

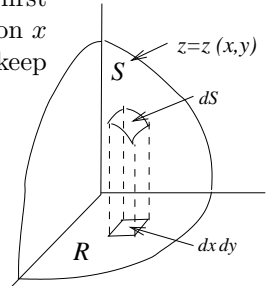
## V9.3-4 Surface Integrals

### 3. Flux through general surfaces.

For a general surface, we will use  $xyz$ -coordinates. It turns out that here it is simpler to calculate the infinitesimal vector  $d\mathbf{S} = \mathbf{n} dS$  directly, rather than calculate  $\mathbf{n}$  and  $dS$  separately and multiply them, as we did in the previous section. Below are the two standard forms for the equation of a surface, and the corresponding expressions for  $d\mathbf{S}$ . In the first we use  $z$  both for the dependent variable and the function which gives its dependence on  $x$  and  $y$ ; you can use  $f(x, y)$  for the function if you prefer, but that's one more letter to keep track of.

$$(11a) \quad z = z(x, y), \quad d\mathbf{S} = (-z_x \mathbf{i} - z_y \mathbf{j} + \mathbf{k}) dx dy \quad (\mathbf{n} \text{ points "up"})$$

$$(11b) \quad F(x, y, z) = c, \quad d\mathbf{S} = \pm \frac{\nabla F}{F_z} dx dy \quad (\text{choose the right sign});$$



#### Derivation of formulas for $d\mathbf{S}$ .

Refer to the pictures at the right. The surface  $S$  lies over its projection  $R$ , a region in the  $xy$ -plane. We divide up  $R$  into infinitesimal rectangles having area  $dx dy$  and sides parallel to the  $xy$ -axes — one of these is shown. Over it lies a piece  $dS$  of the surface, which is approximately a parallelogram, since its sides are approximately parallel.

The infinitesimal vector  $d\mathbf{S} = \mathbf{n} dS$  we are looking for has

*direction:* perpendicular to the surface, in the “up” direction;

*magnitude:* the area  $dS$  of the infinitesimal parallelogram.

This shows our infinitesimal vector is the cross-product

$$d\mathbf{S} = \mathbf{A} \times \mathbf{B}$$

where  $\mathbf{A}$  and  $\mathbf{B}$  are the two infinitesimal vectors forming adjacent sides of the parallelogram. To calculate these vectors, from the definition of the partial derivative, we have

$\mathbf{A}$  lies over the vector  $dx \mathbf{i}$  and has slope  $f_x$  in the  $\mathbf{i}$  direction, so  $\mathbf{A} = dx \mathbf{i} + f_x dx \mathbf{k}$  ;

$\mathbf{B}$  lies over the vector  $dy \mathbf{j}$  and has slope  $f_y$  in the  $\mathbf{j}$  direction, so  $\mathbf{B} = dy \mathbf{j} + f_y dy \mathbf{k}$  .

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ dx & 0 & f_x dx \\ 0 & dy & f_y dy \end{vmatrix} = (-f_x \mathbf{i} - f_y \mathbf{j} + \mathbf{k}) dx dy ,$$

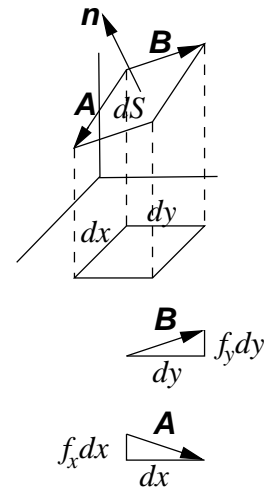
which is (11a).

To get (11b) from (11a), , our surface is given by

$$(12) \quad F(x, y, z) = c, \quad z = z(x, y)$$

where the right-hand equation is the result of solving  $F(x, y, z) = c$  for  $z$  in terms of the independent variables  $x$  and  $y$ . We differentiate the left-hand equation in (12) with respect to the independent variables  $x$  and  $y$ , using the chain rule and remembering that  $z = z(x, y)$ :

$$F(x, y, z) = c \Rightarrow F_x \frac{\partial x}{\partial x} + F_y \frac{\partial y}{\partial x} + F_z \frac{\partial z}{\partial x} = 0 \Rightarrow F_x + F_z \frac{\partial z}{\partial x} = 0$$



from which we get

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \text{and similarly,} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

Therefore by (11a),

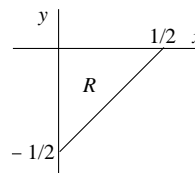
$$d\mathbf{S} = \left( -\frac{\partial z}{\partial x} \mathbf{i} - \frac{\partial z}{\partial y} \mathbf{j} + 1 \right) dx dy = \left( \frac{F_x}{F_z} \mathbf{i} + \frac{F_y}{F_z} \mathbf{j} + 1 \right) dx dy = \frac{\nabla F}{F_z} dx dy,$$

which is (11b).

**Example 3.** The portion of the plane  $2x - 2y + z = 1$  lying in the first octant forms a triangle  $S$ . Find the flux of  $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  through  $S$ ; take the positive side of  $S$  as the one where the normal points “up”.

**Solution.** Writing the plane in the form  $z = 1 - 2x + 2y$ , we get using (11a),

$$\begin{aligned} d\mathbf{S} &= (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) dx dy, \quad \text{so} \\ \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S (2x - 2y + z) dy dx \\ &= \iint_R (2x - 2y + (1 - 2x + 2y)) dy dx, \end{aligned}$$



where  $R$  is the region in the  $xy$ -plane over which  $S$  lies. (Note that since the integration is to be in terms of  $x$  and  $y$ , we had to express  $z$  in terms of  $x$  and  $y$  for this last step.) To see what  $R$  is explicitly, the plane intersects the three coordinate axes respectively at  $x = 1/2$ ,  $y = -1/2$ ,  $z = 1$ . So  $R$  is the region pictured; our integral has integrand 1, so its value is the area of  $R$ , which is  $1/8$ .

**Remark.** When we write  $z = f(x, y)$  or  $z = z(x, y)$ , we are agreeing to parametrize our surface using  $x$  and  $y$  as parameters. Thus the flux integral will be reduced to a double integral over a region  $R$  in the  $xy$ -plane, involving only  $x$  and  $y$ . Therefore you must *get rid of  $z$  by using the relation  $z = z(x, y)$*  after you have calculated the flux integral using (11a). Then determine the region  $R$  (the projection of  $S$  onto the  $xy$ -plane), and supply the limits for the iterated integral over  $R$ .

**Example 4.** Set up a double integral in the  $xy$ -plane which gives the flux of the field  $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  through that portion of the ellipsoid  $4x^2 + y^2 + 4z^2 = 4$  lying in the first octant; take  $\mathbf{n}$  in the “up” direction.

**Solution.** Using (11b), we have  $d\mathbf{S} = \frac{\langle 8x, 2y, 8z \rangle}{8z} dx dy$ . Therefore

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \frac{8x^2 + 2y^2 + 8z^2}{8z} dx dy = \iint_S \frac{1}{z} dx dy = \iint_R \frac{dx dy}{\sqrt{1 - x^2 - (y/2)^2}},$$

where  $R$  is the portion of the ellipse  $4x^2 + y^2 = 4$  lying in the first quadrant.

The double integral would be most simply evaluated by making the change of variable  $u = y/2$ , which would convert it to a double integral over a quarter circle in the  $xu$ -plane easily evaluated by a change to polar coordinates.

**4. General surface integrals.\*** The surface integral  $\iint_S f(x, y, z) dS$  that we introduced at the beginning can be used to calculate things other than flux.

a) **Surface area.** We let the function  $f(x, y, z) = 1$ . Then the area of  $S = \iint_S dS$ .

b) **Mass, moments, charge.** If  $S$  is a thin shell of material, of uniform thickness, and with density (in gms/unit area) given by  $\delta(x, y, z)$ , then

$$(13) \quad \text{mass of } S = \iint_S \delta(x, y, z) dS,$$

$$(14) \quad x\text{-component of center of mass} = \bar{x} = \frac{1}{\text{mass } S} \iint_S x \cdot \delta dS$$

with the  $y$ - and  $z$ -components of the center of mass defined similarly. If  $\delta(x, y, z)$  represents an electric charge density, then the surface integral (13) will give the total charge on  $S$ .

c) **Average value.** The average value of a function  $f(x, y, z)$  over the surface  $S$  can be calculated by a surface integral:

$$(15) \quad \text{average value of } f \text{ on } S = \frac{1}{\text{area } S} \iint_S f(x, y, z) dS.$$

### Calculating general surface integrals; finding $dS$ .

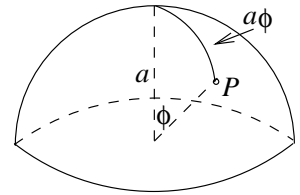
To evaluate general surface integrals we need to know  $dS$  for the surface. For a sphere or cylinder, we can use the methods in section 2 of this chapter.

**Example 5.** Find the average distance along the earth of the points in the northern hemisphere from the North Pole. (Assume the earth is a sphere of radius  $a$ .)

**Solution.** — We use (15) and spherical coordinates, choosing the coordinates so the North Pole is at  $z = a$  on the  $z$ -axis. The distance of the point  $(a, \phi, \theta)$  from  $(a, 0, 0)$  is  $a\phi$ , measured along the great circle, i.e., the longitude line — see the picture). We want to find the average of this function over the upper hemisphere  $S$ . Integrating, and using (9), we get

$$\iint_S a\phi dS = \int_0^{2\pi} \int_0^{\pi/2} a\phi a^2 \sin \phi d\phi d\theta = 2\pi a^3 \int_0^{\pi/2} \phi \sin \phi d\phi = 2\pi a^3.$$

(The last integral used integration by parts.) Since the area of  $S = 2\pi a^2$ , we get using (15) the striking answer: average distance =  $a$ .



For more general surfaces given in  $xyz$ -coordinates, since  $d\mathbf{S} = \mathbf{n} dS$ , the area element  $dS$  is the magnitude of  $d\mathbf{S}$ . Using (11a) and (11b), this tells us

$$(16a) \quad z = z(x, y), \quad dS = \sqrt{z_x^2 + z_y^2 + 1} dx dy$$

$$(16b) \quad F(x, y, z) = c, \quad dS = \frac{|\nabla F|}{|F_z|} dx dy$$

**Example 6.** The area of the piece  $S$  of  $z = xy$  lying over the unit circle  $R$  in the  $xy$ -plane is calculated by (a) above and (16a) to be:

$$\iint_S dS = \iint_R \sqrt{y^2 + x^2 + 1} dx dy = \int_0^{2\pi} \int_0^1 \sqrt{r^2 + 1} r dr d\theta = 2\pi \cdot \frac{1}{3} (r^2 + 1)^{3/2} \Big|_0^1 = \frac{2\pi}{3} (2\sqrt{2} - 1).$$

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