Stability of Networked Systems with Switching Topologies*

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Abstract— **The evolution of many networked systems, such as air transportation, can be modeled using a combination of the network topology and the resultant dynamics. In particular, time-varying networks can be represented by switching between candidate topologies. This paper models such systems as discrete-time, positive Markov Jump Linear Systems. Time-varying, periodic Markovian transition matrices and continuous state resets during discrete-mode transitions are also incorporated. Two notions of stability are considered: Mean Stability and Almost-Sure Stability, and appropriate conditions are derived for both of them. The analysis techniques are demonstrated using models determined from operational air traffic delay data. The results show that air traffic delay networks satisfy the proposed conditions for both mean stability and almost-sure stability, implying that delays tend to decay over time, even though several of the component discrete modes are unstable. Different nodes (airports) are also evaluated in terms of the persistence of delays and their susceptibility to network effects.**

I. INTRODUCTION

Infrastructure systems, such as air transportation, comprise of hundreds of nodes, and several thousands of links between them. As a result, disruptions at one node in the network can easily propagate to many others. For example, 40% of the total departure delay incurred by domestic flights in the United States is caused by the late arrival of the aircraft on its previous leg [1].

Network models have been proposed and studied for a vast range of systems, from disease epidemics [2] and rumor propagation [3], to engineered systems such as power grids [4], [5], the Internet [6], roads [7], public transport [8], railroads [9] and air transportation [10]. Prior research on spreading processes in networked systems has focused primarily on epidemiological models [11]. While these models are representative of network interactions in some systems, they do not encompass some of the behaviors that are important in infrastructure systems. Key amongst these limitations is the representation of the state of a node: Epidemic models such as Susceptible-Infected-Susceptible (SIS) or Susceptible-Infected-Recovered (SIR) models assume that a node is in one of a small set of discrete-states [11]; by contrast, the nodal state in infrastructure systems are better modeled as continuous variables (for example, delays, traffic volume, capacity, etc.). Recent work on the control of epidemics has also focused on SIS models [12]. The type of networks that have been considered in epidemic models have tended to be undirected, unweighted networks; however, the interactions between nodes in many real systems are not all binary (in that nodes either interact or they do not), but are instead weighted and directed (that is, the interactions are not symmetric). Finally, most infrastructure systems exhibit not just spatial patterns due to network interactions, but temporal patterns such as seasonal or daily trends, resulting in time-varying network topologies. Only very recently has there been an analysis on disease spread over switched networks; however, this work has considered undirected networks and SIS models [13].

Prior work on the robustness of networks to external disturbances or perturbations has generally been restricted to undirected (symmetric) networks [14], [15]. These problems are also closely related to those of consensus formation in networked systems [16]. Approaches to analyze robustness have only recently been extended to the case of directed networks, but for fixed topologies [17].

Clustering of similar nodes in networks has also been considered [18], [19], [20]. Community detection algorithms using the notion of modularity [21] can be used to determine groups of nodes (communities) such that there is stronger connectivity between nodes within a particular community than between nodes in different ones [22]. Connectivity in the air transportation system has been traditionally modeled only in terms of operations, that is, flight service between an origin and a destination [23], [24], [25], [26]. By contrast, delay is a measure of the quality of service between two airports. The delay on an edge may not be proportional to the number of operations on that edge, since the capacity of airports and links vary significantly. As a result, two links with similar traffic levels can experience very different levels of delay. In recent work, we have considered potential models of the interaction between delays and traffic flows, but only for fixed network topologies [27]. The analysis of the stability of these models remains an open problem.

A. Contributions of this paper

This paper analyzes the stability of networked systems with time-varying network topologies. In particular, the topologies are assumed to switch randomly between a set of candidate networks. The underlying dynamics within a particular mode (i.e., network topology) are assumed to be linear. Motivated by applications to air traffic systems, we consider discrete-time systems, and weighted and directed

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networks. The state of each node is assumed to lie in a continuous range of values; we allow for the possibility of discontinuous jumps in this state when the discrete mode of the system changes. Under these assumptions, we show that the system can be modeled as a discrete-time, positive, Markov Jump Linear System. We consider two notions of stability: *mean stability* and *almost-sure stability*, and determine conditions under which each form of stability is guaranteed. Finally, we illustrate our results using examples drawn from operational air traffic delay data.

II. MODEL OF DYNAMICS

A. Network topologies

A network is represented as $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, where $\mathscr{V}$ is the set of *vertices* (nodes) and $\mathscr{E}$ is the set of edges between them. The network has $|\mathscr{V}| = V$ nodes. Each edge is represented as an ordered pair, (v_1, v_2), denoting a link from v_1 to v_2. Edge (i, j) has a nonnegative weight w_{ij} associated with it. The *adjacency matrix*, $A = [a_{ij}] \in \mathbb{R}^{V\times V}$ is given by

$$a_{ij} = \begin{cases} w_{ij}, & \text{if } (i,j) \in \mathscr{E}, \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

In general, the adjacency matrix of a directed graph is asymmetric.

B. Continuous state dynamics

Suppose the continuous state of the system is denoted by $\vec{x} \in \mathbb{R}^{n\times 1}$. Then, we assume that the evolution of the continuous state is given by:

$$\vec{x}(t+1) = \Gamma \vec{x}(t), \tag{2}$$

where $\Gamma \in \mathbb{R}^{n\times n}$ is a function of the adjacency matrix, A.

1) Motivating example: We consider the instance of an air traffic network, where the nodes correspond to airports and edges correspond to flight service between them. The weight of each edge at any time is the median departure delay of flights taking off on that edge during that time period. Then, a possible model for the evolution of delays at a node is as follows:

The total outbound delay at node i at time-step $t+1$ is the sum of a component proportional to the outbound delay at that node at the previous time-step t, and one that is proportional to the delay bound to i from the other nodes at time t. The proportionality constants for airport i are denoted α_i and β_i, respectively, and are assumed to be nonnegative. These constants are not necessarily equal to one, since delay may not be conserved due to factors such as the padding of flight schedules (which can help attenuate delays), slack in turnaround times, degree-of-connectivity at airports, etc. The dynamics of delay on the network is modeled by the following discrete-time linear system:

$$d_i^{\text{out}}(t+1) = \alpha_i^{\text{out}} d_i^{\text{out}}(t) + \beta_i^{\text{out}} \sum_{j=1}^{n} \bar{a}_{ij} d_j^{\text{in}}(t) \tag{3}$$

$$d_i^{\text{in}}(t+1) = \alpha_i^{\text{in}} d_i^{\text{in}}(t) + \beta_i^{\text{in}} \sum_{j=1}^{n} \bar{a}_{ji} d_j^{\text{out}}(t) \tag{4}$$

In terms of the adjacency matrix,

$$d_i^{\text{out}}(t+1) = \alpha_i^{\text{out}} d_i^{\text{out}}(t) + \beta_i^{\text{out}} \bar{A}^{\text{T}} d_j^{\text{in}}(t) \tag{5}$$

$$d_i^{\text{in}}(t+1) = \alpha_i^{\text{in}} d_i^{\text{in}}(t) + \beta_i^{\text{in}} \bar{A} d_j^{\text{out}}(t) \tag{6}$$

In (3)-(6), $d_i^{\text{out}}(t)$ and $d_i^{\text{in}}(t)$ are the total outbound and inbound delays at node i in time-step t, and $\bar{a}_{ij}$ represent the elements of the row normalized adjacency matrix $\bar{A}$, that is, $\bar{a}_{ij}$ is the fraction of the total outbound delay at node i that is destined for node j.

Suppose the continuous-state vector of the system at time-step t, denoted $\vec{x}(t) \in \mathbb{R}^{2V\times 1}$ is given by

$$\vec{x}(t) = \begin{bmatrix} \vec{d}^{\text{out}}(t) \\ \vec{d}^{\text{in}}(t) \end{bmatrix}. \tag{7}$$

Then, (3)-(6) can be written as

$$\begin{aligned} \vec{x}(t+1) &= \left(\text{diag}([\vec{\alpha}^{\text{out}}; \vec{\alpha}^{\text{in}}]) + \text{diag}([\vec{\beta}^{\text{out}}; \vec{\beta}^{\text{in}}])\mathscr{A}\right)\vec{x}(t) \\ &= \Gamma \vec{x}(t), \end{aligned} \tag{8}$$

where the matrix $\mathscr{A} = \begin{pmatrix} 0 & \bar{A} \\ \bar{A}^{\text{T}} & 0 \end{pmatrix}$ depends on the network topology.

C. Switching between network topologies

An important aspect of real-world networks is that the interconnectivity between nodes is not static, but instead varies with time. For example, Fig. 1 shows the network of delay connectivities between different airports in the US, at different periods of time. The links are colored based on their weights (median delay on that link), and illustrates that the weights can vary significantly from one time step to another (a nonzero weight corresponds to the presence of a link).

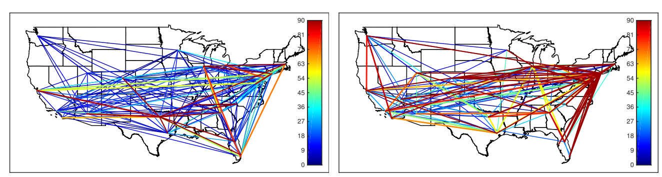

Fig. 1. Network showing weighted connectivity of delays observed between airports (in minutes) at two different times. Although the networks are directed in nature, the links are colored by the average of the weights in the two directions for ease of visualization.

The time-varying nature of the network structure is modeled by switches between M possible topologies from one time to another. The system is assumed to randomly transitions from one network topology to another (for example, these transitions in air traffic network may be caused by weather disruptions, congestion, etc.) These transitions are modeled using a $M \times M$ Markov transition matrix denoted $\Pi \in \mathbb{R}^{M\times M}$, where M is the number of discrete modes. Element π_{ij} represents the probability of transitioning from mode i to mode j at any time. That is,

$$\Pi = [\pi_{ij}] = \Pr[m(t+1) = j | m(t) = i]. \tag{9}$$

D. Markov Jump Linear Systems

The resultant dynamics can be described using a discrete-time switched linear system [28], with evolution in each discrete mode (i.e., network topology) of the form shown in (2), and with Markovian jumps between different discrete modes. Such a system is known as a Markov Jump Linear System or MJLS [29], [30]. The air traffic delay network transitions from one discrete mode to another when there are disruptions to the system or mitigation strategies put in place. These discrete mode transitions may therefore also be accompanied by resets or jumps in the continuous state, which are assumed to be linear functions of the continuous state just prior to the jump. Denoting the discrete mode at time t by $m(t)$, the dynamics of this system can be expressed as:

$$\begin{aligned}\vec{x}(t+1) &= \Gamma_{m(t)}\vec{x}(t)\\ \pi_{ij} &= \Pr[m(t+1)=j|m(t)=i]\\ \vec{x}(t+1) &= J_{ij}\Gamma_i\vec{x}(t),\ \text{if } m(t)=i \text{ and } m(t+1)=j.\end{aligned}$$

Equivalently, the system is governed by the MJLS:

$$\vec{x}(t+1) = J_{m(t),m(t+1)}\Gamma_{m(t)}\vec{x}(t);\ \ \vec{x}(0)=\vec{x}_0 \tag{10}$$

$$\pi_{ij} = \Pr[m(t+1)=j|m(t)=i];\ m(0)\sim\bar{m}_0 \tag{11}$$

where $J_{ij}=I$ if $i=j$, and the initial discrete mode is randomly generated from a distribution $\bar{m}_0$.

We also note that all elements of J_{ij} and Γ_i are nonnegative (for all discrete modes, i and j). Therefore, the system is a *positive* Markov Jump Linear System, that is, one in which the values of the continuous states remain nonnegative, for any nonnegative initial values.

III. STABILITY ANALYSIS

In analyzing networked systems, it is important to understand how the state evolves, and in particular, whether it is likely to remain bounded. In the case of air traffic networks in which the state vector corresponds to delays, stability can provide guarantees that the delays will not grow in an unbounded manner, and that they will decay.

Two notions of stability that are relevant in the context of air traffic delay networks are those of *Mean Stability* and *Almost-Sure Stability*. These are defined below.

Definition 1 (Mean Stability): A system is said to be *mean stable* if the expected value of its state tends to zero as time tends to infinity, that is,

$$\lim_{k\to\infty}\mathbb{E}[\|\vec{x}(k)\|]=0 \tag{12}$$

for any nonnegative initial condition $\vec{x}(0)$.
The above equation corresponds to the stability of the 1st-mean of the state vector, but for simplicity, it is referred to as just mean stability. This definition is aligned with the stability of homogeneous linear systems, and is well-suited to positive stochastic systems. While mean-square stability (that is, the guaranteed convergence of the second mean of the state vector) is commonly used in the analysis of general MJLS [30], when restricted to positive systems, mean stability is a more suitable notion of stability.

A stronger notion of stability is that of *exponential mean stability*, which is defined as follows [31]:

Definition 2: (Exponential Mean Stability) A discrete-time system is said to be *exponentially mean stable* if there exist positive scalars c and $r<1$ such that:

$$\mathbb{E}[\|\vec{x}(k)\|] < c\,r^k\|\vec{x}(0)\|, \tag{13}$$

for any nonnegative initial conditions $\vec{x}(0)$ and $\bar{m}(0)$.
From the above definitions, we see that exponential mean stability implies mean stability (but not vice versa).

Almost-sure stability is a more useful concept in practice, since it deals with the convergence of almost all sample paths of the system to zero.

Definition 3 (Almost-Sure Stability): A system is said to be *almost-surely stable* if the state tends to zero as time tends to infinity with probability 1, that is,

$$\Pr[\lim_{k\to\infty}\|\vec{x}(k)\|=0]=1, \tag{14}$$

for any nonnegative initial condition $\vec{x}(0)$.
In other words, this definition of stability considers the convergence to zero of almost all trajectories of the state, which is a more practical definition of stability for the case of positive MJLS. Recent work has shown that exponential mean stability implies almost-sure stability for continuous-time, positive MJLS [31]; this paper shows that the same holds for discrete-time, positive MJLS as well.

A. Stability of systems with switching network topologies

The stability of the linear system governed by 2 is determined by the maximum eigenvalue of the matrix Γ. It follows from linear systems theory that the system is stable if and only if the absolute value of the largest eigenvalue of Γ is less than 1. However, we are interested in systems in which the network topology is not constant, but changes as the system switches from one discrete mode to another. The stability analysis of switched systems (or hybrid systems) needs to consider more than just the stability of the component discrete modes. It has been well-established that a switched system can be unstable in spite of all its discrete modes being stable; similarly, a system can comprise entirely of unstable modes and still be stable due to switching [32]. For a system governed by (10)-(11), in addition to the network topologies that govern the continuous state dynamics of each mode, the Markov transition matrices for the discrete modes (Π) and the state reset matrices (J) also play a significant role in deciding system stability.

In considering the system given by (10)-(11), one must consider both the continuous-state and the discrete-mode. We therefore model the augmented state vector, $\vec{q}(t)\in\mathbb{R}^{Mn\times 1}$ of the MJLS by:

$$\vec{q}(t)=\begin{bmatrix}\vec{q}_1(t)\\ \vdots\\ \vec{q}_M(t)\end{bmatrix}, \tag{15}$$

where

$$\vec{q}_j(t) = \mathbb{E}[\vec{x}(t)\,\mathbb{1}_{m(t)=j}]. \quad (16)$$

$$\Longrightarrow \mathbb{E}[\vec{x}(t)] = \sum_{j=1}^{M} \vec{q}_j(t). \quad (17)$$

In the above, the indicator function $\mathbb{1}_{m(t)=j}$ equals 1 if the mode at time t is j, and 0 otherwise.

Using (10)-(11), we get

$$\begin{aligned}
\vec{q}_j(t+1) &= \mathbb{E}[\vec{x}(t+1)\,\mathbb{1}_{m(t+1)=j}] \\
&= \vec{x}(t+1)|_{m(t+1)=j}\,\Pr[m(t+1)=j] \\
&= \sum_{i=1}^{M} J_{ij}\Gamma_i\vec{x}(t)|_{m(t)=i}\,\pi_{ij}\Pr[m(t)=i] \\
&= \sum_{i=1}^{M} \pi_{ij}J_{ij}\Gamma_i\vec{q}_i(t). \\
\Longrightarrow \vec{q}(t+1) &= [J_{ji}]\Box\big((\Pi^{\mathrm{T}}\otimes I_n)\mathrm{diag}(\Gamma_i)\big)\vec{q}(t) \\
&= \mathscr{B}\vec{q}(t), \quad (18)
\end{aligned}$$

where $[J_{ji}]$ is the block-partitioned matrix built from the matrices J_{ji}, Π^{T} is the transpose of the Markov transition matrix, $\mathrm{diag}(\Gamma_i)$ is the block diagonal matrix built from the Γ corresponding to the different modes, $\vec{x}(t)|_{m(t)=i}$ is the value of $\vec{x}(t)$ when the mode is i, and I_n is the $n\times n$ identity matrix. In the above expressions, $\Box$ denotes the block Hadamard product, and $\otimes$ denotes the block Kronecker product [33]. In other words, the augmented state $q(t)$ follows linear dynamics, as shown in (18). Solving these dynamics, we see that:

$$\vec{q}(t) = \mathscr{B}^t\,\vec{q}(0) \;\le\; (r_\sigma(\mathscr{B}))^t\,\vec{q}(0), \quad (19)$$

where $r_\sigma(\mathscr{B})$ denotes the spectral radius (largest eigenvalue) of $\mathscr{B}$.

1) Mean stability of discrete-time MJLS: In order to determine mean stability of MJLS models, we need to assess conditions under which (12) holds. Since we are interested in mean stability, the notation $\|\cdot\|$ in the discussion below will denote the 1-norm of a vector. From (10), (11) and (17), we see that for a given initial condition $\vec{x}(0)$ and *any* distribution $\bar{m}_0$:

$$\mathbb{E}[\vec{x}(0)] = \vec{x}(0) = \sum_{j=1}^{M} \vec{q}_j(0) \quad (20)$$

$$\Longrightarrow \|\vec{q}(0)\| = \|\vec{x}(0)\| \quad (21)$$

Theorem 1 (Exponential Mean Stability): The discrete-time positive MJLS governed by (10)-(11) is exponentially mean stable if and only if the largest eigenvalue of the matrix $\mathscr{B} = [J_{ji}]\Box\big((\Pi^{\mathrm{T}}\otimes I_n)\mathrm{diag}(\Gamma_i)\big)$ is less than 1.
Proof– From (17), we have that:

$$\begin{aligned}
\mathbb{E}[\|\vec{x}(t)\|] &= \mathbb{E}[\sum_{i=1}^{n} x_i(t)] = \|\vec{q}(t)\| \quad (22)\\
&\le (r_\sigma(\mathscr{B}))^t\|\vec{q}(0)\| \text{ (from (19))} \quad (23)\\
\Longleftrightarrow \mathbb{E}[\|\vec{x}(t)\|] &\le (r_\sigma(\mathscr{B}))^t\|\vec{x}(0)\| \text{ (from (21))} \quad (24)
\end{aligned}$$

Comparing (13) and (24), we see that the system governed by (10)-(11) is exponentially mean stable if and only if $r = r_\sigma(\mathscr{B}) < 1$. ■

We note that if the system is exponentially mean stable, then the system is also mean stable, since

$$\begin{aligned}
\lim_{t\to\infty}\mathbb{E}[\|\vec{x}(t)\|] &\le \lim_{t\to\infty}(r_\sigma(\mathscr{B}))^t\|\vec{x}(0)\| \\
&= 0, \text{ (if } r<1, \text{ for a positive system).}
\end{aligned}$$

2) Almost-sure stability of discrete-time MJLS: In order to determine almost-sure stability, we need to consider (14).

Theorem 2: (Almost-Sure Stability) The discrete-time positive MJLS governed by (10)-(11) is almost-surely stable if it is exponentially mean stable.
Proof– For any element of $\vec{q}(t)$, we have

$$\begin{aligned}
q_{j_i}(t) &= \mathbb{E}[x_i(t)\,\mathbb{1}_{m(t)=j}] \\
\Longrightarrow \quad q_{j_i}(t) &\le \|\vec{q}(t)\| \quad (25)\\
&\le r^t\|\vec{q}(0)\| \text{ (from (19))} \\
\Longrightarrow \quad \max x_i(t) &\le r^t\,\|\vec{x}(0)\| \quad (26)\\
\Longrightarrow \quad \|\vec{x}(t)\| &\le c\,r^t\,\|\vec{x}(0)\| \quad (27)\\
\Longrightarrow \quad \lim_{t\to\infty}\|\vec{x}(t)\| &\le \lim_{t\to\infty} c\,r^t\,\|\vec{x}(0)\| \quad (28)\\
&= 0, \text{ if } r<1. \quad (29)\\
\Longrightarrow \Pr[\lim_{t\to\infty}\|\vec{x}(t)\|=0] &= 1. \quad (30)
\end{aligned}$$

By (25), although (26) may fail to hold on some sample paths $\vec{x}(t)$, these sample paths will necessarily fall into a set of probability 0. This implies that (28)-(29) hold with probability 1. In other words, if the discrete-time positive MJLS is exponentially mean stable, then every trajectory of the system converges to zero as t tends to infinity with probability 1. Therefore, it is almost-surely stable. ■

B. Time-varying, periodic Markovian transition matrices

Infrastructure systems such as the air traffic network also exhibit temporal patterns. Air traffic delays show strong time-of-day patterns [22] that can be incorporated into our models using periodic transition matrices. In other words, we modify the system dynamics given by (10)-(11) to be of the form:

$$\begin{aligned}
\vec{x}(t+1) &= J_{m(t),m(t+1)}\Gamma_{m(t)}\vec{x}(t);\;\; \vec{x}(0)=\vec{x}_0 \quad (31)\\
\pi_{ij}(t) &= \Pr[m(t+1)=j|m(t)=i];\; m(0)\sim\bar{m}_0 \quad (32)
\end{aligned}$$

where the Markovian transition matrix $\Pi_t = [\pi_{ij}(t)]$ is assumed to be periodic with fundamental period $K\in\mathbb{Z}^+$, i.e.,

$$\Pi_{t+nK} = \Pi_t,\ \forall n\in\mathbb{Z}.$$

For systems such as the air traffic network that exhibit daily temporal patterns, $K=24$ (assuming that each time-step is 1 hour long).

Using a rationale similar to the one used to derive (18), we can show that the dynamics of the augmented state vector $\vec{q}(t)$ is now given by:

$$\begin{aligned}
\vec{q}(t+1) &= \mathscr{B}_t\,\vec{q}(t) \quad (33)\\
\text{where } \mathscr{B}_t &= [J_{ji}]\Box\big((\Pi_t^{\mathrm{T}}\otimes I_n)\mathrm{diag}(\Gamma_i)\big) \quad (34)\\
\Longrightarrow \vec{q}(t) &= \mathscr{B}_{t-1}\mathscr{B}_{t-2}\cdots\mathscr{B}_0\,\vec{q}(0) \quad (35)\\
\Longrightarrow \|\vec{q}(t)\| &\le (r_\sigma(\mathscr{B}_{t-1}\mathscr{B}_{t-2}\cdots\mathscr{B}_0))\,\|\vec{q}(0)\|, \quad (36)
\end{aligned}$$

where, as before, $r_\sigma(\mathscr{B})$ denotes the spectral radius (largest eigenvalue) of $\mathscr{B}$. Suppose we denote $\mathscr{D} = \mathscr{B}_{K-1}\mathscr{B}_{K-2}\cdots\mathscr{B}_0$. Then, we have:

$$\begin{aligned}
\vec{q}(k+nK) &= \mathscr{B}_{k-1}\cdots\mathscr{B}_0\vec{q}(nK), \forall n \in \mathbb{Z}^+; k \in [0,K] \\
\implies \vec{q}(K) &= \mathscr{D}\vec{q}(0) \qquad (37)\\
\implies \vec{q}(nK) &= \mathscr{D}^n\vec{q}(0), \forall n \in \mathbb{Z}^+ \qquad (38)\\
\implies \|\vec{q}(nK)\| &\leq r_D^n\|\vec{q}(0)\|, \qquad (39)
\end{aligned}$$

where r_D is the spectral radius of $\mathscr{D}$.

1) Mean stability with periodic transition matrices: We now derive conditions under which a discrete-time MJLS with periodic Markovian transition matrices is exponentially mean stable (and therefore mean stable).

Theorem 3: (Exponential Mean Stability with periodic transition matrices) The discrete-time positive MJLS with Markovian transition matrices that are periodic with time period K, governed by (31)-(32), is exponentially mean stable if the largest eigenvalue of the matrix $\mathscr{D} = \mathscr{B}_K\mathscr{B}_{K-1}\cdots\mathscr{B}_0$ is less than 1.

Proof– Suppose $r_\sigma(\mathscr{D}) < 1$. Then (39) implies:

$$\|\vec{q}(nK)\| \leq r_D^n\|\vec{q}(0)\| = (r_D^{\frac{1}{K}})^{nK}\|\vec{q}(0)\| = (\tilde{r}_D)^{nK}\|\vec{q}(0)\|. \qquad (40)$$

In other words, we know that $\vec{q}(t)$ is exponentially bounded at values of t that are integer multiples of K. However, it is not necessary that $\vec{q}(t)$ be a monotonic function, as shown in Figure 2 for a 1-dimensional system. Suppose we denote the maximum amplification between $t = 0$ and $t = K$ by $a_{\max}$. That is,

$$\max_{t=0\cdots K}\|\vec{q}(t)\| = a_{\max}\|\vec{q}(0)\|. \qquad (41)$$

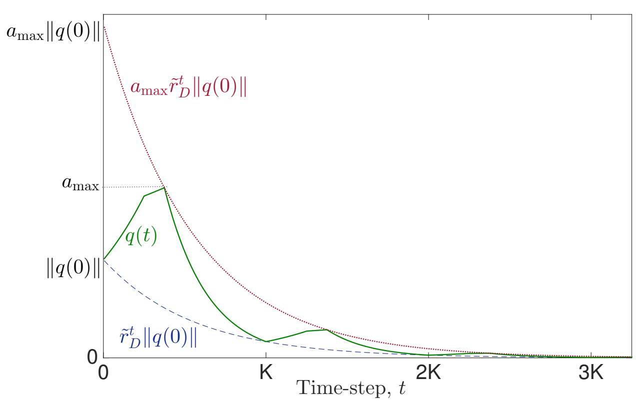

Fig. 2. Illustration of exponential bounds for scenario with periodic Markovian transition matrices.

Then, for $nK \leq t \leq (n+1)K$, we get:

$$\begin{aligned}
\vec{q}(t) &= \mathscr{B}_{t-nK-1}\mathscr{B}_{t-nK-2}\cdots\mathscr{B}_0\vec{q}(nK) \qquad (42)\\
&= \mathscr{B}_{k-1}\mathscr{B}_{k-2}\cdots\mathscr{B}_0\vec{q}(nK),\ k = t-NK \\
\implies \max_t\|\vec{q}(t)\| &= a_{\max}\|\vec{q}(nK)\| \text{ (from (41))} \qquad (43)\\
\implies \|\vec{q}(t)\| &\leq a_{\max}\tilde{r}^t\|\vec{q}(0)\|, \tilde{r} = r_D^{\frac{1}{K}} \qquad (44)\\
\implies \mathbb{E}[\|\vec{x}(t)\|] &\leq a_{\max}\tilde{r}^t\|\vec{x}(0)\|. \qquad (45)
\end{aligned}$$

Therefore, comparing to (13), if $r_\sigma(\mathscr{D}) < 1$, then (45) implies that the system is exponentially mean stable. ■

Theorem 4: (Exponential Mean Stability with periodic transition matrices) The discrete-time positive MJLS with Markovian transition matrices that are periodic with time period K, governed by (31)-(32), is exponentially mean stable if and only if the largest eigenvalue of the matrix $\mathscr{B}_{k+K}\mathscr{B}_{k+K-1}\cdots\mathscr{B}_k$ is less than 1, for *some* integer $k \in [0,K]$.

Proof– Considering the proof for Theorem 3, we note that the choice of start time $k = 0$ is arbitrary, and that we could have picked any start time, k. Without loss of generality, let us consider a start time $k \in [0, K-1]$. Then, a K time-step "cycle" would correspond to the time-steps $k, k+1, \cdots, k+K-1$. Suppose we denote $\tilde{\mathscr{D}} = \mathscr{B}_{k+K-1}\mathscr{B}_{k+K-2}\cdots\mathscr{B}_0\mathscr{B}_1\cdots\mathscr{B}_k$. In general, we note that $\tilde{\mathscr{D}}$ may note be equal to $\mathscr{D}$, since the $\mathscr{B}$ matrices may not commute. However, using similar arguments as before, we can show that if the spectral radius of $\tilde{\mathscr{D}}$ is less than 1 for some value of k, then the system is exponentially mean stable, and vice versa. ■

2) Almost-sure stability with periodic transition matrices: As shown in Theorem 2, exponential mean stability implies almost-sure stability for discrete-time, positive MJLS (with constant transition matrices). We can use an identical argument (replacing (25) with (44)) to show that the same property holds for the case of periodic transition matrices as well. Therefore, Theorem 3 provides a sufficient condition for almost-sure stability.

IV. ILLUSTRATIVE AIR TRAFFIC DELAY NETWORK EXAMPLES

Air traffic delay networks can be modeled as weighted, directed graphs, with airports as nodes, and links (flight service) between airports as edges. Time is discretized into 1-hour intervals. The model is constructed using Bureau of Transportation Statistics (BTS) data from the years 2011-2012 [1]. The network is restricted to links that serve at least 5 flights/day on average, which results in 158 airports (nodes) and 1,107 edges.

The discrete modes are identified from operational data by clustering the networks (adjacency matrices) corresponding to different time-steps [22]. The principal eigenvector of $\mathscr{A}$ (from (8)) is reflective of the network structure: It corresponds to the *hub* and *authority* scores of the network, which are an extension of the concept of eigencentrality to directed networks [34]. In prior work, we considered air traffic delay networks (i.e., the edge weights were the delay levels on that link), and clustered them using the delay-weighted hub and authority scores as features [22]. In this manner, six characteristic delay modes were identified: (1) San Francisco (SFO) high delays or *SFO*; (2) National Aispace System (NAS) wide high delays, also denoted *High NAS*; (3) System-wide low delays or *Low NAS*; (4) Atlanta high delays or *ATL*; (5) Chicago high delays or *ORD*; (6) System-wide moderate delays or *Med NAS*. We further categorize each of these delay modes by whether the system delays are increasing or decreasing, resulting in a total of 12 discrete modes.

Having identified the 12 discrete modes, the system (that is, the network) is classified as being in one of the 12 modes (that is, the closest in terms of the feature vector) in each time period over the two years (2011-2012). Considering each hour of the day, the Markovian transition matrices between discrete modes (Π_t) are also estimated. By considering the evolution of the continuous state for all time-steps when the system is in a given discrete mode, the matrix Γ_i for each mode i is determined. Finally, by analyzing transitions between each pair of discrete modes, the jump matrices J_{ij} are also estimated. In modeling the dynamics of the continuous states, we restrict our models to the network formed by the 30 largest airports in the US (also known as the "Core 30" airports [35]), resulting in a continuous state vector $\vec{x}(t) \in \mathbb{R}^{60\times 1}$.

A. Stability of air traffic delay networks

We now apply the analysis presented in Section III to the air traffic delay networks described above.

1) Stability of individual discrete modes: Fig. 3 shows the spectral radius (largest eigenvalue) of the Γ_i corresponding to each of the 12 discrete modes.

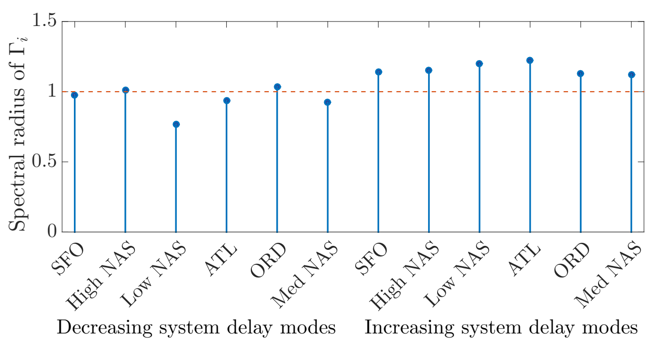

Fig. 3. Spectral radius of the system matrix, Γ_i, corresponding to each discrete mode.

An individual discrete-mode is stable if and only if the spectral radius of the corresponding Γ is less than 1. Examining Fig. 3, we see that the discrete modes with increasing system delays are all unstable. We also see that with the exception of the "Low NAS/ decreasing system delay" mode, the other decreasing delay modes have spectral radii that are quite close to 1 (i.e., the stability boundary), and two of them, the "High NAS/ decreasing system delay" and "ORD decreasing system delay" modes are marginally unstable.

2) Instantaneous temporal variation of MJLS stability: As mentioned before, stability (or instability) of individual modes does not imply the stability (or lack thereof) of the MJLS. From (33)-(34), we can evaluate the "instantaneous" temporal variation of system stability over the course of a day by considering the spectral radius of $\mathscr{B}_t$. This matrix describes the evolution of the augmented state at that time, and considers the dynamics in individual modes, as well as the effects of transitioning to different modes, and the probability of these transitions.

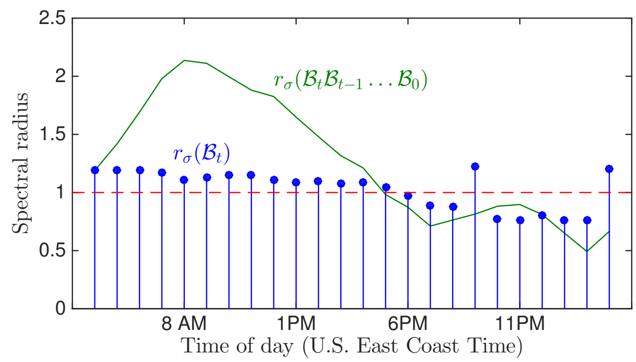

Fig. 4. Spectral radius of $\mathscr{B}_t$ and $\mathscr{B}_t\mathscr{B}_{t-1}\mathscr{B}_0$ as a function of the time-of-day, t. 4AM (i.e. $t = 0$) is considered to be our start of day.

Fig. 4 shows that the $\mathscr{B}$ matrices are generally unstable in the morning (when delays can increase, and also the probability of transitioning to an unstable mode is high), while they tend to become stable after ~7PM Eastern Time, when demand starts to decrease at many airports and the probability to transitioning to decreasing delay/stable modes is higher. There is a temporary exception at around 9PM Eastern Time, potentially driven by factors on the West Coast of the US, where it is only 6PM and traffic is still high. Finally, the $\mathscr{B}$ matrix becomes unstable again at the end of the operational day (3AM Eastern Time), because the system has a high probability of transitioning to an increasing delay mode at the start of the next morning.

3) Evolution of augmented state during a day: Next, we consider (35), which describes the evolution of the augmented state, $\vec{q}(t)$:

$$\vec{q}(t) = \mathscr{B}_{t-1}\mathscr{B}_{t-2}\cdots\mathscr{B}_0\,\vec{q}(0)$$

This equation relates the augmented state at any time to the augmented state at $t = 0$, which we consider to be the start of the day. Then, as shown in (36), the "amplification" of the augmented state, that is, the ratio of the value of its norm at time t to its value at time zero, is upper-bounded by the spectral radius of the matrix product $\mathscr{B}_{t-1}\mathscr{B}_{t-2}\cdots\mathscr{B}_0$. This quantity is reflective of the stability of the system between the start of the day and time-step t, and is plotted in Fig. 4.

4) Stability of air traffic delays with switching network topologies and periodic transition matrices: From Theorem 3, we know that the system is exponentially mean stable if and only if $\mathscr{D} = \mathscr{B}_K\mathscr{B}_{K-1}\cdots\mathscr{B}_0 < 1$. We can see from Fig. 4 that at the end of the day, the spectral radius of the product of the $\mathscr{B}_t$ matrices is indeed less than 1. Therefore, the system (with periodic time-varying transition matrices) is exponentially mean stable, and by Theorem 2, almost-surely stable. It is worth mentioning that one could have alternatively considered a single Markovian transition matrix over the entire day (that is, by ignoring the temporal variations, and just considering the probability of transitioning between modes at any time). Such an approach yields a $\mathscr{B}$ matrix that has a spectral radius of 1.061, which suggests that the system is marginally not exponentially mean stable.

However, this approach ignores the temporal variation of the transition matrices that may help stabilize the system: For example, transitions into the low NAS (and decreasing delay) mode is much more likely in the evening than in the mornings, which has a stabilizing effect.

B. Amplification of continuous state

So far, we have considered the evolution of the augmented state $\vec{q}$, and used it to guarantee the convergence of the continuous state vector, $\vec{x}$. We now consider the expected amplification of $\vec{x}(t)$, by using (17):

$$\mathbb{E}[\vec{x}(t)] = \sum_{j=1}^{M} \vec{q}_j(t).$$

First, we construct a variant of Fig. 4, which shows the amplification of $\mathbb{E}[\vec{x}(t)]$ at each time.

First we consider a specific version of Fig. 4 (which shows the amplification of the augmented state at each time) with the initial condition $m(0)$ as the *Low NAS Increasing* mode and $\|\vec{q}(0)\| = 1$. We use (17) to determine the corresponding values of the continuous state, and find that it peaks around 7PM Eastern Time (it is interesting to note that the peak in $\mathbb{E}[\|\vec{x}(t)\|]$ occurs at a different time from the peak in $\|\vec{q}(t)\|$, which occurs at 9AM). We calculate the corresponding principal eigenvector and use that as the initial condition to determine the expected value of the continuous state, $\mathbb{E}[\vec{x}(t)]$, at each hour of the day. The norm of this vector is plotted in Fig. 5. It shows that the maximum amplification factor is 5.9 (achieved at 7PM Eastern Time), relative to the norm of the continuous state at the start of the day.

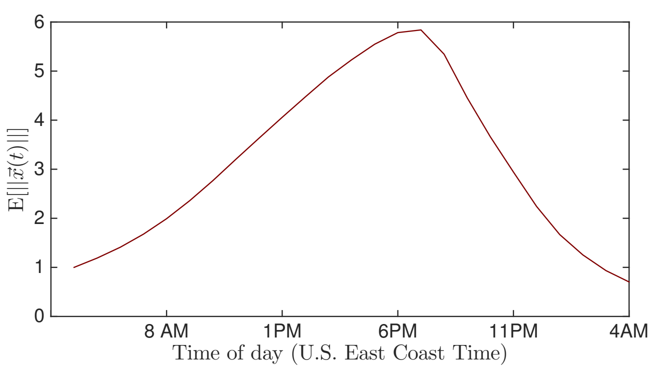

Fig. 5. Expected value of the norm of the continuous state vector, $\mathbb{E}[\|\vec{x}(t)\|]$ vs. time of day.

C. Persistence of delays and resistance to network effects

From (2) and (8), we see that the off-diagonal elements of Γ reflect the dependence of delays at a node on delays at other nodes (i.e., the network effects), while the diagonal elements reflect the dependence of delays at a node on the delays at that same node in the previous time-step. In particular, considering (3)-(4), we see that the larger the value of α_i^{out} (or α_i^{in}), the more the persistence of any outbound (or inbound, respectively) delays there, and the greater its resistance to network effects. In other words, one would need higher delays at other airports before airport i was also impacted by delays. Similarly, smaller values of α_i^{out} and α_i^{in} mean that the airport is relatively more sensitive to network effects; however, any delays at i will decay faster.

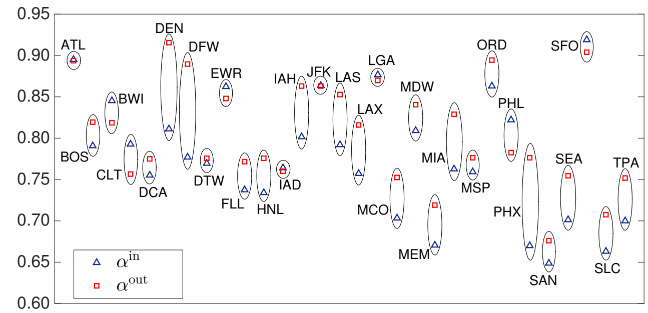

Fig. 6. Average values of α^{out} and α^{in} for different airports. The values are averaged over the different discrete modes.

Fig. 6 shows the α^{out} and α^{in} values for different airports, averaged over the 12 discrete modes. We see that many of the major airports (especially the airline hubs – Atlanta (ATL), San Francisco (SFO), Chicago O'Hare (ORD)) have among the largest values of these parameters, suggesting a persistence of delays. We also see that Denver (DEN) and Dallas/ Fort Worth (DFW) have a high value of α^{out}, but a smaller value of α^{in}. This suggests an asymmetry in the persistence of outbound and inbound delays, with the former being more persistent (and less susceptible to network effects). We also notice that the three New York area airports (EWR, JFK and LGA) have reasonably high (and similar) values of these parameters. We see that some of the smaller airports (e.g., Memphis (MEM), San Diego (SAN) and Salt Lake City (SLC)) have lower values of these parameters, especially with regards to inbound delays. The reason for this phenomenon could be that there is typically sufficient capacity at these airports to satisfy the levels of demand seen at these airports; as a result, there is little congestion. We believe that this analysis is a first step towards evaluating the resilience of different airports to disruptions.

V. CONCLUSIONS

This paper presented and analyzed the stability of a discrete-time, positive MJLS model of air traffic delay networks. It showed that the proposed model could account for time-varying networks using random switches between topologies. Periodic Markovian transition matrices and continuous state resets were also incorporated. We derived conditions under which such a system is exponentially mean stable and almost-surely stable. We demonstrated these conditions on models derived from operational air traffic delay data, and showed that air traffic delay networks are both exponentially mean stable and almost-surely stable. Finally, we analyzed the models to evaluate the persistence of delays at different airports, and their susceptibility to delays from other airports due to network connectivity.

These results present an important first step toward understanding the behavior of networked systems with switching topologies. They also present opportunities for designing novel controllers that improve the performance of such systems, both through feedback of the continuous state, and by leveraging the ability to switch networks.

REFERENCES

[1] Bureau of Transportation Statistics, "Airline On-Time Statistics and Delay Causes," 2015, http://www.transtats.bts.gov/.

[2] M. Salathe, M. Kazandjieva, J. W. Lee, P. Levis, M. W. Feldman, and J. H. Jones, "A high resolution human contact network for infectious disease transmission," *Proceedings of the National Academy of Sciences*, vol. 107, no. 51, pp. 22 020–22 025, 2010.

[3] K. Dietz, "Epidemics and rumours: A survey," *Journal of the Royal Statistical Society. Series A (General)*, vol. 130, no. 4, pp. 505–528, 1967.

[4] R. Albert, I. Albert, and G. Nakarado, "Structural vulnerability of the North American power grid," *Physical Review E*, vol. 69, p. 025103, 2004.

[5] P. Crucitti, V. Latora, and M. Marchiori, "A topological analysis of the Italian electric power grid," *Physica A: Statistical Mechanics and its Applications*, vol. 338, pp. 92–97, 2004.

[6] M. Crovella and B. Krishnamurthy, *Internet Measurement: Infrastructure, Traffic and Applications*, 2006.

[7] V. Kalapala, V. Sanwalani, A. Clauset, and C. Moore, "Scale invariance in road networks," *Physical Review E*, vol. 73, p. 026130, 2006.

[8] C. von Ferber, Y. H. T. Holovatch, and V. Palchykov, "Public transport networks: empirical analysis and modeling," *The European Physical Journal B*, vol. 68, pp. 261–275, 2009.

[9] J. Manitz, "Statistical Inference for Propagation Processes on Complex Networks," Ph.D. dissertation, Georg-August-Universität Göttingen, 2014.

[10] M. Zanin and F. Lillo, "Modelling the air transport with complex networks: A short review," *The European Physical Journal Special Topics*, vol. 215, pp. 5–21, 2013.

[11] C. Nowzari, V. Preciado, and G. Pappas, "Analysis and Control of Epidemics: A Survey of Spreading Processes on Complex Networks," *IEEE Control Systems Magazine*, February 2016.

[12] C. Ho, M. J. Kochenderfer, V. Mehta, and R. S. Caceres, "Control of epidemics on graphs," in *Proceedings of the Conference on Decision and Control*, 2015.

[13] M. Ogura and V. M. Preciado, "Disease spread over randomly switched large-scale networks," in *Proceedings of the American Control Conference*, 2015.

[14] D. Acemoglu, V. M. Carvalho, A. Ozdaglar, and A. Tahbaz-Salehi, "The network origins of aggregate fluctuations," *Econometrica*, vol. 80, no. 5, pp. 1977–2016, 2012.

[15] I. Herman, D. Martinec, Z. Hurak, and M. Sebek, "Nonzero Bound on Fiedler Eigenvalue Causes Exponential Growth of H-Infinity Norm of Vehicular Platoon," *IEEE Transactions on Automatic Control*, vol. 60, no. 8, pp. 2248–2253, 2015.

[16] R. Olfati-Saber, J. Fax, and R. Murray, "Consensus and Cooperation in Networked Multi-Agent Systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[17] T. Sarkar, M. Roozbehani, and M. Dahleh, "Robustness scaling in large networks," in *American Control Conference*, 2016.

[18] P. Fleurquin, J. Ramasco, and V. Eguiluz, "Systemic delay propagation in the US airport network," *Scientific Reports*, p. 1159, 2013.

[19] ——, "Data-driven modeling of systemic delay propagation under severe meteorological conditions," in *11th USA/Europe Air Traffic Management Research and Development Seminar (ATM2011)*, Lisbon, Portugal, June 2015.

[20] M. E. J. Newman, "Fast algorithm for detecting community structure in networks," *Physical Review E*, no. 6, 2004.

[21] M. Girvan and M. E. J. Newman, "Community structure in social and biological networks," *Proceedings of the National Academy of Sciences*, pp. 7821–7826, 2002.

[22] K. Gopalakrishnan, H. Balakrishnan, and R. Jordan, "Clusters and Communities in Air Traffic Delay Networks," in *American Control Conference*, July 2016.

[23] G. Spiers, P. Wei, and D. Sun, "Algebraic connectivity maximization of the air transportation network," in *American Control Conference*, 2012.

[24] P. Wei, G. Spiers, and D. Sun, "Algebraic connectivity maximization for air transportation networks," *IEEE Transactions on Intelligent Transportation Systems*, no. 2, pp. 685–698, 2014.

[25] O. Lordan, J. M. Sallan, P. Simo, and D. Gonzalez-Prieto, "Robustness of the air transport network," *Transportation Research Part E*, pp. 155–163, 2014.

[26] C. Nicolaides, L. Cueto-Felgueroso, M. C. Gonzalez, and R. Juanes, "A Metric of Influential Spreading during Contagion Dynamics through the Air Transportation Network," *PLoS One*, vol. 7, no. 7, 2012.

[27] K. Gopalakrishnan, H. Balakrishnan, and R. Jordan, "Deconstructing Delay Dynamics: An air traffic network example," in *International Conference on Research in Air Transportation (ICRAT)*, 2016.

[28] H. Lin and P. J. Antsaklis, "Stability and Stabilizability of Switched Linear Systems: A Survey of Recent Results," *IEEE Transactions on Automatic Control*, vol. 54, no. 2, pp. 308–322, 2009.

[29] O. Costa and M. Fragoso, "Stability Results for Discrete-Time Linear Systems with Markovian Jumping Parameters," *Journal of Mathematical Analysis and Applications*, vol. 179, pp. 154–178, 1993.

[30] O. Costa, M. Fragoso, and R. Marques, *Discrete-Time Markov Jump Linear Systems*, 1st ed. USA: Springer-Verlag, 2005.

[31] P. Bolzern, P. Colaneri, and G. D. Nicolao, "Stochastic stability of positive markov jump linear systems," *Automatica*, vol. 50, no. 4, pp. 1181–1187, 2014. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0005109814000545

[32] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Control Systems Magazine*, vol. 19, pp. 59–70, 1999.

[33] M. Günther and L. Klotz, "Schur's theorem for a block hadamard product," *Linear Algebra and its Applications*, vol. 437, no. 3, pp. 948–956, 2012. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0024379512002650

[34] M. Benzi, E. Estrada, and C. Klymko, "Ranking hubs and authorities using matrix functions," *Linear Algebra and its Applications*, no. 5, pp. 2447–2474, 2013.

[35] Federal Aviation Administration, "Core 30," `http://aspmhelp.faa.gov/index.php/Core_30`, 2011.