Quadratic Approximation

Quadratic approximation is an extension of linear approximation – we're adding one more term, which is related to the second derivative. The formula for the quadratic approximation of a function $f(x)$ for values of x near x_0 is:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 \quad (x \approx x_0)$$

Compare this to our old formula for the linear approximation of f:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \quad (x \approx x_0).$$

These are more complicated and so are only used when higher accuracy is needed.

Let's look at the quadratic version of our estimate of $\ln(1.1)$. The formula for the quadratic approximation turns out to be $\ln(1 + x) \approx x - \frac{x^2}{2}$, and so $\ln(1.1) = \ln(1 + \frac{1}{10}) \approx \frac{1}{10} - \frac{1}{2}(\frac{1}{10})^2 = 0.095$. This is not the value 0.1 that we got from the linear approximation, but it's pretty close (and slightly more accurate).

We'll save the derivation of the formula for later; right now we're going to find formulas for quadratic approximations of the functions for which we have a library of linear approximations.

0.0.1 Basic Quadratic Approximations:

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2 \quad (x \approx 0)$$

$f(x)$	$f'(x)$	$f''(x)$	$f(0)$	$f'(0)$	$f''(0)$
$\sin x$	$\cos x$	$-\sin x$	0	1	0
$\cos x$	$-\sin x$	$-\cos x$	1	0	-1
e^x	e^x	3^x	1	1	1
$\ln(1+x)$	$\frac{1}{1+x}$	$\frac{-1}{(1+x)^2}$	0	1	-1
$(1+x)^r$	$r(1+x)^{r-1}$	$r(r-1)(1+x)^{r-2}$	1	r	$r(r-1)$

1. $\sin x \approx x \quad (\text{if } x \approx 0)$

2. $\cos x \approx 1 - \dfrac{x^2}{2} \quad (\text{if } x \approx 0)$

3. $e^x \approx 1 + x + \dfrac{1}{2}x^2 \quad (\text{if } x \approx 0)$

4. $\ln(1 + x) \approx x - \dfrac{1}{2}x^2 \quad (\text{if } x \approx 0)$

5. $(1 + x)^r \approx 1 + rx + \dfrac{r(r-1)}{2}x^2 \quad (\text{if } x \approx 0)$

We've computed some formulas; now let's think about where they come from.

Geometric significance (of the quadratic term): A quadratic approximation gives a best-fit parabola to a function. For example, let's consider $f(x) = \cos(x)$ (see Figure 1).

The linear approximation of $\cos x$ near $x_0 = 0$ approximates the graph of the cosine function by the straight horizontal line $y = 1$. This doesn't look like a very good approximation.

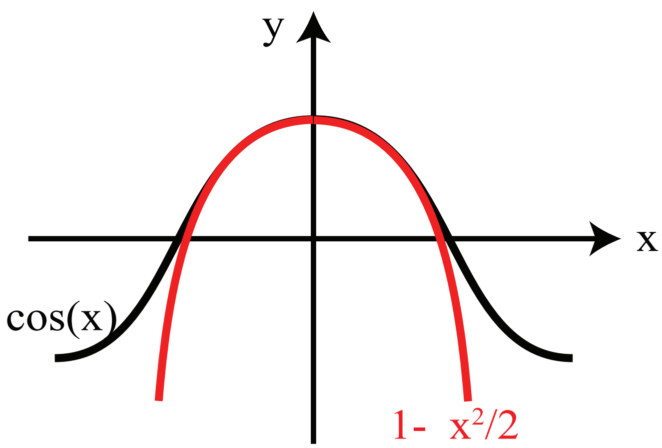

Figure 1: Quadratic approximation to $\cos(x)$.

The quadratic approximation to the graph of $\cos(x)$ is a parabola that opens downward; this is much closer to the shape of the graph at $x_0 = 0$ than the line $y = 1$. To find the equation of this quadratic approximation we start by assuming $x_0 = 0$, then perform the following calculations:

$$\begin{aligned}
f(x) &= \cos(x) \quad \Longrightarrow \quad f(0) = \cos(0) = 1 \\
f'(x) &= -\sin(x) \quad \Longrightarrow \quad f'(0) = -\sin(0) = 0 \\
f''(x) &= -\cos(x) \quad \Longrightarrow \quad f''(0) = -\cos(0) = -1
\end{aligned}$$

We conclude that:

$$\cos(x) \approx 1 + 0 \cdot x - \frac{1}{2}x^2 = 1 - \frac{1}{2}x^2.$$

This is the closest (or "best fit") parabola to the graph of $\cos(x)$ when x is near 0.