Spherical Coordinates

Last class, we said that \( \iiint \, da = 4\pi r^2 \) for a sphere because that’s the total surface area. Let’s go prove that using spherical coordinates. Take a small area element \( da \). The “height” of the patch is an arc with arc length \( r \, d\theta \), and the “width” is an arc with arc length \( r \sin \phi \, d\phi \). (In physics, \( \theta \) is the angle from the y-axis and \( \phi \) is the polar angle in the plane; this is opposite the mathematical convention.) Therefore, the area element is \( da = r^2 \sin \theta \, d\theta \, d\phi \) and the volume element is \( dv = r^2 \sin \theta \, dr \, d\theta \, d\phi \). (If you need a refresher on spherical coordinates, try to show that \( x = r \sin \theta \cos \phi \), \( y = r \sin \theta \sin \phi \), \( z = r \cos \theta \), and \( r^2 = x^2 + y^2 + z^2 \).)

Using this area element, we can verify:

\[
\iiint \, da = \int_0^{2\pi} \int_0^\pi \int_0^r r^2 \sin \theta \, d\theta \, d\phi \, dr = 2\pi \int_0^\pi \sin \theta \, d\theta \int_0^r r \, dr = 4\pi r^2 = A
\]

When finding the enclosed charge, it is also useful to know the following:

\[
\iiint \, dv = \int_0^{2\pi} \int_0^\pi \int_0^r r^2 \sin \theta \, dr \, d\theta \, d\phi = \frac{1}{3} \pi r^3 \int_0^{2\pi} \int_0^\pi \sin \theta \, d\theta \, d\phi = \frac{1}{3} \pi r^3 (4\pi) = \frac{4}{3} \pi r^3 = V
\]

Proof of Gauss’s Law

Now, for a proof of Gauss’s Law:

1. Take a point charge at the center of a Gaussian sphere. Assuming Coulomb’s law, we can write \( \iiint \vec{E} \cdot d\vec{a} = \int (q_i / r^2) da = (q_i / r^2) 4\pi r^2 = 4\pi q_i \).

2. Now take a point charge outside the Gaussian sphere. Draw a “wedge” from the charge to the sphere. Taking the solid angle \( d\Omega = \sin \theta \, d\theta \, d\phi \) (the solid angle of an entire sphere is \( \Omega = 4\pi \)), we can write \( da = r^2 \, d\omega \). Thus, \( \iiint \vec{E} \cdot d\vec{a} = -E_1 da_1 + E_2 da_2 = -(q_i / r^2) \int_0^\pi \int_0^{2\pi} \sin \theta \, d\theta \, d\phi = \frac{1}{3} \pi r^3 (4\pi) = \frac{4}{3} \pi r^3 = 0 \), since the solid angle is constant.

3. For a non-spherical surface, we can use a deformation argument. Take a small area element and draw the tangent plane. The flux through this element is \( \vec{E}_s \cdot d\vec{a}_s \). If we rotate this plane by an angle \( \theta \), then \( \vec{E}_s = \vec{E}_1 \) and \( d\vec{a}_s = d\vec{a}_1 / \cos \theta \). But \( \vec{E}_s \cdot d\vec{a}_s = E_s da_s \cos \theta = E_1 da_1 \). Thus, the flux is still the same, even if the Gaussian surface is not spherical.

4. Now take an arbitrary Gaussian surface \( S \) and an arbitrary continuous source distribution. Say that the Gaussian surface contains part of the source. We can represent the source distribution as a collection of point charges, and then instantly neglect the part of the source outside our surface. Now surround each point charge inside the surface with a small sphere. For each sphere, the flux through the sphere is \( \phi_i = 4\pi q_i \). Let \( S_i \) be the Gaussian surface excluding the sphere around \( q_i \). Any flux caused by \( q_i \) that enters \( S_i \) later exits \( S_i \), so \( q_i \) contributes no flux to \( S_i \). Thus, the contribution of \( q_i \) to the flux of \( S \) is \( 4\pi q_i = 0 = 4\pi q_i \). Summing over all charges, we find that \( \sum \phi = 4\pi \sum q \).

Often, students confuse the source surface and the Gaussian surface. The source surface is a physical surface defined by the presence of charges. The Gaussian surface doesn’t exist — it’s just a mathematical construct used to probe the field, just like a test charge can probe an electric field at a point. Gauss’s law is always true, but it’s easiest for us to evaluate when \( |E| \) is constant along a surface, and when \( \vec{E} \) is parallel or perpendicular to \( \vec{n} \). For instance, a ring has lots of symmetry, but we can’t apply Gauss’s law because it doesn’t follow these guidelines. There is “not enough symmetry.” A finite cylinder also doesn’t work because of edge effects, but an infinite cylinder does. We know this because we can argue \( \text{a priori} \) by symmetry that the only possible \( \vec{E} \) field has to point radially outward; this means we can simply enclose the infinite cylinder with a larger (but finite) cylinder.

Homework: When the cylinder has radius \( R \), find the electric field at \( r < R \) and \( r > R \).
Homework

Let $r \leq R$. Draw as a Gaussian surface a cylinder of radius $r$ and length $l$. The electric field is constant and outward along this cylinder, so the flux is $E A = E (2 \pi r l)$. The charge enclosed is $\rho V = \rho (\pi r^2 l)$. Gauss’s Law tells us that $2 \pi r l E = 4 \pi \rho r^2 l$, so $E = 2 \pi r \rho$. When $r > R$, we modify the enclosed charge formula to $\rho (\pi R^2 l)$, so $E = 2 \pi R^2 \rho / r$. 