Simple Folding is Strongly NP-Complete

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Deciding whether a crease pattern can fold into a non-self-intersecting flat-folded state is known to be strongly NP-complete [**?**], but if the folding must be formed using a sequence of simple folds with prescribed mountain/valley assignment, the decision problem has only been shown weakly NP-complete [**?**] via reduction from Partition. We close this gap by proving the latter problem strongly NP-complete, even for orthogonal polygons with only orthogonal valley creases.

Theorem 1. *The problem of deciding simple foldability of an orthogonal piece of paper with an orthogonal mountain-valley pattern is strongly NP-complete.*

The reduction is from 3-Partition. Given an instance of 3-Partition with integers $A = \{a_1, \ldots, a_n\}$ to be partitioned into $n/3$ triples each with sum $\sum A/(n/3) = t$, construct an orthogonal polygon with orthogonal valley creases as shown in Figure **??**. We assume that the a_i are all close to $t/3$: if not, add a large number to all a_is so that they are.

There are four main functional sections of the polygon. On the left is the Bar, a $2 : 2\infty$ rectangle of paper without creases that is very long ($\infty = 10nt$). The Staircase is the next section that encodes the a_is in order in a series of steps of height twice their value. Each step i contains two creases c_{2i-1}, c_{2i} that when both folded will raise the Bar by exactly $2a_i$. The Wrapper section is a horizontal rectangle of length $2n/3$ with vertical valley creases d_i (d_1 being the right most crease) dividing the Wrapper into unit squares. The Cage on the right bounds a closed polygon.

The construction forces the Bar to wrap through the Cage $n/3$ times, each time shifted up by distance $2t$ (note that ∞ is chosen large to ensure that the Staircase does not intersect the Cage polygon when wrapping). To prove the claim, we prove that the Wrapper must fold its vertical creases in order from right to left. If this were not the case, then there exists some first crease d_i to be folded whose right neighbor d_{i-1} has not yet been folded. But d_i has at least two squares of unfolded paper to its left that will cover d_{i-1}, making d_{i-1} impossible to fold using simple folds without having to unfold or violate the mountain/valley assignment, which contradicts our model. Because the Wrapper executes its folds from right to left, the Bar must pass through the Cage $n/3$ times sequentially from the rightmost slot to the left, with each subsequent slot shifted up by $2t$.

If the 3-Partition instance has a positive solution, then the polygon has a simple folding: just fold the creases corresponding to the a_is in one of the satisfying triples, then fold the Bar through the Cage along Wrapper creases, and repeat. Further, if the polygon has a simple folding, the 3-Partition instance has a positive solution because the Staircase must be folded on both creases from exactly three a_is between each wrap. To see this, all a_is are close to $t/3$, so exactly three a_i sections must be flipped from their original orientation to shift by $2t$, achievable by folding at most six creases. Since all $2n$ horizontal creases must be used to bring the Bar up to the final slot, and because there are $n/3$ slots, exactly six creases must be used to raise up the Bar each time: three pairs of creases bounding one section each. Now that we have creases that must be folded in adjacent pairs, and because all creases are used, a parity argument guarantees that pairs of creases are from the same a_i section.

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The reduction is polynomial because the entire constructed polygon is bounded by a $3\infty \times 4n$ rectangle. Lastly, the problem is in NP because given a certificate of the crease folding order, each fold can be simulated and checked in polynomial time. $\square$

Because this reduction only ever folds through one layer at a time, the reduction works in the one-layer, some-layers, and all-layers models of simple folds. Our reduction extends to folding a square polygon with creases constrained to angles at multiples of 45° using the same construction as described in [**?**]. We also adapt this reduction to crease patterns without mountain/valley assignment under the some-layers and all-layers simple fold model. Determining NP-hardness, weak or strong, remains open for unassigned mountain/valley assignment under the one-layer simple folds model.

References

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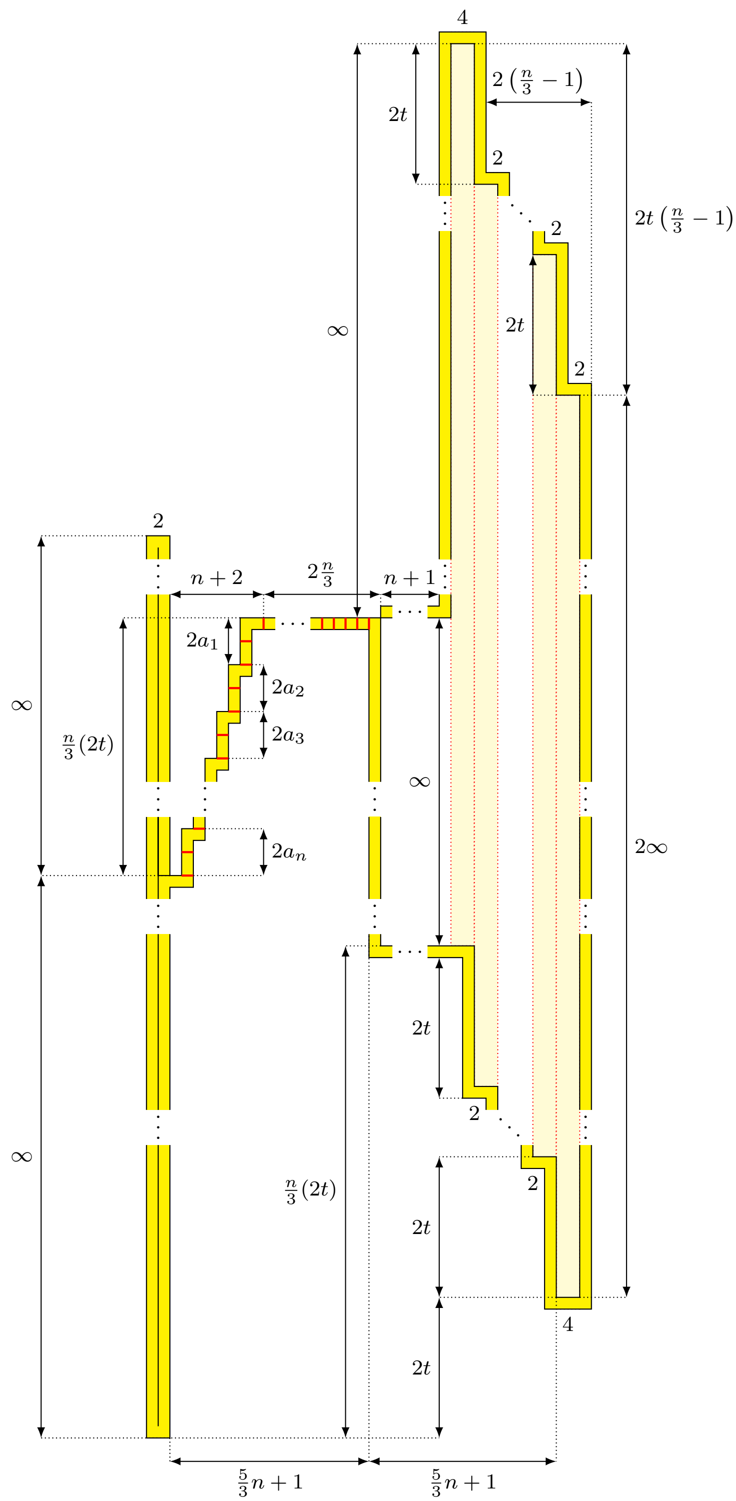

Figure 1: An orthogonal simple polygon with orthogonally aligned valley creases (in red) constructed from an instance of 3-Partition that can be folded using simple folds if and only if the instance of 3-Partition is valid.