac{1}{h^H(V_{b\tau}(z))} \;+\; \frac{1}{h^L(V_{b\tau}(z))}\right),$$

and therefore

$$V_{b\tau}(z) \;=\; V_{b\tau}(z_0) \;+\; (\overline{G})^{-1}\int_{z_0}^{z}\left(\frac{y}{1-y}\,\frac{1}{h^H(V_{b\tau}(y))} \;+\; \frac{1}{h^L(V_{b\tau}(y))}\right)dy$$

for any $z_0 \in (0,1)$. A direct application of l'Hopital's rule implies that

$$\frac{1}{h^H(x)} \;=\; \frac{G^H(x)}{\psi^H(x)} \;\to\; \alpha^{-1}$$

as $x \to +\infty$. Using the identity

$$\frac{G^H(x)}{\psi^H(x)} \;-\; \alpha^{-1} \;=\; e^{\alpha x}\,\frac{\int_x^{+\infty} e^{-\alpha y}((y/x)^{\gamma} e^{\alpha y} y^{-\gamma}\psi^H(y) - e^{\alpha x} x^{-\gamma}\psi^H(x))\,dy}{e^{\alpha x} x^{-\gamma}\,\psi^H(x)},$$

it is possible to show that this will converge to zero at least as fast as $x^{-\gamma}$. Indeed, condition (87) implies that we can replace $e^{\alpha y} y^{-\gamma}\psi^H(y)$ by its limit value c_τ as the difference will be asymptotically negligible. Thus, it remains to consider

$$e^{\alpha x}\int_x^{+\infty} e^{-\alpha y}((y/x)^{\gamma} - 1)\,dy \;=\; \int_0^{\infty} e^{-\alpha y}((1+y/x)^{\gamma} - 1)\,dy \;\le\; x^{-\gamma}\int_0^{\infty} e^{-\alpha y} y^{\gamma}\,dy.$$

Therefore, we can write

$$\begin{aligned}
V_{b\tau}(z) \;&= V_{b\tau}(z_0) \;+\; (\overline{G})^{-1}\int_{z_0}^{z}\left(\frac{y}{1-y}\,\frac{1}{h^H(V_{b\tau}(y))} \;+\; \frac{1}{h^L(V_{b\tau}(y))}\right)dy \\
&=\; V_{b\tau}(z_0) \;+\; \frac{1}{\overline{G}\alpha}(-z \;-\; \log(1-z) - (-z_0 - \log(1-z_0))) \\
&\quad +\; \frac{1}{\overline{G}}\int_{z_0}^{z}\left(\frac{y}{1-y}\left(\frac{1}{h^H(V_{b\tau}(y))} - \frac{1}{\alpha}\right) \;+\; \frac{1}{h^L(V_{b\tau}(y))}\right)dy.
\end{aligned} \tag{123}$$

Consequently, when $z \uparrow 1$,

$$V_{b\tau}(z) \;\sim\; \frac{1}{\overline{G}\alpha}\log\frac{1}{1-z} \;+\; K(\varepsilon),$$

where

$$\begin{aligned}
K(\varepsilon) \;&=\; V_{b\tau}(z_0) \;+\; \frac{1}{\overline{G}\alpha}(-1 + z_0 + \log(1-z_0)) \\
&\quad +\; \frac{1}{\overline{G}}\int_{z_0}^{1}\left(\frac{y}{1-y}\left(\frac{1}{h^H(V_{b\tau}(y))} - \frac{1}{\alpha}\right) \;+\; \frac{1}{h^L(V_{b\tau}(y))}\right)dy,
\end{aligned} \tag{124}$$

and the claim follows. ■

Lemma L.11 *When $\overline{G}$ becomes large, $K(\varepsilon)$ converges to*

$$K \;=\; A - \int_{-\infty}^{A} \alpha^{-1}\, h^H(x)\, dx \;+\; \int_{A}^{+\infty} (1 - h^H(x)/\alpha)\, dx\,.$$

Proof. Based on the change of variables

$$V_{b\tau}(y) = x\ ,\ dy = B'_\tau(x)\, dx = \overline{G}\left(\frac{B_\tau(x)}{1-B_\tau(x)}\,\frac{1}{h^H(x)} \;+\; \frac{1}{h^L(x)}\right)^{-1},$$

we have

$$\begin{aligned}
&\frac{1}{\overline{\overline{G}}}\int_{z_0}^{1}\left(\frac{y}{1-y}\,\left(\frac{1}{h^H(V_{b\tau}(y))} - \frac{1}{\alpha}\right) \;+\; \frac{1}{h^L(V_{b\tau}(y))}\right) dy \\
&\quad = \int_{V_{b\tau}(z_0)}^{+\infty} \frac{\frac{B_\tau(x)}{1-B_\tau(x)}\left(\frac{1}{h^H(x)} - \frac{1}{\alpha}\right) \;+\; \frac{1}{h^L(x)}}{\frac{B_\tau(x)}{1-B_\tau(x)}\,\frac{1}{h^H(x)} \;+\; \frac{1}{h^L(x)}}\, dx.
\end{aligned} \tag{125}$$

When $\overline{G} \to \infty$, $B_\tau(x) \to v^H = 1$. Hence the leading asymptotic of the integrand is given by $1 - h^H(x)/\alpha$. Therefore, for any $A > 0$,

$$\begin{aligned}
&\int_{V_{b\tau}(z_0)}^{+\infty} \frac{\frac{B_\tau(x)}{1-B_\tau(x)}\left(\frac{1}{h^H(x)} - \frac{1}{\alpha}\right) \;+\; \frac{1}{h^L(x)}}{\frac{B_\tau(x)}{1-B_\tau(x)}\,\frac{1}{h^H(x)} \;+\; \frac{1}{h^L(x)}}\, dx \\
&\quad\approx \int_{V_{b\tau}(z_0)}^{+\infty} (1 - h^H(x)/\alpha)\, dx \\
&\quad= A - V_{b\tau}(z_0) \;-\; \int_{V_{b\tau}(z_0)}^{A} \alpha^{-1}\, h^H(x)\, dx \;+\; \int_{A}^{+\infty} (1 - h^H(x)/\alpha)\, dx \\
&\quad\approx A - V_{b\tau}(z_0) \;-\; \int_{-\infty}^{A} \alpha^{-1}\, h^H(x)\, dx \;+\; \int_{A}^{+\infty} (1 - h^H(x)/\alpha)\, dx,
\end{aligned} \tag{126}$$

and the claim follows. ■

Lemma L.12 *When $\theta \to +\infty$ and $\overline{G} \to \infty$ in such a way that $\theta - \log\varepsilon/(\alpha+1) \to +\infty$, we have*

$$u_{s,0,\tau}(N_{\max}, \theta) - u_{s,0,\tau}(N_{\min}, \theta) \;\sim\; e^{-(\alpha+1)\frac{\theta}{\overline{G}\alpha - 1}}\, |\theta|^{\gamma_\tau}\, Z$$

and

$$\frac{\partial}{\partial\theta}\,\left(u_{s,0,\tau}(N_{\max}, \theta) - u_{s,0,\tau}(N_{\min}, \theta)\right) \;\sim\; -\frac{\alpha+1}{\overline{G}\alpha - 1} e^{-(\alpha+1)\frac{\theta}{\overline{G}\alpha - 1}}\, |\theta|^{\gamma_\tau}\, Z$$

for some constant $Z > 0$.

Proof. When $y \to \infty$ we have $S_\tau(y) \to 1$. Thus,

$$\begin{aligned} y \;=\; V_{s\tau}(S_\tau(y)) \;=\; \log \frac{S_\tau(y)}{1 - S_\tau(y)} \;-\; V_{b\tau}(S_\tau(y)) \\ \sim\; \left(1 - \frac{1}{\overline{G}\alpha}\right) \log \frac{1}{1 - S_\tau(y)} \;-\; K(\varepsilon)\,. \end{aligned} \tag{127}$$

Therefore,

$$1 \;-\; S_\tau(y+\theta) \;\sim\; e^{-(y+\theta+K(\varepsilon))/\left(1-\frac{1}{\overline{G}\alpha}\right)}$$

when $\theta \to \infty$ and

$$V_{b\tau}(S_\tau(y+\theta)) \;\sim\; \frac{y+\theta+K(\varepsilon)}{1-(\overline{G}\alpha)^{-1}} - (y+\theta).$$

Hence

$$G^H_{b\tau}(V_{b\tau}(S_\tau(y+\theta))) \;\sim\; \frac{c_\tau}{\alpha}\,|\theta|^{\gamma_\tau}\, e^{-\alpha\frac{y+\theta+\overline{G}\alpha K}{\overline{G}\alpha-1}}.$$

Therefore, in the high state, we get

$$\begin{aligned} &\int_{\mathbb{R}} (S_\tau(y+\theta)-1)\, G^H_{b\tau}(V_{b\tau}(S_\tau(y+\theta)))\, R^{H,N}_{t,\tau}(\theta, y+\theta)\, dy \\ &\sim\; -\frac{c_\tau}{\alpha}\,|\theta|^{\gamma_\tau} \int_{\mathbb{R}} e^{-(y+\theta+K(\varepsilon))/\left(1-\frac{1}{\overline{G}\alpha}\right)}\, e^{-\alpha\frac{y+\theta+\overline{G}\alpha K}{\overline{G}\alpha-1}}\, R^{H,N}_{t,\tau}(\theta, y+\theta)\, dy \\ &=\; -|\theta|^{\gamma_\tau} e^{-(\alpha+1)\frac{\overline{G}\alpha}{\overline{G}\alpha-1}K(\varepsilon)}\, e^{-\theta\left(\frac{\alpha(\overline{G}+1)}{\alpha\overline{G}-1}\right)}\, \frac{c_\tau}{\alpha} \int_{\mathbb{R}} e^{-(\alpha+1)\frac{y}{\overline{G}\alpha-1}} e^{-y}\, R^{H,N}_{t,\tau}(\theta, y+\theta)\, dy. \end{aligned} \tag{128}$$

In the low state, using $S_\tau(y+\theta) - v_s \sim \overline{G}$, we get

$$\begin{aligned} &\int_{\mathbb{R}} (S_\tau(y+\theta)-v_s)\, G^L_{b\tau}(V_{b\tau}(S_\tau(y+\theta)))\, R^{L,N}_{t,\tau}(\theta, y+\theta)\, dy \\ &\quad\sim\; \overline{G}\,|\theta|^{\gamma_\tau}\, \frac{c_\tau}{\alpha+1} \int_{\mathbb{R}} e^{-(\alpha+1)\frac{y+\theta+\overline{G}\alpha K}{\overline{G}\alpha-1}}\, R^{L,N}_{t,\tau}(\theta, y+\theta)\, dy \\ &\quad\sim\; \overline{G}\, e^{-(\alpha+1)\frac{\theta+\overline{G}\alpha K}{\overline{G}\alpha-1}}\, |\theta|^{\gamma_\tau}\, \frac{c_\tau}{\alpha+1} \int_{\mathbb{R}} e^{-(\alpha+1)\frac{y}{\overline{G}\alpha-1}}\, R^{L,N}_{t,\tau}(\theta, y+\theta)\, dy. \end{aligned} \tag{129}$$

Thus, in the limit as $\overline{G} \to 0$, the gain in the low state from acquiring information satisfies

$$\begin{aligned} &e^{-(\alpha+1)\frac{\theta+\overline{G}\alpha K}{\overline{G}\alpha-1}}\, |\theta|^{\gamma_\tau}\, \frac{\overline{G}c_\tau}{\alpha+1} \int_{\mathbb{R}} e^{-(\alpha+1)\frac{y}{\overline{G}\alpha-1}} (R^{L,N_{\max}}_{t,\tau}(\theta, y+\theta) - R^{L,N_{\min}}_{t,\tau}(\theta, y+\theta))\, dy \\ &\sim\; \frac{1}{\alpha} e^{-(\alpha+1)K}\, e^{-(\alpha+1)\frac{\theta}{\overline{G}\alpha-1}}\, |\theta|^{\gamma_\tau}\, c_\tau \int_{\mathbb{R}} (-y)(\eta^L_{N_{\max}} - \eta^L_{N_{\min}}) * h^L_{t,\tau}(y)\, dy, \end{aligned} \tag{130}$$

whereas the loss in the H state is asymptotically negligible because the additional factor $\overline{G}$ is missing. ■

The following lemma completes the proof of Theorem K.1.

Lemma L.13 *There exist $g, A > 0$ such that a threshold equilibrium exists whenever $\overline{G} > g$ and $\pi < e^{-A\overline{G}}$. In any such equilibrium, conditions (88) and (89) hold as $\pi^{-1}, \overline{G} \to \infty$.*

Proof. Fix a threshold acquisition policy $\{\overline{X}_{it}, \underline{X}_{it}\}_{i=b,s,t\geq 1}$ of all the agents in the market. It follows from the above (Lemmas L.1, L.3, L.5 and L.12) that there exist constants $a, g, B > 0$ such that the gains from information acquisition are monotone decreasing in $|\theta|$ when $|\theta| > B$, $\overline{G} > g$ and

$$\min\{\min_{i,t} |\overline{X}_{it}|, \min_{i,t} |\underline{X}_{it}|\} > a\overline{G}\,.$$

Therefore, the optimal acquisition policy for any agent is also of threshold type, given by $\{\tilde{\overline{X}}_{it}, \tilde{\underline{X}}_{it}\}$, whenever π is sufficiently small. It follows from the proofs of Lemmas L.5 and L.12 that, in fact, there exists an $A > 0$ such that $\pi < e^{-A\overline{G}}$ is sufficient for this. Clearly, choosing $A > 0$ sufficiently big, we can achieve that

$$\min\{\min_{i,t} |\tilde{\overline{X}}_{it}|, \min_{i,t} |\tilde{\underline{X}}_{it}|\} > a\overline{G}\,.$$

Making the change of variables $\theta \to Re^{\theta}/(1 + Re^{\theta})$, we immediately get that the mapping from $\{\overline{X}_{it}, \underline{X}_{it}\}_{i=b,s,t\geq 1}$ to $\{\tilde{\overline{X}}_{it}, \tilde{\underline{X}}_{it}\}_{i=b,s,t\geq 1}$ maps bounded convex set into itself. Therefore, existence of a threshold equilibrium follows by the Brower fixed point Theorem. The fact that any equilibrium satisfies (88) and (89) follows by a careful examination of alternative cases, is very lengthy and is therefore omitted. ■

We can now calculate approximations for the optimal acquisition thresholds. Though we cannot prove that an equilibrium is unique, the next result implies that the equilibrium is *asymptotically unique*, in the sense that the asymptotic behavior of the equilibrium thresholds is the same for any equilibrium.

Lemma L.14 *For any equilibrium, in the limit when $\overline{G} \to \infty$ and $\pi \to 0$ in such a way that $\pi < e^{-A\overline{G}}$, the optimal information acquisition thresholds satisfy*

1.

$$\begin{aligned}(\alpha+1)\overline{X}_{bt} \approx \overline{K}_{b,t,\tau} + \left(m_{t,T}^{Q} - \frac{\alpha}{\alpha+1} m_T^{\psi}\right)\log\lambda + \log(\pi^{-1}) - \frac{2\alpha+1}{\alpha+1}\log\overline{G} \\ +\ (\gamma_{t,T}^{Q} + N_{\max})\log(\log(\pi^{-1}\overline{G}^{-\frac{2\alpha+1}{\alpha+1}})) - \frac{\alpha}{\alpha+1}\gamma_T^{\psi}\log\log\overline{G}.\end{aligned} \quad (131)$$

2.

$$-(\alpha+1)\underline{X}_{bt} \;\approx\; \underline{K}_{b,t,\tau} \;+\log R \;+\; (\log(\pi^{-1})+(\gamma^{Q}_{t,T}+N_{\max})\log(\log(\pi^{-1}))) \\ +\; \frac{\alpha}{\alpha-1}\log\overline{G} \;+\; \left(\gamma^{Q}_{t,T}+N_{\max}+\frac{\alpha}{\alpha-1}\gamma^{\psi}_{T}\right)\log\log\overline{G}. \tag{132}$$

3.

$$(\alpha+1)\overline{X}_{st} \;\approx\; (\overline{G}\alpha-1) \\ \left(\log(\pi^{-1})+\overline{K}_{s,0,T}+\gamma_T\log\left(\frac{\overline{G}\alpha-1}{\alpha+1}(\log(\pi^{-1})+\overline{K}_{s,0,T})\right)\right). \tag{133}$$

4.

$$-(\alpha+1)\underline{X}_{st} \;\approx\; \underline{K}_{s,0,\tau} \;+\; \left(m^{Q}_{t,T}-\frac{\alpha}{\alpha+1}m^{\psi}_{T}\right)\log\lambda \;+\; \log(\pi^{-1}) \\ -\; \frac{\alpha}{\alpha+1}\log\overline{G} \;+\; (\gamma^{Q}_{t,T}+N_{\max})\log(\log(\pi^{-1}\overline{G}^{-\frac{\alpha}{\alpha+1}})) \\ -\; \frac{\alpha}{\alpha+1}\gamma^{\psi}_{T}\log\log\overline{G}, \tag{134}$$

where $\gamma^{\psi}_{T}=(2^{T+1}-1)N_{\max}-1$ *and* $\gamma^{Q}_{t,T}=(2^{T+1}-2^{t+1})N_{\max}-1\,.$

Proof. The proof follows directly from Lemma L.5 and Lemmas L.6-L.12. ■

Proof of Theorem K.2. This theorem follows from substituting the asymptotic expressions of Lemma L.14 into the asymptotic formulae of Lemma L.1 for the tail behaviour of the densities of type distributions. ■

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