M275 Notes

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Lecture notes from M275 at SJSU, a graduate-level course at SJSU in Algebraic Topology taught by Professor Richard Kulbelka. These notes were taken using Vim and/or GVim equipped with LaTeX-Suite, which sped up typesetting significantly; `latexmk` with the flag `-pvc` was useful for compiling these notes in real-time.

1 August 22, 2012

1.1 The Question

Question. Given two topological spaces, can we determine if they are homeomorphic?

In short, this is very hard.

Exercise. Show that $\mathbb{R} \not\approx [0, 1]$ with the usual topology.

Solution. Let $I = [0, 1]$. If we remove 0 from I, then I is still connected. On the other hand, no point in $\mathbb{R}$ has this property, so we lose, since homeomorphism preserves connectedness. $\square$

Definition. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle.

1.2 Category Theory

Definition. A category $\mathcal{C}$ will consist of a "class" of objects $ob(\mathcal{C})$, as well as a "class" of morphisms $mor(\mathcal{C})$, such that for every pair of objects there is associated a (possibly empty) set of morphisms $mor(X, Y)$.

Example. The category $\mathcal{G}$ of groups, with $ob(\mathcal{G})$ being "the set of all groups", except this leads to a contradiction, so we call it a class instead. In fact, if the objects of a category are a set, it is called a "small category" or "kitty-gory".

Example. $\mathcal{A}$ is the category of abelian groups, category $\mathbb{V}$ of vector spaces, category $\mathbb{R}$ of rings.

Definition. Give $X, Y \in ob(\mathcal{C})$, then $mor(X, Y)$ is the set of morphisms "from X to Y".

Given two groups $G_1, G_2 \in ob(\mathcal{G})$, then $mor(G_1, G_2)$ is the set of all homomorphisms from G_1 to G_2. For vector spaces $V_1, V_2 \in ob(\mathcal{V})$, then $mor(V_1, V_2)$ is the set of all linear maps from V_1 to V_2.

Finally, Top is the category of topological spaces. $ob(Top)$ is the set of topological spaces, and $Mor(Top)$ is the set of morphisms $Mor(X, Y)$, which are continuous functions from X to Y.

Note that morphism sets $Mor(X, Y)$ can be small or even empty. For example, there are very few homomorphisms from $\mathbb{Z}/3\mathbb{Z}$ to $\mathbb{Z}$.

Suppose we have a morphism $\varphi : X \to Y$ and a morphism $\psi : Y \to Z$. As one might expect, we define composition to yield another morphism. Also $mor(X, X)$ must contain the identity.

We define morphisms $\varphi : X \to Y$ and $\zeta : Y \to X$ such that $\varphi \circ \zeta = \zeta \circ \varphi = id$, then we say they are inverses, and both of them are called "isomorphs."

1.3 Outsourcing

In algebraic topology, we say topology is too hard and start with the category Top and the category of groups $\mathcal{G}$.

Definition. A *functor* F between categories sends an object $X \in ob(Top)$ to a group $F(X) \in ob(\mathcal{G})$.

Given $\varphi \in Mor(X, Y)$, we want F to take φ to $F(\varphi) \in Mor(F(X), F(Y))$. So a functor is a function on two levels: it sends topological spaces are groups, and morphisms between spaces to homomorphisms between these groups.

We will be studying the fundamental group functor π_1. It will associate a space X to a group $\pi_1(X)$, and we want to send $f : X \to Y$ to $f_* : \pi_1(X) \to \pi_1(Y)$. We also want it to preserve compositions: if $f : X \to Y$ and $g : Y \to Z$, then we want $\pi_1(g \circ f) = (g \circ f)_*$.

Given three functions $f : X \to Y$, $g : Y \to Z$ and $h : X \to Z$. The diagram is said to be commutative when $h \equiv g \circ f$. We want the functor to preserve this (aren't we just restating?): we need

$$(g \circ f)_* = g_* \circ f_*$$

Fact. If f is an isomorph, then f_* must be an isomorphism.

This is useful if, say, $\pi_1(X) \ncong \pi_1(Y)$, this implies $X \not\approx Y$. A nice success is that manifolds are completely characterized by the number of "holes" (in a loose sense).

2 August 27, 2012

2.1 Announcements and Such

Category theory handout has been updated.

Homework may or may not be due on Wednesday.

2.2 Setting Out

Given a space X, we can either begin algebraic topology by first

(i) Define homology groups $H_1(X), H_2(X), \cdots$.

(ii) Define homotopy groups $\pi_1(x), \pi_2(X), \cdots$.

The issue is as follows: homology groups are harder to define, but are relatively easy to compute. Homotopy groups are easy to define and hard to compute.

Definition. S^n denotes an n-dimensional sphere; that is, $S^n \subseteq \mathbb{R}^{n+1}$ can be defined as

$$S^n = \{(x_0, \cdots, x_n) : \sum x_i^2 = 1\}$$

Example. $S^1 \subset \mathbb{R}^2$ is a unit circle. $S^2 \subset \mathbb{R}^3$ is the surface of a sphere.

One can also think of S^n as the equator of S^{n+1}.

Anyways, understanding $\pi_i(S^n)$ is analogous to understanding the symmetric groups S_n; by Cayley's Theorem, this will suffice to understand all of finite group theory.

Definition. $\pi_1(X)$ is called the fundamental group of X.

Theorem 1 (Brouwer Fixed Point Theorem). *If $f : D^2 \to D^2$ is continuous, then $\exists x_0 \in D^2$ which is fixed by f.*

Corresponding geometric idea: crumple a piece of paper, and lay it on a copy of itself. Then there are two corresponding points which are directly above each other. (We can use paper (rectangles) instead of discs, since $D^2 \approx I \times I$.)

Other spaces which satisfy Brouwer Fixed Point Theorem are said to find the *fixed point property.* This is strong; for example S_1 does not satisfy this property (just rotate it $90°$; now no points are fixed. Remember that S_1 is a circle, not a disk.)

We will prove later that $\exists \varphi : D^2 \to S^1$ such that $\varphi \mid_{S^1} = \mathrm{id}_{S^1}$ (that is, no continuous function from $D^2 \to S^1$ which fixes the circumference.)

2.3 Homotopy Groups

A spider lives in a topological space X. He lives at a base point $x_0 \in X$. Any web from x_0 to x_0 can be reeled back in; that is, any path is nulhomotopic.

On the other hand, a punctured plane (a plane minus a point) does not have the property, since if the web encloses the hole, then the web gets stuck.

Consider a bagel/torus. Once again, it does not have this property.

OK, formal nonsense. The terms map, function, etc. are used interchangeably, and $I = [0, 1]$.

Definition. A *path* in Y is a continuous map $\varphi : I \to Y$, where I has the standard topology it inherits as a subspace of $\mathbb{R}$. The initial point is $y_0 = \varphi(0)$, and the ending point is $y_1 = \varphi(1)$.

Pictorially, we can draw the image of the path $\varphi(I)$. This does not contain all the information (e.g. speed).

Since I is compact, $\varphi(I)$ cannot go off to infinity or oscillate around a point, or do any other silly things.

Definition. A *path-connected* space is one where any two points are connected by a path.

Definition. A *loop* in X at x_0 is path $\gamma : I \to X$ such that $x_0 = \gamma(0) = \gamma(1)$. It is *simple* if it is not self-intersecting (other than at the endpoint).

Definition. Given two continuous functions $f, f' : X \to Y$, we say that f is *homotopic* to f' if there exists a continuous function $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$.

Definition. If A is a subspace of X and $f = f'$ on A, (i.e. $f(a) = f'(a) \forall a \in A$), then we say that f is homotopic to f' relative to A if $\exists f : X \times I \to Y$ such that $F(x, 0) = f(x)$, $F(x, 1) = f'(x)$ AND $F(a, t) = f(a)$.

A special case we will use a lot is if we have two paths in Y that both start and end at the same point.

Definition. Paths $f, f' : I \to Y$ in Y satisfy $y_0 = f(0) = f'(0)$ and $y_1 = f(1) = f'(1)$. Then f and f' are *path homotopic* if they are homotopic relative to $\{0, 1\}$.

In other words, $f, f' : I \to Y$ are path homotopic if $\exists F : I^2 \to I$ continuous with

$$\begin{aligned}
F(0, t) &= f(0) = f'(0) \\
F(1, t) &= f(1) = f'(1) \\
F(s, 0) &= f(s) \\
F(0, s) &= f(s)
\end{aligned}$$

These three definitions give equivalence relations.

Consider the homotopy classes of loops beginning at a specified point. We define the product $*$ to be the "composition" of the paths, in the sense that we glue one path to the end of the other. Under this operation, the classes form a group at that point.

Theorem 2. *All such groups (for a fixed space X) are isomorphic.*

It is not hard to check that these operations are all equivalence relations. We write them as $f \simeq f'$, $f \simeq_A f'$ and $f \simeq_p f'$. This is left as an exercise, since it's actually not hard.

Since apparently this is confusing people:

Exercise. Let $f : X \to Y$ be continuous, and define $F : X \times I \to Y$ by $F(x, t) = f(x)$. Then F is continuous.

Proof. Let V be an open subset in Y. $F^{-1}(Y) = f^{-1}(X) \times I$, however; this is the product of open sets, and hence itself open. □

3 August 29, 2012

3.1 Questions

Definition. Given spaces X and Y and a continuous map $f : X \to Y$, we say f is *nulhomotopic* if f is homotopic to a constant map; i.e. if there exists $y_0 \in Y$ such that when we define the constant map, $c_{y_0} : X \to Y$ by $x \to y_0$, then $f \simeq c_{y_0}$.

In other words, f is nulhomotopic if $\exists y_0 \in Y$ and $F : X \times I \to Y$ continuous for which $F(x, 0) = f(x)$, $F(x, 1) = y_0$.

3.2 Straight-Line Homotopy

In $\mathbb{R}^n$, one useful way to construct a homotopy is to consider $\vec{R}(t) = (1-t)\vec{x} + t\vec{y}$.

Example. Given 2 paths in the plane $\gamma_1, \gamma_2 : [0, 1] \to \mathbb{R}^2$ by

$$\gamma_1(s) = e^{i\pi s} \quad \text{and} \quad \gamma_2(s) = e^{-i\pi s}$$

Show that γ_1 and γ_2 are path-homotopic.

Solution. If we want to show $\gamma_1 \simeq_p \gamma_2$, we need $F : I \times I \to \mathbb{R}^2$ with $F(s, 0) = \gamma_1(s)$, $F(s, 1) = \gamma_2(s)$. So we can construct the convenient homotopy

$$F(s, t) = (1-t)\gamma_1(s) + t\gamma_2(s)$$

$\square$

Musing: for any two paths p_1 and p_2 in X, can we find a larger space Y for which they are path-homotopic? Probably yes.

This, of course, generalizes. For example, given a loop at the North Pole of S^2 not passing through the South Pole, we can show that it is homotopic to the constant function of the North Pole. Consider $\gamma : I \to S^2$. Of course we can't simply use the map $H(s, t) = (1-t)\gamma(s) + t \cdot \langle 0, 0, 1\rangle$, since this will in general not be on the sphere, but we can recover this by dividing by the magnitude, forcing it to lie on the sphere; hence, the desired homotopy is simply $F(s, t) = H(s, t) \cdot \frac{1}{|H(s,t)|}$.

3.3 Product of Paths

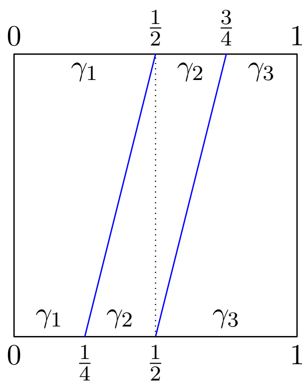

Figure 1: Not Associative, but homotopic

Fix a space X. Given a path γ_1 from x_1 to x_2, and a path γ_2 from x_2 to x_3, we can get a path $\gamma_1 * \gamma_2$ from x_1 to x_3. Specifically, the function is defined as follows.

Definition. Given $\gamma_1, \gamma_2 : I \to X$, define $\gamma_1 * \gamma_2 : I \to X$ by

$$(\gamma_1 * \gamma_2)(s) = \begin{cases} \gamma_1(2s) & s \in \left[0, \frac{1}{2}\right] \\ \gamma_2(2s-1) & s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

It is not hard to check that this is well-defined. It is easy to verify that $\gamma_1(2s)$ and $\gamma_2(2s-1)$ are continuous, and they agree at the common point $s = \frac{1}{2}$, so this implies that the product is continuous by the pasting lemma.

The operation $*$ is not associative; an easy way to see that $(\gamma_1 * \gamma_2) * \gamma_3 \neq \gamma_1 * (\gamma_2 * \gamma_3)$ by evaluating at $s = 1/4$. However, it is associative with respect to homotopy classes. The easy special case is to show that the above quantities are homotopic... by distorting time linearly, we can establish the homotopy by a rather irritating computation. Explicitly, the homotopy $F : I^2 \to X$ is

$$F(s,t) = \begin{cases} \gamma_1\left(\frac{4s}{t+1}\right) & 0 \le s \le \frac{t+1}{4} \\ \gamma_2\left(4s - t - 1\right) & \frac{t+1}{4} \le s \le \frac{t+2}{4} \\ \gamma_3\left(\frac{4s-t-2}{2-t}\right) & \frac{t+2}{4} \le s \le 1 \end{cases}$$

and this is continuous by the Pasting Lemma and composition. (Since $t \in I$, no denominators are zero.)

4 September 5, 2012

4.1 Homeomorphism

Definition. X is homeomorphic to Y, denoted $X \approx Y$, if $\exists f : X \to Y$ a homeomorphism.

Definition. A homeomorphism is a continuous functions $f : X \to Y$ such that $\exists g : Y \to X$ is a continuous inverse; that is, $g \circ f = f \circ g = \mathrm{id}$.

Definition. We say X is of the same homotopy type as Y, written $X \simeq Y$, if $\exists f, g$ with $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

This is an equivalence relation. Clearly, this is weaker than homeomorphism.

Example. $\mathbb{R}^2 \setminus \{\vec{0}\} \simeq S_1$

Proof. Identifying $\mathbb{R}^2$ with the corresponding complex number in $\mathbb{C}$, define $f : \mathbb{C} \to S_1$ by $f(z) = \frac{z}{|z|}$, and let $g : S_1 \to \mathbb{C}$ be an "identity" (e.g. inclusion). Evidently, $f \circ g : S_1 \to S_1$ is actually equal to the identity in S_1. On the other hand, $g \circ f : \mathbb{C} \to \mathbb{C}$ is basically equal to f. □

4.2 The Fundamental Group

Given X and $x_0 \in X$, define the fundamental group of X based at x_0, denoted $\pi_1(X, x_0)$, as the set of path homotopy classes of loops based at x_0. (A loop is a path with the same start/end.) The operation of the group is essentially $*$, the concatenation of two paths. Formally, if $f, g : I \to X$ are paths, then $[f] \cdot [g] = [f * g]$.

Question. Is this operation well-defined?

Answer. Yes. This isn't hard to verify, just follow through.

Question. Is this a group operation?

Answer. Yes. Again, not hard to verify. The constant loop is the identity, and associativity is evident: more generally, if f, g, h are paths, we showed last time that $f * (g * h) \simeq_p (f * g) * h$. The inverse of $[f]$ is $[\bar{f}]$, where $\bar{f}(t) = f(1 - t)$ for each t.

These groups are, in general, *hard* to compute. Unlike homology classes, which are rather easy to compute.

Question. Can we calculate some simple homotopy groups?

Question. Does this group depend on x_0?

Anyways, here is an actual example.

Example. $\pi_1(\mathbb{R}, 0)$ is the trivial group.

Proof. Well, everything is nulhomotopic, so... heh.

In general, any contractible space has trivial fundamental groups. The converse is not true. □

Example. $\pi_1(S^1, 1) \cong \mathbb{Z}$.

Note that these loops are restricted to path components. If X is not path-connected, then the path components are completely unrelated in terms of their fundamental groups. So, it really only makes sense to discuss fundamental groups of path-connected spaces.

Now the triumph:

Theorem 3. *If X is path-connected, then $\pi_1(X, x_1)$ is isomorphic to $\pi_(X, x_2)$ for any points $x_1, x_2 \in X$.*

Proof. In the book. □

Define an equivalence relation on I as follows. Then x y iff $x = y$ or $\{x, y\} = \{0, 1\}$. In other words, this just glues the endpoints together. Now look at the quotient topology $I/ \approx S_1$. We can now rephrase loops, which are functions $f : I \to X$ with $f(0) = f(1)$, to functions $f : S^1 \to X$. So we can view homotopy groups, then, as the homotopy classes of maps from S^1 into X, denoted $[S^1, X]$.

Why π_1? It turns out that π_n represents the homotopy classes of maps from S^n into X.

Comment: π_1 is a group, but not abelian. On the other hand, π_n is abelian whenever $n \geq 2$

5 September 10, 2012

5.1 Homomorphisms Induced by Functions

So far we've seen $\pi_1(X, x_0)$, the homotopy classes of loops in X based at x_0. If $f : (X, x_0) \to (Y, y_0)$,[1], this yields a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $[\gamma] \mapsto [f \circ \gamma]$. Indeed, one can check that $f_*([\gamma][\zeta]) = f_*([\gamma])f_*([\zeta])$.

If $f : X \to Y$ and $g : Y \to Z$, then $(g \circ f)_* : \pi_1(X) \to \pi_1(Z)$ is $g_* \circ f_*$. This is powerful. That is, given a commutative diagram of spaces, we get for free a commutative diagram of groups. On another note, this is also completely trivial to prove; it follows immediately by using definitions.

This is one of the **functorial properties** of $*$. The other functorial property is that if $\mathrm{id}_X : X \to X$ is the identity, then $(\mathrm{id}_X)_*$ is the identity at the group level.

5.2 Some More Properties of Fundamental Groups

Claim. If X and Y are homeomorphic, then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.

Proof. Since X and Y are homeomorphic, $\exists f : X \to Y, g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Now, we have homomorphisms f_* and g_* which are inverses between the fundamental groups. This implies the conclusion. □

Fact. If X is contractible, then $\pi_1(X, x_0) = 0$ is the trivial group.[2]

Proof. All loops at x_0 can be compressed, lol. More formally, since X is contractible, we know that $\mathrm{id}_X \simeq c_{x_0}$. Now consider a loop γ. Then,

$$\mathrm{id}_X \circ \gamma \simeq c_{x_0} \circ \gamma$$

But the left-hand side is γ and the right-hand side is c_{x_0}, implying the loop is itself nulhomotopic and yielding the conclusion. □

5.3 Homework Problem

Problem. Let A be a subspace of $\mathbb{R}^n$. Let $h : (A, a_0) \to (Y, y_0)$. Show that if h is extendable to a continuous map of $\mathbb{R}^n$ into Y, then h_* is the trivial homomorphism.

Proof. We are given that $\exists g : \mathbb{R}^n \to Y$ with $g|_A = H$. Let j be the inclusion. Then $g \circ j = h$.

Now we use functorial properties. We have $h_* = g_* \circ j_*$. But this is a map

$$\pi_1(A, a_0) \xrightarrow{j_*} \pi_1(\mathbb{R}^n, a_0) \xrightarrow{g_*} \pi_1(Y, y_0).$$

But $\pi_1(\mathbb{R}^n, a_0) = 0$. So $j_*(A) = e$, the identity, is trivial. So $g_* \circ j_*$ is trivial, implying the conclusion. □

What does this imply? For example, for a non-trivial function $\pi_1(S^1, 1) \xrightarrow{h_*} \pi_1(S^1, 1)$, we can't extend it to $\mathbb{R}^2$.

[1] That is, $f : X \to Y$ and $f(x_0) = y_0$. This generalizes to subsets in place of x_0 and y_0

[2] This is a terrible abuse of notation. But "everyone does it", so whatever.

5.4 Hat Homomorphisms

We saw last time that if α is a path from x_0 to x_1, then this gives a homomorphism $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by $[\gamma] \mapsto [\bar{\alpha} * \gamma * \alpha]$. It turns out this is an isomorphism! Hence, in a path-connected space, the fundamental groups are all isomorphic. (When not, they are completely unrelated.)

However, we still pay some attention to the base point, because the isomorphisms are not necessarily natural.

Now notice that we can do the same thing if α is a loop! This gives an automorphism from $\pi_1(X, x_0)$ to itself. But $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_0)$, and now $\alpha \in \pi_1(X, x_0)$. Now the map is

$$\hat{\alpha} : [\gamma] \mapsto [\bar{\alpha} * \gamma * \alpha] = [\alpha]^{-1}[\gamma][\alpha].$$

Suddenly we find that for any loop, $\hat{\alpha}$ is an inner automorphism!

5.5 Even Covering

So far, the fundamental groups are all trivial. Let's find some more interesting ones.

Definition. Given a function $P : E \to B$, select a base point b_0 and U an open neighborhood of b_0. We call U *evenly covered* (by P) if $P^{-1}(U)$ is a disjoint union of open sets (possibly infinite) such that P restricted to any of these sets is a homeomorphism. P is then called a *covering projection* if this is possible for any b_0. Informally, we will say "E is a covering projection of B" if every point has an evenly covered neighborhood.

Note that P must be surjective.

Example. Let $[n]$ denote $\{1, 2, \cdots, n\}$ with the discrete topology. Take $P : B \times [n] \to B$ by $(b, k) \to b$. Then this is a simple example of a covering projection.

This is not interesting because $B \times [n]$ is path-connected.

Example. Take $P : \mathbb{R} \to S^1$ by $\theta \mapsto e^{2\pi i\theta}$. This is essentially wrapping the real line into a single helix and projecting it down.

This is a covering projection. For example, if U is the arc joining cis $\frac{\pi}{2012}$ to cis $-\frac{\pi}{2012}$, then the pre-image are intervals of the form $\left(-\frac{1}{4024}, \frac{1}{4024}\right) + \mathbb{Z}$. For each interval we have a homeomorphism, but this is not true for the entire function.

Note that if we take the neighborhood too big, then we may destroy our homeomorphism structure.

The idea is that locally, the pre-image of U looks like a product of U and what is called a *fiber*; in the example above, the fiber looks like $\mathbb{Z}$.

6 September 12, 2012

6.1 Homework Stuff

Hey cool no points lost.

Comment: remember that if you want to show S has one element, it is not sufficient to show that all elements of S are the same. You do need to additionally check that S is not empty.

6.2 Covering Spaces and Projections

Consider a function $p : E \to B$. Suppose that $\forall b \in B$, $\exists U$ with $b \in U$ evenly covered by p. Then p is called a covering projection. The key example is the map $p : \mathbb{R} \to S^1$ given by $\theta \mapsto e^{2\pi i\theta}$.

Example. Take $p : S^1 \to S^1$ by $z \mapsto z^3$. This is also a covering projection (check this). Indeed, the inverse image of a little open interval around 1 is three little intervals centered at $e^{\frac{2\pi i}{3}}$, $e^{\frac{4\pi i}{3}}$, and 1, each three times shorter. We can replace the 3 with any integer.

Definition. The *fiber* of a point b_0 is the inverse image $p^{-1}(\{b_0\})$.

Sidenote: the covering of a space Y by $Y \times A$, where A has the discrete topology, is called trivial. (See last class's example.) It turns out that coverings of contractible spaces are essentially trivial.

Fact. If $E_1 \xrightarrow{p_1} B_1$ and $E_2 \xrightarrow{p_1} B_2$ are covering projections, then so is $E_1 \times E_2 \xrightarrow{p_1 \times p_2} B_1 \times B_2$.

In figure 53.4, we can view $S^1 \times S^1$ as a torus. Take the covering projection $p \times p$, where $p : \theta \mapsto e^{2\pi i\theta}$. The fiber of $(1, 1) \in S^1 \times S^1$ is precisely $\mathbb{Z}^2$, and the inverse of a small patch on the torus is small patches at each lattice point.

If B is connected, then the cardinality of the fiber at any point in B is constant over all the points in B.

The map $(z, w) \mapsto (z^m, w^n)$ taking $S^1 \times S^1$ to $S^1 \times S^1$ produces a wrap for a bagel!

Covering projections must be onto since neighborhoods around a point with an empty fiber are not going to be nonempty by continuity and such. ⟨/handwave⟩.

Fact. If $p : E \to B$ is a covering projection, then p is an open mapping, i.e. $p(U)$ is open in B if $U \subseteq E$ is open.

Since p is surjective, then p is a quotient map. That is, if we put an equivalence relation $\sim$ on E, then $B = E/\sim = \{\text{equiv classes}\}$. This is corollary 22.3 in the book.

p is also a local homeomorphism, i.e. given $e \in E$, $\exists V$ a neighborhood of e such that $p|_V : V \to p(V)$ is a homeomorphism. The proof is not hard; just take an evenly covered neighborhood of $p(e)$, and use the definition of a projection map. However, not all local homeomorphisms are covering projections.

Example. Take $(0, 2) \to S^1$ by $\theta \mapsto e^{2\pi i\theta}$. This is a local homeomorphism. But now take a small neighborhood around 1 in S^1, say the arc joining $e^{-2\pi i\epsilon}$ to $e^{2\pi i\epsilon}$. We get small intervals $(0, \epsilon)$, $(1-\epsilon, 1+\epsilon)$ and $(2-\epsilon, 2)$. But this map is not a homeomorphism between $(0, \epsilon)$ to the arc. So local homeomorphisms are not necessarily covering projections.

6.3 Group Actions

Definition. Let G be a group and let X be a set. A left action of G on X is a function (almost a "binary operation") $G \times X \xrightarrow{\mu} X$ by $(g, x) \mapsto \mu(g, x)$, which will be abbreviated gx. This action must obey (i) $g_1(g_2x) = (g_1 \cdot g_2)x$ for all $g_1, g_2 \in G$ and $x \in X$, and (ii) $1x = x$ for all $x \in X$.

We want to put a topology on $G \times X$, with the discrete topology on G, with μ a continuous function. For our purposes, G will in general be finite.

Note that if we fix $g = g_0$, then $x \mapsto g_0x$ is a bijection (its inverse is $x \mapsto g_0^{-1}x$).

Example. Let $G = \mathbb{Z}$ (additively), $X = \mathbb{R}$, and an action $n \cdot x = x + n$. It's not hard to check this is a group action. Take the equivalence relation whose equivalence classes are Gx, the orbits of x. If we mod out by this equivalence relation, then this becomes the covering projection $\mathbb{R} \to S^1$ discussed before.

7 September 17, 2012

7.1 Group Actions

If G is a group and X is a set (space), a right action of G on X is a function $X \times G \to G$ that send $(x, g) \mapsto xg$, such that $(xg_1)g_2 = x(g_1g_2)$ for all $g_1, g_2 \in G, x \in X$ and $x1 = x$ for every X.

For example, $X = \mathbb{R}$ and $G = (\mathbb{Z}, +)$, then take the action as $(x, n) \mapsto x + n$. We can look at the orbit of a given $x \in X$.

Example. Let $\mathbb{Z}/n\mathbb{Z} = \langle t \mid t^n = 1 \rangle$ act on the circle S^1 by $S^1 \times G \mapsto S : (z, t^i) \mapsto z \cdot \zeta^i$, where ζ is a primitive nth root of unity. Each orbit has size n.

Example. Since $(S^1, \cdot)$ is itself a group, it acts on itself by $S^1 \times S^1 \to S^1 : (z, w) \mapsto zw$. Now the orbit is the entire circle.

Definition. An action is transitive if there is only one orbit.

Given an action, we can define an equivalence relation $\sim$ on X by $x_1 \sim x_2$ if they are in the same orbit. Obviously $\sim$ is terribly uninteresting when the action is transitive. This partitions X into equivalence classes.

Given an action and the corresponding equivalence relation $\sim$, define

$$\begin{aligned} X' &= X/\sim \\ &= \{[x] : x \in X\} \\ &= \{\text{orbits}\} \end{aligned}$$

We have a function that x takes to its equivalence class, as usual.

7.2 Covering Projections

If X has a topological structure, and we require that the map $X \times G \to X$ is continuous where G has the discrete topology, then for any fixed $g \in G$, we get a homeomorphism from X to Xg (since it has an inverse).

This often provides a covering projection from X to $X/\sim$.

Let us consider the example $G = (\mathbb{Z}, +)$ again. It is not hard to show that $[0, 1)$ is a complete set of representatives. This is called the *fundamental domain* for the action. Now $X/\sim \approx S_1$. We verify this is a covering projection for this particular instance. Each little interval on S^1 has a pre-image composed of several little intervals.

Let's take $n = 4$ in $G = \mathbb{Z}/4\mathbb{Z}$, $X = S^1$ as above. A fundamental domain is half-open arc joining 1 to i. The resulting covering projection is the map $S^1 \to S^1 : z \mapsto z^4$.

Fact. Product of group actions are group actions. If G_i acts on X_i, then $G_1 \times \cdots \times G_n$ acts on $X_1 \times \cdots \times X_n$ by $(x_1, \cdots, x_n)(g_1, \cdots, g_n) = (x_1g_1, x_2g_2, \cdots, x_ng_n)$.

Example. Take $X = \mathbb{R}^2$ and $G = \mathbb{Z} \times \mathbb{Z}$ and consider the right action $(x, y)(n, m) = (x + n, y + m)$. The fundamental domain is $[0, 1)^2$, which ends up as a box. This is a torus; we now have a covering projection from $\mathbb{R}^2$ to the torus, which can the be viewed as $S^1 \times S^1$.

This hold in general, as follows.

Fact. $(X_1 \times X_2 \cdots \times X_n)' = X_1' \times X_2' \times \cdots X_n'$.

It is not necessarily the case that we always get a covering projection. Consider the action $S^1 \times S^1 \to S^1$ by $(z, w) \mapsto zw$. This action is transitive, so $S^1/\sim$ is a group with one element. Its pre-image is the entire S^1. However, we need to write this as a disjoint union of homeomorphic "spaces" (which are in this case points). But these points are not open in S^1.

Coverings which arise from group actions are called *regular*. In regular coverings $E \to B$, the coverings correspond to the normal subgroups of $\pi_1(B, b_0)$.

Definition. A topological group $(G, \cdot)$ is a group G with a topology τ for which $\cdot$ is a continuous function.

In group actions, if G is a topological group we need to impose the discrete topology on the action $X \times G \to X$.

7.3 Liftings

Suppose $X \xrightarrow{f} B$ is continuous and $E \xrightarrow{p} B$ is a covering projection, then a mapping $X \xrightarrow{\tilde{f}} E$ is called a *lifting* if $p \circ \tilde{f} = f$.[3]

Liftings don't always exist. However, suppose we pick X to be the interval $I = [0, 1]$, and suppose $f(0) = b_0 \in B$. Select a e_0 in the fiber $p^{-1}(\{b_0\})$. If B is path-connected, then there exists a *unique* lifting $\tilde{f}$ with $\tilde{f}(0) = e_0$.

To prove this, we use the following lemma.

Definition. If (X, d) is a metric space and $\{O_i\}$ is an open covering for X, then q is a *Lebesgue number* for the covering if any set of diameter at most q is contained entirely in one of the O_i.

Definition. In a metric set X, the diameter of a set $A \subseteq X$ is defined by $\sup\{d(x, y) \mid x, y \in A\}$.

Lemma 4 (Lebesgue Number Lemma). *A Lebesgue number exists for every covering whenever X is a compact metric space.*

Let $\{U_i\}$ be a collection of evenly-covered open subsets of B. Look at the collection of open subsets $\{f^{-1}(U_i)\}$, which is an open covering of I. Also, this covers I. Applying the Lebesgue Number Lemma. Let n be a large integer with $1/n < \frac{\delta}{2012}$, where δ is the Lebesgue number. This chops I into n pieces, each of which is contained in some evenly covered neighborhood.

Now we can just construct our lifting greedily. For each interval $I_k = [k/n, (k+1)/n]$ just define $f(I_k)$ by using the inverse of the homeomorphism determined by the definition of even covering. Then apply the pasting lemma on f.

We can even lift homotopies. By replacing I with $I \times I$, we can do the same thing since $I \times I$ is metric and compact.

[3] The lifting problem, in category theory, is the dual of the inclusion problem.

8 September 19, 2012

8.1 Path homotopies lifted

If $I \xrightarrow{\gamma} B$ is a path is a path $\gamma(0) = b_0$, there exists a unique lifting of γ starting at e_0.

Furthermore, if $I \times I \xrightarrow{F} B$ is a homotopy with $F(0,0) = b_0$, then there exists a unique lifting of F to $\tilde{F}$ with $\tilde{F}(0,0) = e_0$.

If $F : I \times I \to B$ is a path homotopy, then $\tilde{F}$ is also a path homotopy. By uniqueness of lifting, it is not hard to show that $\tilde{F}(0,t)$ and $\tilde{F}(1,t)$ are constant functions in t. This implies that path homotopies also lifts to path homotopies.

8.2 Fundamental Lifting Correspondence

Suppose $E \xrightarrow{P} B$ is a covering projection, and fixed $e_0 \in E$ is in the fiber of some fixed $b_0 \in B$.

Claim (Fundamental Lifting Correspondence)**.** We have a function[4] $\Phi : \pi_1(B, b_0) \to p^{-1}(\{b_0\})$ with $[\gamma] \mapsto \tilde{\gamma}(1)$, where $p \circ \tilde{\gamma} = \gamma$ and $\gamma(1) = \gamma(1) = b_0$ and $\tilde{\gamma}(0) = e_0$.

Proving this claim invokes the fact

Fact. $[\gamma_1] = [\gamma_2] \Rightarrow \gamma_1 \simeq_p \gamma_2$.

The fact that this function is well defined now follows from $\tilde{\gamma}_1 \simeq_p \gamma_2 \Rightarrow \tilde{\gamma}_1(1) = \tilde{\gamma}_2(2)$.

Exercise. If E is path connected, then Φ is surjective.

Proof. Suppose $e_1 \in p^{-1}(\{b_0\})$. Suffices to prove $\exists[\eta]$ with $\Phi([\eta]) = e_1$.

We know there is a path $\hat{\eta}$ joining e_0 to e_1. The function $p \circ \hat{\eta}$ is now a loop in B. We claim that $\eta = p \circ \hat{\eta}$ is the desired loop. Indeed, by uniqueness of lifting, $\tilde{\eta} = \hat{\eta}$! (All we did was project down and lift back up). Now we're done. ☐

Exercise. If E is simply connected, then Φ is one-to-one.

Proof. Suffices to prove that if $\Phi([\gamma_1]) = \Phi([\gamma_2])$, then $[\gamma_1] = [\gamma_2]$.

Lift to $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. In E we get $\tilde{\gamma}_1 \simeq_p \tilde{\gamma}_2$ since their endpoints coincide. Project down! If $\tilde{F}$ is a path-homotopy between $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, then $F = p \circ \tilde{F}$ is now a path-homotopy between γ_1 and γ_2. ☐

In particular, we have a one-to-one correspondence between the cardinality of the fiber as well as the cardinality of the fundamental group of B.

8.3 Making Headway on Fundamental Groups

Recall our standard example $\mathbb{R} \xrightarrow{p} S^1$. The fiber above $1 \in S^1$ is the integers. Now since $\mathbb{R}$ is contractible and thus simply connected, we find that $\pi_1(S^1, 1)$ is in a one-to-one correspondence between the integers $\mathbb{Z}$. (This doesn't imply the group is actually the integers, just that it's countable and not finite.)

Now consider 1 in S^2. Let $G = \mathbb{Z}_2 = \langle T \mid T^2 = 1 \rangle$. Now G acts on S^2 by $T \cdot \vec{x} = -\vec{x}$. This essentially takes a point to the point diametrically opposite it. Modding out by that action, we get $S^2/\sim$ as a hemisphere. The result is $S^2/\sim = \mathbb{R}P^2$, called real projective 2-space.

[4]This is actually just a function, since it takes a group. There is no sense of "continuity" here, etc.

This gives a covering projection $S^2 \xrightarrow{p} \mathbb{R}P^2$ (the inverse image of a patch is two patches diametrically opposite it). But we know that $\pi_1(\mathbb{R}P^2, [1])$ is in a one-to-one correspondence with $\mathbb{Z}_2$. Since there are not that many groups of order two, we find that $\pi(\mathbb{R}P^2, [1]) \cong \mathbb{Z}_2$. Finally a nontrivial fundamental group!

Let's try and understand $\mathbb{R}P^2$. It seems to be the northern hemisphere, with half the equator included. If we are Santa and look down, we see this is a disk with half the boundary included. We call it a *manifold* since locally it looks like Euclidean space.

In this image of a disk, the nontrivial loop is just a path from 1 to -1.

Fact. The fundamental group of $X \times Y$ is isomorphic to the fundamental groups of X and Y.

Since there exist spaces L_n with $\pi_(L_n) \cong \mathbb{Z}_n$ for each n, via the Fundamental Theorem of Abelian Groups, we can construct spaces having a fundamental group of any arbitrary finite abelian group, or even a finitely generated abelian group once we know $\pi(S_1)$ below.

8.4 Fundamental Group of the Circle

We showed earlier that there is a one-to-one correspondence $\pi_1(S^1, 1) \overset{\Phi}{\leftrightarrow} \mathbb{Z}$. We now want to show that Φ is a group homomorphism (somehow); since we have tho correspondence, we will be done.

That is, keeping in mind that Φ returns integers, we want to show that

$$\Phi([\gamma_1] \cdot [\gamma_2]) = \Phi([\gamma_1]) + \Phi(\gamma_2)$$

Suppose that $\Phi([\gamma_1]) = n = \tilde{\gamma}_1(1)$, where we view $n \in \mathbb{R}$, and $\Phi([\gamma_2]) = m = \tilde{\gamma}_2(1)$. But $[\gamma_1] \cdot [\gamma_2] = [\gamma_1 * \gamma_2]$. Take a lifting of $\gamma_1 * \gamma_2$ starting at zero defined by

$$\tilde{\zeta} : I \to \mathbb{R} \quad \text{by} \quad \tilde{\zeta}(s) = \begin{cases} \tilde{\gamma}_1(2s) & 0 \le s \le \frac{1}{2} \\ n + \tilde{\gamma}_2(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}.$$

It is trivial to check that this is well-defined and continuous. But one can check that this is a lifting. Also, we have easily enough that

$$\begin{aligned} p(\tilde{\gamma}_1(2s)) &= \gamma_1(2s) \\ p(n + \tilde{\gamma}_2(2s-1) &= p(\tilde{\gamma}_2(2s-1)) \\ &= \gamma_2(2s-1) \end{aligned}$$

Hence, $\tilde{\zeta}(s)$ is a lifting. Since liftings are unique, we get $\Phi([\gamma_1 * \gamma_2]) = n + m$, implying the conclusion.

9 September 24, 2012

Poincare's Conjecture is algebraic topology. Most of algebraic topology is 20th century.

Names: Hopf, Alexandrov, Eilenberg-Steenrod. Euler is sort of considered the father of topology. As a discipline, topology is quite recent.

9.1 Covering Projections

Same setup as usual. Select $b_0 \in B$ and a covering projection $E \xrightarrow{p} B$. Up in E we have $p^{-1}(\{b_0\})$ which we will abbreviate to just $p^{-1}(b_0)$. Then we pick some $e_0 \in p^{-1}(b_0)$.

Claim. The group $\pi_1(B, b_0)$ acts on $p^{-1}(b_0)$ from the right with $p^{-1}(b_0) \times \pi_1(B, b_0) \to p^{-1}(b_0)$ via

$$(e, [\gamma]) \mapsto \hat{\gamma}(1)$$

where $\hat{\gamma}$ is the unique lifting of γ into E.

We need only show that $(e[\gamma])[\eta] = (e[\gamma][\eta])$. This is left to the diligent student.

9.2 Orbits

What is the orbit? $e\pi_1(B, b_0)$. Any time we have an orbit xG, then this is essentially (i.e. can be modelled by) as the set of right cosets $\{Hg : g \in G\}$ with G acting on the right by $(Hg)g' = H(gg')$. Here $H = G_x$ is the isotropy subgroup, or the stabilizer, of x.

We need to find which elements of $\pi_1(B, b_0)$ fix e.

We want to characterize e with $e[\gamma] = e$. If $\hat{\gamma}(1) = e$, then $\hat{\gamma}$ is actually a loop. But $\hat{\gamma}$ lifts γ; i.e. $[p \circ \hat{\gamma}] = [\gamma]$. This is $p_*([\hat{\gamma}])$.

Claim. $[\gamma]$ fixes e if and only if $[\gamma] \in p_*(\pi_1(E, e))$

Hence, the stabilizer of e is precisely $p_*(\pi_1(E, e))$ and the orbits are right cosets of this set.

However, we saw last time that if E is path-connected, then the action is transitive. That is, $e\pi_1(B, b_0) = p^{-1}(b_0)$. So the entire fiber can be modelled as cosets of $p_*(\pi_1(E, e))$.

(Based on the restrictions of E, it's connected iff path-connected.)

Now recall:

Theorem 5 (Lagrange's Theorem)**.** *The number of cosets of G divides $|G|$ for finite G.*

So if we know that $\pi_1(B, b_0)$ is finite, then we know that the fiber $p^{-1}(b_0)$ is pretty limited (given E is path-connected). Remember, we have a transitive action on a finite group on a finite set.

Claim. If $E \xrightarrow{p} B$ is a covering, then $\pi_1(E, e) \xrightarrow{p_*} \pi_1(B, b_0)$, p_* is an injective homomorphism.

In some sense, the smaller the fiber, the closer E is to being B.

9.3 Using the Fundamental Group of the Circle

Let j be inclusion; for $A \subseteq X$, the map $A \xrightarrow{j} X$ is called inclusion.

Definition. We say that A is a *retract* of X via the retraction r if $\exists r : X \to A$ such that $r \mid_A = \mathrm{id}_A \Leftrightarrow r \circ j = \mathrm{id}_A$; that is $r(a) = a$ for all $a \in A$.

Example. Let $A = \mathbb{R} \times \{0\}$ be the x-axis in $X = \mathbb{R}^2$. The retraction r will be just projecting down.

We can generalize this to $A = Y \times \{z_0\} \subseteq Y \times Z = X$. This is just because A is homeomorphic to Y. In a loose sense, "projecting down into a component is a retraction".

Example. A Moebius strip can be projected down into its equator.

Bringing stars back in, note that $r_* \circ j_*$ is the identity $\pi_1(A, a_0) \xrightarrow{j_*} \pi_1(X, a_0) \xrightarrow{r_*} \pi(A, a_0)$. Hence j_* is injective and r_* is surjective.

If $j \circ r$ is homeomorphic to the identity, then something interesting happens; in a sense, you can collapse in a controlled fashion.

Theorem 6 (No-Retraction Theorem). *Consider the two-ball $B^2 = \{z \in \mathbb{C} \mid \|z\| \le 1\}$ and its boundary $S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$. (Note that $S^1 \subseteq B^2$). Then S^1 is not a retract of B^2.*

Proof. If we had such a retraction, then $\pi_1(S^1, 1) \xrightarrow{j_*} \pi_1(B^2, 1)$ would have to be injective. But this is a map from $\mathbb{Z}$ to $\{0\}$. $\square$

10 September 26, 2012

10.1 Meteorology

Theorem 7. *Suppose we have two continuous functions $f, g : S^2 \to \mathbb{R}$. Then we can consider a function $\Phi : S^2 \to \mathbb{R}^2$ as $\Phi = f \times g$. Then, there exists $z_0 \in S^2$ with $\Phi(z_0) = \Phi(-z_0)$ by a theorem.*

If we let S^2 be the earth, and f, g be temperature and barometric pressure. Then the theorem implies that there exists two antipodal points on the Earth where the temperature and pressure coincide. This is a homework problem.

Catastrophe Theory (René Thon)

10.2 Topology

B^2 does not retract onto its boundary S^1; i.e.

$$\nexists r : B^2 \to S^1 : r \mid_{S^1} = \mathrm{id}_{S^1}$$

We proved this using the fact that B has a trivial fundamental group, but S^1 does not.

The most well-known consequence of this is

Theorem 8 (Brower Fixed Point Theorem)**.** *If $f : B^2 \to B^2$ is continuous, then $\exists x_0 \in B^2$ such that $f(x_0) = x_0$; i.e. x_0 is a fixed point of f.*

Munkres uses a proof using vector fields. The proof that most people usually see doesn't rely on this.

Proof. Suppose on the contrary that $f : B^2 \to B^2$ has no fixed points. Define $r : B^2 \to S^1$ by taking the ray starting at $f(x)$, passing through x, and intersecting that with its boundary. This is $r(x)$. This function is well-defined since $f(x) \neq x$.

One can check that r is continuous. (Handout on website). But r is clearly a retraction, and this is a contradiction. $\square$

This generalizes to any B^n.

Remark. We say that X has the fixed point property if $\forall f : X \to X$ continuous, f has a fixed point.

(i) B^n has the fixed point property for every n.

(ii) S^1 doesn't have the fixed point property (consider rotation). $\mathbb{R}$ doesn't either via translation. Neither does $S^1 \times S^1$ or $\mathbb{R}^n$.

(iii) If $X \approx Y$ and Y has the fixed point property, then so does X. In other words, the property is a topological invariant. If we have $h : X \to Y$ a homomorphism, then for any $f : X \to X$ we can construct $\tilde{f} = h \circ f \circ h^{-1}$. We know that $\tilde{f}(y_0) = y_0$ for some y_0. Then $x_0 = h^{-1}(y_0)$ is a fixed point of f.

We only talk about a fixed point when we have a function to itself.

10.3 Vector Fields

If $\mathcal{O}$ is a non-retarded subset of $\mathbb{R}^n$, then a vector field on O is a function from $\mathcal{O}$ to $\mathbb{R}^n$.

A "nowhere vanishing vector field" is a v such that $v(\vec{x}) \neq \vec{0}$ for every $\vec{x} \in \mathcal{O}$. In that case we can have $v : \mathcal{O} \to \mathbb{R}^n \setminus \{\vec{0}\}$.

Theorem 9. *If v is a nowhere vanishing vector field on B^2 and w is its restriction to $S^1 \subseteq B^2$, then there exists x_{out} and x_{in} in S^1 for which $w(x_{out})$ is outward-pointing and $w(x_{in})$ is inward-pointing.*

For x, we say x is outward-pointing if $w(x) = kx$ for some scalar $k > 0$. If $k < 0$ then instead we say it is inward-pointing. This result implies the Brower fixed point theorem.

Vector fields can be viewed as e.g. a part of the ocean, where the vector corresponds to the strength and direction of the current, etc.

A tangent-vector field is trickier; the output of $v(x)$ needs to be "tangent" to the point (the definition of tangent depends on the space). In other words, the domain depends on f.

Theorem 10 (Hairy Ball Theorem). *S^2 does not have a nowhere vanishing tangent vector field.*

This is of course not true if we replace S^1 by S^2.

Fact. There exists a non-vanishing tangent vector fields on S^n if and only if n is odd.

Since $S^{2n+1} \subseteq \mathbb{C}^{n+1}$; we can write it as

$$S^{2n+1} = \{(z_0, \cdots, z_n) \mid z_0^2 + \cdots + z_n^2 = 1\}$$

so we can get a tangent vector field by taking $(z_0', \cdots, z_n')$ where $z_k' = -y + xi$ for $z_k = x + yi$.

Which S^n are parallelizable? Something like $\{S^1, S^3, S^7\}$. Prove only in early 1960's by Adams.

10.4 Algebra

$\mathbb{R}^1$ is a field. $\mathbb{R}^2$ can be thought of as $\mathbb{C}$, which form a field.

For $\mathbb{R}^3$ we don't have what we want.

However, $\mathbb{R}^4$ has a quaternion structure $\mathbb{H}$. We get a division algebra (almost a field, except non-commutative multiplication). In $\mathbb{R}^*$ we have the octonians $\mathbb{O}$, but we lose associativity.

In $\mathbb{R}^{16}$ we have the Cayley numbers, which loses even more. These things are called algebras. It turns out that we can't have algebra structures for sufficiently large $\mathbb{R}^n$ which is closely related to the parallelizable spheres above.

Punk hairy ball theorem.

11 October 1, 2012

11.1 Something

If $X \xrightarrow{f} Y$ is a function of sets X, Y, and we have an equivalence relation $\sim$ which is preserved under f, then we have an "induced" function $\hat{f} : X/\sim \to Y$ from defined by $\hat{f}([x]) = f(x)$. This function is generally continuous when you want it to be (by composition).

Claim. If $h : S^1 \to X$ is nulhomotopic, then there exists an continuous extension $k : B^2 \to X$ of H.

Proof. We are given that there is a homotopy $S^1 \times I \xrightarrow{F} X$ with $F(z,0) = h(z)$ and $F(z,1) = c_{x_0}(z)$ for all z (where c_{x_0} is the constant map sending everything to x_0.) Define *sim* on $S^1 \times I$ by $(z,t) \sim (z,t)$ for $t < 1$ and

$$(z,1) \sim (z',1) \forall z, z' \in S^1.$$

This is clearly an equivalence relation. Now define $(S^1 \times I)/\sim \xrightarrow{\hat{F}} X$ as above. Since $z \sim z' \Rightarrow F(z,t) = F(z',t)$ is evident, we see that $\hat{F}$ is a continuous function. But we claim that $B^2 \approx (S^1 \times I)/\sim$. Indeed, it's like a cone, which can be flattened to a disk. This proves the claim. $\square$

Any time we have $Z \xrightarrow{h} X$ nulhomotopic, we can try the same trick to get an extension $CZ \xrightarrow{k} X$, a "cone"-like version of Z. The cone on Z is actually contractible, since we can pull everything towards the tip. As a particular example, $CS^{n-1} = B^n$; we have invoked $n = 2$.

The converse of this turns out to be true.

11.2 Spheres to Spheres

Let $[X, Y]$ denote the homotopy (equivalence) classes of maps $X \to Y$. It turns out that $\pi_1(Y) = [S^1, Y]$, and $\pi_n(Y) = [S^n, Y]$. If you understand $\pi_n(S^m)$ for all n and m, then you completely understand $[X, Y]$ for reasonable X and Y. This means that it's sort of a lost cause, but we do have some "easy" facts:

$$\pi_n(S^m) = \begin{cases} \text{trivial} & \text{if } n < m \\ \mathbb{Z} & \text{if } n = m \\ ? & \text{if } n > m \end{cases}$$

For instance, it turns out that $\pi_3(S^2) = \mathbb{Z}$, but $\pi_{n+1}(S^n) = \mathbb{Z}_2$ when $n > 2$.

11.3 Antipodes

Definition. If $h : S^n \to S^m$, then we call h *antipode-preserving* if $h(-x) = -h(x)$.

It does not make sense to talk about antipodes in most spaces.

Claim. If $h : S^1 \to S^1$ is antipode-preserving (and continuous), then it is not nulhomotopic.

Proof. Let $q : S^1 \to S^1$ by $q(z) = z^2$. Let $g = q \circ h$. (See diagram on pg 357).

We want to show that if $q(x_1) = q(x_2) \Leftrightarrow x_1^2 = x_2^2$, then $g(x_1) = g(x_2)$. But this is obvious; h is antipode-preserving! So, we get an induced function $k : S^1 \to S^1$ with $k \circ q = q \circ h$ by $k(x^2) = g(x)$; this is well defined by the property we have just shown.

Now recall that q is a covering projection. Therefore, $q_* : \pi_1(S^1) \to \pi_1(S^1)$ is a injective (in general, $p_* : \pi_1(E) \to \pi_1(B)$.

Consider $\pi_1(S^1, 1) \xrightarrow{k_*} \pi_1(S^1, 1)$. Consider a path $\gamma : I \to S^1$ with $\gamma(0) = 1$ and $\gamma(1) = -1$. First, we claim that $[q \circ \gamma]$, the class of a loop at S^1, is nontrivial. It cannot be trivial because this would imply that $q \circ \gamma$ was nulhomotopic; but the lifting of this is γ which is not a loop. But now,

$$k_* : [q \circ \gamma] \mapsto k_*([q \circ \gamma]) = [k \circ q \circ \gamma] = [q \circ h \circ \gamma]$$

But we claim that $[q \circ h \circ \gamma]$ is nontrivial. Note that $q \circ (h \circ \gamma)$ lifts to $h \circ \gamma$, which is not a loop; indeed, $h(\gamma(0)) = h(1) \neq -h(1) = h(-1) = h(\gamma(1))$. We conclude that k_* is nontrivial.

Finally: the homomorphism k_* is injective, since it is a nontrivial homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ (check this is sufficient). Also, q_* is injective as mentioned earlier. But

$$q_* \circ h_* = k_* \circ q_*$$

from which we find that h_* is injective. $\square$

Remark. In fact, $q_* : \pi_1(S^1) \to \pi_1(S^1)$ has $1 \mapsto 2$. Indeed, view q as a covering projection; we have that the fiber (which has cardinality 2) is the right index.

Next week: ham sandwich theorem.

12 October 3, 2012

12.1 Something

Want to show the identity $i : S^n \to S^n$ is not nulhomotopic.

If $\nexists r : B^{n+1} \to S^n$ a retraction, then $f : S^n \to X$ is nulhomotopic is equivalent to F extending to $\tilde{f}$.

$$\begin{tikzcd}
S^n \arrow[r, "f"] \arrow[d, "j"'] & X \\
B^{n+1} \arrow[ur, "\tilde{f}"] &
\end{tikzcd}$$

If the identity were homotopic, then upon taking f the identity, $\tilde{f} \circ j$ is the identity, i.e. a retraction.

12.2 Geography

You can't "map", i.e. embed S^2 into $\mathbb{R}^2$. That is, there is no $S^2 \xrightarrow{f} \mathbb{R}^2$ which is homeomorphic to its image, because the Borsuk-Ulam Theorem implies $\exists x_0 \in S^2$ with $f(x_0) = f(-x_0)$; in particular, f cannot be injective.

For all n, if $f : S^n \to \mathbb{R}^n$, Borsuk-Ulam $\exists x_0 \in S^n$ such that $f(-x_0) = f(x_0)$. That is, Borsuk-Ulam generalizes to arbitrary n instead of just $n = 2$.

12.3 Ham Sandwich Theorem

Theorem 11 (Ham Sandwich Theorem)**.** *Given two slices of bread in $\mathbb{R}^3$ with as lice of ham in between, it's possible to slice the sandwich in such a way that both silecs of bread and the ham slice are divided in half (volume-wise).*

The proof is analogous to the one given in the book of the bisection theorem.

Proof. Consider $\mathbb{R}^3 \times \{3\} \subseteq \mathbb{R}^4$ which contain the pieces of the sandwich.

We want to construct a continuous map $F : S^3 \to \mathbb{R}^3$ because Borsuk-Ulam will say $\exists x_0 \in S^3$ such that $F(x_0) = F(-x_0)$.

Given $\vec{u} \in S^3$, there is a unique hyperplane ($\approx \mathbb{R}^3$) which passes through the origin and is perpendicular to $\vec{u}$. Denote this hyperplane by $P_{\vec{u}}$.

Let $i = 1, 2, 3$ correspond to the first slice, ham, and second slice respectively. Let $f_i(\vec{u})$ be the portion of the volume of the i-th slice of the food that lies on the same side as $\vec{u}$.

If $\vec{u} = \langle 0, 0, 0, 1 \rangle$, then $f_i(\vec{u}) = \mathrm{Vol}(A_i)$. On the other hand, $f_i(\langle 0, 0, 0, -1 \rangle) = 0$, since we see no sandwich looking down. On the other hand, for any other vector, $P_{\vec{u}}$ intersects $\mathbb{R}^3 \times \{3\}$ in some two-dimensional plane. On the other hand, $f_i(\vec{u}) + f_i(-\vec{u}) = \mathrm{Vol}(A_i)$, since this adds up the sandwich on both sides.

Waving our hands, we get that f_i is continuous for the slices. Now, define $F : S^3 \to \mathbb{R}^3$ by

$$F(\vec{u}) = (f_1(\vec{u}), f_2(\vec{u}), f_3(\vec{u}))$$

Now Borsuk-Ulam says that $\exists \vec{u}_0 : F(\vec{u}_0) = F(-\vec{u}_0)$. This says that

$$\forall i : f_i(\vec{u}_0) = f_i(-\vec{u}_0)$$

But we know that their sum is $\mathrm{Vol}(A_i)$, so for each i we have $f_i(\vec{u}_0) = \frac{1}{2}\,\mathrm{Vol}(a_i)$. $\square$

12.4 Retracts

We say $A \subseteq X$ is a retract if there exists a retraction; i.e. a function $r : X \to A$ such that $r \circ j = \mathrm{id}_A$, where $j : A \to X$ is the inclusion.

In other words, r is a "left-inverse" function for j. If r were a right-inverse for j; i.e. $j \circ r = \mathrm{id}_X$, then $A = X$ and this is quite stupid. Let's weaken the condition that r is a right-inverse, as follows

Definition. If r is a retraction onto $A \subseteq X$ such that $j \circ r \simeq \mathrm{id}_X$ we call r a *deformation retraction* and say that A is a deformation retract of X.

Munkres actually requires something stronger; not all books agree on a definition here.

Example. $A = \{\vec{0}\} \subseteq \mathbb{R}^n = X$ is a deformation retract of X. Let r be a constant map to zero; we can check that $j \circ r \simeq \mathrm{id}_X$ simply because $\mathbb{R}^n$ is contractible.

Example. In $\mathbb{R}^n$, the map $r : \langle x_1, x_2, \cdots, x_n \rangle = \langle x_1, 0, 0, \cdots, 0 \rangle$ is a defromation retraction.

Now for a less trivial example:

Example. The topologist's comb (also called the harmonic comb) is the set $X \subseteq \mathbb{R}^2$ defined by

$$X = \bigcup_{n \ge 1} \left(\{\frac{1}{n}\} \times I \right) \times (\{0\} \times I) \cup (I \times \{0\})$$

If $A = \{(0,1)\}$, then A is a deformation retraction of f. We can deform it continuously by smashing the teeth onto the base, pushing that to $(0,0)$ and the brining the whole thing up to $(0,1)$.

Example. The topologist's fan (also called the harmonic fan) is the set $X \subseteq \mathbb{R}^2$ defined by joining the segments $(0,0)$ to $(1, \frac{1}{n})$ for each integer n, as well as $(1,0)$. Again $A = \{(1,0)\}$ is a deformation retraction

These aren't terribly useful deformation retractions, so the alternate definition called by some authors as a strong deformation retraction is given by

Definition. A *strong deformation retraction* $r : X \to A$ is a deformation retraction such that the homotopy F between $j \circ r$ to id_X fixes A; i.e. the homotopy doesn't fix A.

Both the examples above fail under this stronger definition; on the other hand, the earlier two examples (contractible spaces and projecting onto the x-axis) still are okay. Munkres just calls this a deformation retraction, so we will do so as well.

12.5 Homotopy Equivalence

The punch line is as follows: spaces are homeomorphic if $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} X$ satisfy $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. Retractions weaken one of these to $g \circ f \simeq \mathrm{id}_X$.

It turns out that if we weaken both conditions, we get an equivalence relation called *homotopy equivalence*. That is, spaces X and Y are homotopy equivalent if $\exists f : X \to Y, g : Y \to X$ is $f \circ g \simeq \mathrm{id}_Y$, $g \circ f \simeq \mathrm{id}_X$. For example, a point is homotopy equivalent to $\mathbb{R}^n$.

It turns out that algebraic topology is not able to distinguish between homotopy equivalent spaces; homology groups, co-homology groups, and the fundamental groups $\pi_1(X)$ and even $\pi_2(X)$, all coincide.

Thus, algebraic topology cannot distinguish between a point and $\mathbb{R}^n$. Also, it cannot distinguish between the male and female symbols, since both are homotopy equivalent to S^1.

13 October 8, 2012

13.1 Homotopy Equivalence

Last time we had an equivalence relation, the homotopy type.

Definition. A map $f : X \to Y$ is called a *homotopy equivalence* if $\exists g : Y \to X$ such that

$$g \circ f \simeq \mathrm{id}_X$$
$$f \circ g \simeq \mathrm{id}_Y$$

We say X is *homotopy equivalent* to Y, denoted $X \simeq Y$, if there exists a homotopy equivalence between them. We say that X and Y have the same homotopy type

For example, $\{\vec{0}\} \simeq \mathbb{R}^n$, since $\mathbb{R}^n$ is contractible.

Last time we said that if $X \simeq Y$, then $\pi_n(X) \cong \pi_n(Y)$ for each n, and the homology groups and co-homology groups are all isomorphic as well. That is, most of the constructs of algebraic topology cannot distinguish between homotopic groups.

This is not *too* crippling, though:

Fact. If $n \neq m$, then $\mathbb{R}^n \not\approx \mathbb{R}^m$.

Proof. We may assume WLOG that $n \geq m + 1$. Recall $S^m \subseteq \mathbb{R}^n$. If we had a homeomorphism

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "h"] & \mathbb{R}^m \\
S^m \arrow[u, phantom, "\bigcup"] \arrow[ur, "\tilde{h}"] &
\end{tikzcd}$$

So $\tilde{h}$, the restriction of h to S^m, must be injective. But the Borsuk-Ulam Theorem implies that this is not possible. $\square$

So there are ways to show spaces X and Y are different even if they are homotopy equivalent.

13.2 Retraction

Recall that $A \subseteq X$ is a (strong) deformation retract if $\exists r : X \to A$ such that $r \circ j$ is the identity on A and $j \circ r \simeq_A \mathrm{id}_X$. (See the first lecture).

It turns out that if you take the category of topological spaces, you can make a new category hTop which are equivalence classes of homotopy classes. In Top, the objects are topological spaces and the morphisms are continuous functions. In hTop, the objects are homotopy classes of maps, and the morphisms are *equivalence classes of maps*. Hence, the arrows are not actually functions between objects. Hence, the space hTop is not *concrete*.

If A is a deformation retract of X, then A is the same homotopy type as X.

Example. The space $\mathbb{C} \setminus \{-1, 1\}$ is homotopy equivalent to pair of tangent circles; i.e. the figure eight. These spaces are both deformation retracts of $\mathbb{C} \setminus \{-1, 1\}$, but neither is a deformation retract of the other; however they are of the same homotopy type.

It turns out that $f : X \to Y$ is a homotopy equivalence, we can build a space M_f such that $X, Y \subseteq M_f$ are deformation retracts of M_f.

Consider a "cylinder"-like $X \times I$ and Y. Take $X \times I + Y$ the direct sum topology (that is, just put the spaces next to each other).

Take $M_f = [(X \times I) + Y]/\sim$, where $(x, 1) \sim f(x)$. Think a top hat.

Now $X \approx X \times \{0\}$. So we can collapse everything to Y by compressing the hat down; so Y is a deformation retraction of this new space. (This does not use the fact that f is a homotopy.)

This idea is called a mapping cylinder. (Mapping cones are like witch's hats.)

13.3 More About Categories

If $f : X \to Y$ and $g : Y \to X$ are inverses, then we can get $f_* : \pi_1(x) \to \pi_1(Y)$ and g_* similarly.

Lemma 12. *Consider two maps $(X, x_0) \xrightarrow{f,g} (Y, y_0)$. If $f \simeq g$ via a homotopy H and $H(x_0, t) = y_0$ for all t, then*

$$\pi_1(X, x_0) \xrightarrow{f_*, g_*} \pi_1(Y, y_0)$$

are homomorphisms and $f_* = g_*$.

Proof. If $[\gamma] \in \pi_1(X, x_0)$, then we need $f_*([\gamma]) = g_*([\gamma])$. But the left-hand side is $[f \circ \gamma]$ and the right-hand side is $[g \circ \gamma]$. We want these to be path-homotopic; that is,

$$f \circ \gamma \simeq_p g \circ \gamma$$

Now define $K : I \times I \to Y$ by

$$K(s, t) = H(\gamma(s), t)$$

and this does exactly what we want (check this). ☐

Now, what if $X \xrightarrow{f,g} Y$ but $f(x_0) = y_0$ and $g(x_0) = y_1$? If $f \simeq g$ by a homotopy $H : X \times I \to Y \ldots$

Let's look at $\alpha(t) = H(x_0, t)$. Now $\alpha(0) = f(x_0) = y_0$, and $\alpha(1) = g(x_0) = y_1$. So, we get for free a path α from y_0 to y_1.

So we get a commutative diagram

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[rr, "f_*"] \arrow[dr, "g_*"'] & & \pi_1(Y, y_0) \arrow[dl, "\hat{\alpha}"] \\
& \pi_1(Y, y_1) &
\end{tikzcd}$$

Then f_* and g_* are most certainly not the same (their domains are different), but if we consider a loop $[\zeta] \in \pi_1(Y, y_0)$, then we have $\hat{\alpha}([\zeta]) = [\bar{\alpha} * \zeta * \alpha]$. Remember that α is an isomorphism.

It turns out that $g_* = \hat{\alpha} \circ f_*$. In particular, f_* is surjective/injective/trivial iff g_* is surjective/injective/trivial, etc.

In particular, if $X \xrightarrow{h} Y$ is nulhomotopic with $h(x_0) = y_0$, then $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is trivial.

Anyways,

Theorem 13. *If $f : (X, x_0) \to (Y, y_0)$ is a homotopy equivalence, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.*

Why isn't this obvious? We could potentially get

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f} (Y, y_1)$$

that is,

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "(f_{x_0})_*"] & \pi_1(Y, y_0) \arrow[dl, "g_*"'] \\
\pi_1(X, x_1) \arrow[r, "(f_{x_1})_*"] & \pi_1(Y, y_1)
\end{tikzcd}$$

But because of our alpha,

$$g_* \circ f_* = (g \circ f)_* = \hat{\alpha} \circ (\mathrm{id}_X)_* = \hat{\alpha}$$

So we still have an isomorphism, but g_* is not the inverse; it goes to the wrong place!

Theorem 14 (Seifert-van Kampen Theorem)**.** *If $X = U \cup V$ where U and V are open sets, then $\pi_1(X, x_0)$ is related to $\pi_1(U, x_0)$, $\pi_1(V, x_0)$. More details later.*

14 October 10, 2012

How to get a fundamental group of a space from the fundamental groups of some pieces from the space?

If $U, V \subseteq X$, are open, $U \cap V$ is path-connected (and nonempty), and U, V are simply connected, then $U \cup V = X$, then $U \cup V$ is simply connected.

This was a homework problem. It was quite good. More generally, consider open $U, V \subseteq X$ with $x_0 \in U \cap V$ path-connected and $X = U \cap V$. In essence, $\pi_1(X, x_0)$ can be written as a "product" of $\pi_1(U, x_0)$ and $\pi_1(V, y_0)$. What product might this be?

14.1 Free Product

Consider groups $H, K \le G$ and $H \cap K = \{1\}$. Define a product

$$H * K := \langle H, K \rangle \le G$$

as the smallest subgroup of G containing both H and K, also known as the *subgroup of G generated by H and K*. This consists of *words* of elements of H and K.

This is called the (internal) free product of H and K.

Here is an example of an external free product:

Example. Let $H = \langle a \rangle \cong \mathbb{Z}$ and $K = \langle b \rangle \cong \mathbb{Z}$. Then

$$H * K = \langle \text{words of } a \text{ and } b \rangle$$

that is, the elements all look like $a^{n_1} b^{m_1} \cdots a^{n_k} b^{m_k}$, where $n_i, m_i \in \mathbb{Z}$.

This is the free group on two generators.

Example. Let $H = \langle a \mid a^2 = 1 \rangle$ and $K = \langle b \mid b^2 = 1 \rangle$. Then $H * K$ is the set of words with letters a and b such that neither the pattern aa nor bb appears.

In general, external free products are huge.

14.2 Von Kampen

If $U \cap V$ is simply connected, then we *almost* have $\pi_1(X, x_0) \cong \pi_1(U, x_0) * \pi_1(V, x_0)$.

Consider the inclusions:

$$\begin{tikzcd}
& U \arrow[dr, "j_U"] & \\
x_0 \in U \cap V \arrow[ur, "i_U"] \arrow[dr, "i_V"'] & & X \\
& V \arrow[ur, "j_V"'] &
\end{tikzcd}$$

Where i and j are inclusions. We have homomorphisms

$$(j_U)_* : \pi_1(U, x_0) \to \pi_1(X, x_0)$$
$$(j_V)_* : \pi_(1V, x_0) \to \pi_1(X, x_0)$$

which preserve the base-point x_0.

So the precise statement is that if $U \cap V$ is simply connected, then

$$\pi_1(X, x_0) \cong (j_U)_*(\pi_1(U, x_0)) * (j_V)_*((\pi_1(V, x_0))).$$

and the main idea is basically the same as the specific problem we did.

Example. Consider the figure eight. Consider two copies of S_1 called U and V with $X_8 = U \cap V$, $U \cap V = \{x_0\}$. Hence the intersection (which is quite small!) is simply connected. But U and V are not open in the union! So we need a slight modification: add a little bit of U and V, open, to the set (so U looks like a fish with the vertical segment erased).

Now, $\pi_1(U, x_0) = S^1$. With some handwaving we can get $(j_U)_*(\pi_1(U, x_0)) = \mathbb{Z}$ as well.

Therefore, by Von Kampen we obtain that $\pi_1(X, x_0)$ is the free group on two generators.

So now, we have our first non-abelian fundamental group.

Exercise. Show that the space X_8 is a deformation retraction of the punctured torus.

The fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$, and the space X_8 is now the abelianization of the free group F_2.

15 October 15, 2012

15.1 von Kampen Again

Open U and V in a space X satisfy $X = U \cup V$ and $U \cap V$ path connected and containing some element x_0.

Using the Lebesgue number lemma, we decompose any loop $[\gamma] \in \pi_1(X, x_0)$ as $\gamma_1 * \gamma_2 * \cdots * \gamma_n$, where each γ_i is a path from x_{i-1} to x_i lying entirely in either U and V, and $x_i \in U \cap V$ for all i.

Now construct a path in $U \cap V$ called α_i joining x_i to x_{i-1} for each i. Now we see that we have a composition of loops

$$\gamma = (\gamma_1 * \bar{\alpha}_1) * (\alpha_1 * \bar{\alpha}_2) * (\alpha_2 * \gamma_3 * \bar{\alpha}_3) * \cdots * (\alpha_{n-2} * \gamma_{n-1} * \bar{\alpha}_{n-1}) * (\alpha_{n-1} * \gamma_n) \quad (1)$$

Now the loops alternate between lying in U and V.

However, we are really considering a loop contained in U as a loop in $X = U \cup V$; as in the inclusion diagram:

$$\begin{tikzcd}
& U \arrow[dr, "j_U"] & \\
U \cap V \arrow[ur, "i_U"] \arrow[dr, "i_V"'] & & U \cup V \\
& V \arrow[ur, "j_V"'] &
\end{tikzcd}$$

For example, consider a space B^2 a disk of radius 3. Let V denote the disk with radius 2 and U denote the annulus with outer radius 3 and inner radius 1.

If we take a $[\eta] \in \pi_1(U, x_0) \approx \mathbb{Z}$, we get something that's not trivial if we consider it in the larger space $B^2 = U \cap V$. In other words, $(j_U)_*([\eta]) \in \pi_1(B^2, x_0)$ is trivial even though $[\eta] \in \pi_1(U, x_0)$ is not. In still other words, $(j_u)_*$ need not be one-to-one.

15.2 Seifert-Van Kampen Special Cases

The first special case was U, V simply connected implying $U \cup V$ being simply connected.

If $U \cap V$ is simply connected, then

$$\pi_1(U \cup V, x_0) \cong \pi_1(U, x_0) * \pi_1(V, x_0)$$

The difference here is that if the intersection is simply connected, then "what you see what you get" in 1.

For example, if $\pi_1(X_8, x_0)$, where X_8 is the figure eight, gives the free group $F_2 = \langle a, b \rangle$.

Generally, let's look at $(j_U)_*([\gamma_1]) * (j_V)_*([\gamma_2])$ where $[\gamma_1] \in \pi_1(U, x_0)$ and $[\gamma_2] \in \pi_1(V, x_0)$.

But if $[\eta] \in \pi_1(U \cap V, x_0)$ such that $[\gamma_1] = (i_U)_*([\eta])$ and $[\gamma_2] = (i_V)_*([\eta]^{-1})$, then we have trouble. It could turn out that we have $a \in \pi_1(U, x_0)$ and $b \in \pi_1(V, x_0)$ but $ab = 1$ in the larger group $\pi_1(X, x_0)$.

The full description is

$$\pi_1(U \cup V, x_0) = ([(j_U)_* \pi_1(U, x_0)] * [(j_V)_* \pi_1(V, x_0)])/N$$

where N is the normal closure of (i.e. the smallest normal subgroup which contains) $((i_U)_*[\eta])^{-1} \cdot (i_V)_*[\eta]$ for every $[\eta] \in \pi_1(U \cap V, x_0)$.

This is called the amalgamated product, written

$$\pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0).$$

In general, if K is contained in groups G_1 and G_2 we write the amalgamated product as $G_1 *_K G_2$.

15.3 Killing Homotopy

Suppose we have a continuous function $f : S^1 \to Y$, we define the *mapping cone* C_f as follows.

We consider the disjoint union $Y \cup (S^1 \times I)$ (literally a cylinder floating above I), and take

$$(Y \cup (S^1 \times I))/\sim$$

where $\sim$ identifies $S^1 \times \{0\}$ to a single point, and $(z, 1)$ with $f(z)$ for each z.

The result is what appears to be a witch's hat.

We now split this into two pieces: let U be a $S^1 \times [0, \frac{1006}{2011}]$; that is, a little more than the upper half of the hat. The second piece V is the union of Y and $S^1 \times [\frac{1004}{2011}, 1]$; the base of the hat plus a little above half the hat.

Claim. The fundamental group of $C_f = U \cap V$ is isomorphic to $\pi_1(V)/\sim$.

Proof. By our theorem,

$$\pi_1(C_f, x_0) = \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0).$$

But $U \sim B^2$ so $\pi_1(U, x_0)$ is trivial.

As for V, we just push down part of the hat, so $\pi_1(V) = \pi_1(Y)$.

Finally, what have we identified when we amalgamated $\pi_(U \cap V, x_0)$? We have $U \cap V$ homotopic to S^1, whose fundamental group is $\mathbb{Z}$, generated by some loop η.

But η is trivial in U. So $\sim$ kills loops that can be deformed from η. $\square$

16 October 17, 2012

16.1 Something

Let Σ_3 denote the permutations of the set $\{1, 2, 3\}$. We know Σ_3 is generated by the transpositions

$$\tau_1 = (2\,3) \quad \tau_2 = (3\,1) \quad \tau_3 = (1\,2).$$

This does not imply that Σ_3 is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$. In fact, $\tau_2\tau_3\tau_2 = \tau_1$ so not all the generators are necessary.

16.2 Torus

Consider the frame of a torus T (the two loops with generate $\pi_1(S^1 \times S^1)$). It is not hard to see that this is a figure-eight.

Denote by U a square patch open in T. Let V be all of T minus some smaller co-centric square patch contained entirely in U. Then $U \cup V = T$, and $U \cap V$ is isomorphic to S^1.

Then $\pi_1(T, x_0) = \pi_1(U \cup V, x_0) \cong [\pi_1(U, x_0) * \pi_1(V, x_0)]/N$ for some N.

Now once again we look at the torus as the orbits of $\mathbb{Z}^2$ acting on $\mathbb{R}^2$ by addition. We can take as representatives the square given by $[-\frac{1}{2}, \frac{1}{2})^2$.

Claim. V is homotopy equivalent to the frame X_8.

Proof. Push out the hole. $\square$

So, we get $\pi_1(V) \cong \langle a, b \rangle = F_2$. But although it's not trivial in the figure eight, we find that $aba^{-1}b^{-1}$ becomes trivial in T.

In other words, in the torus, $ab = ba$.

That is; we find now that $\pi_1(U \cup V, x_0) = \langle a, b \mid ab = ba \rangle \cong \mathbb{Z}^2$.

Actually, consider a torus with a rectangle cut out. We can consider patching it with a *cone pointing outwards*; so it looks like a torus with a spike on it. This is a mapping cone!

16.3 Group Presentations

Hey cool I already know all this!

$$D_8 = \left\langle r, s \mid r^4 = s^2 = 1, rs = s^{-1}r \right\rangle.$$

Now we can consider a free group $\langle \tilde{r}, \tilde{s} \rangle$ and the canonical homomorphism φ to D_8. Denote by K the kernel of φ.[5]

Zorn's Lemma states that any poset such that every totally ordered subset has an upper bound, then there is a maximal element in the whole poset (that is, no element which exceeds it). Once again, this is equivalent to the axiom of choice.

Consider the following subgroup of $\langle a, b \rangle$:

$$\langle bab, babab, bababab, \cdots \rangle$$

This is an infinitely generated subgroup of a finitely generated group.

Definition. If G and K are called finitely generated we call G *finitely presented.*

[5]Completely random sidenote: every vector has a basis if and only if the Axiom of Choice holds.

In our particular example, $G = \langle r, s \rangle$ and $K = \left\langle r^4, s^2, rsr^{-1}s \right\rangle_N$ (where the subscript N includes the rest of the elements that makes the group normal). These are called *relators*; any relation can be converted to a relator by moving everything to one side. We get presentations at this point as well, namely $D_8 = \langle r, s \mid r^4, s^2, rsr^{-1}s \rangle$.

Go back to the figure eight with $\pi_1(X_8) = \langle r, s \rangle$. We will mold this into the dihedral group D_8 as follows:

- Take a disk and map its boundary onto the loop $[r]$ by $z \mapsto z^4$. This forces $r^4 = 1$.
- Take another disk and map its boundary onto the loop $[s]$ by $z \mapsto z^2$. This forces $s^2 = 1$.
- Take a third disk and map it in this super troll way that causes $rsr^{-1}s = 1$.

16.4 For Next Time

Universal covers, and unwrapping spaces (reverse of covering projections) such that each space above a given space X corresponds to as subgroup of $\pi_1(X)$, with the universal subgroup corresponding to the trivial subgroup.

17 October 22, 2012

17.1 Something

$$\begin{tikzcd}
E_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\
& B &
\end{tikzcd}$$

φ is a morphism of covering spaces if the diagram commutes; that is, $p_2 \circ \varphi = p_1$.

In particular, ϕ must take fibers to fibers.

If φ is one-to-one and onto, and φ and φ^{-1} are both continuous, then φ is an "isomorphism".

17.2 van Kampen again

Last time, we did stuff with a torus. If you consider a square with the sides identified (i.e. a torus), we have $aba^{-1}b^{-1} = 1$, but not so with the patch.

Now, let's consider identifying the edges of a square in opposite directions to get a space X.

$$\begin{tikzcd}
\triangle \arrow[r, "a"] & \square \arrow[d, "b"] \\
\square \arrow[u, "b"] & \triangle \arrow[l, "a"]
\end{tikzcd}$$

When we identify the edges a, we get a Möbius strip. At this point, ba is a loop. The Möbius strip deformation retracts onto the "core" circle and thus is homotopy equivalent to S^1. Hence, the fundamental group of the Möbius strip is $\pi_1(M) = \mathbb{Z}$, generated by ba.

What happens when we make the second attachment? First, delete a patch in the center. Now $baba$ is a nontrivial loop which corresponds to 2 in $\pi_1(M)$. But when we fill in the patch, suddenly $baba = 1$.

So let's call U be a large band-aid, and V be X minus a patch. Then,

$$\pi_1(X) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) = \{1\} *_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}_2.$$

17.3 n-Cells

We've actually seen this space X before! All we are doing is identifying antipodal points on the boundary, which is our much earlier space S^2/A, where antipodal points are mapped. Now if we look downwards, we get our space (except circular).

This is also equivalent to applying a mapping cone on S^1, but with the loop we are applying it to being 2. The whole process is called attaching a 2-cell to Y; we are attaching the northern hemisphere of B^2 onto the space Y.

We can generalize this to n-cells by replacing "half of B^2" without "half ofB^n".

In this way, we can build up spaces form the bottom; starting with a bunch of points, we attach 1-cells. Now $\partial B^1 = [-1, 1]$ and S^0 is $\{-1, 1\}$. Hence, adding 1-cells let us add "loops" and "edges". Then we go on to n-cells, etc.

These lead to things called *CW complexes*.

One last example: we can build S^1 by considering two points, adding two 1-cells to create S^1, then two 2-cells to build S^2. Or we can be lazy, so that we start with a point, then just throw on S^2. In other words, give a space, there are multiple ways to produce CW complexes.

17.4 Manifolds

A manifold is a space which locally looks like a flat Euclidean sphere; that is, every point has a neighborhood homeomorphic to Euclidean space. For example, S^2 is a two-dimensional manifold.

These manifolds are called *closed*, meaning it is compact and without boundary.

Definition. Given a closed manifold M a M, a morse function is a "nice" function $f : M \to R$, which is an "altitude function".

We assume that M is smooth (a C^∞ manifold). For example, if $M \subseteq \mathbb{R}^3$, then an altitude function might just be $(x, y, z) \mapsto z$.

Suppose $g(x, y)$ is a function. The gradient $\nabla g = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right\rangle$. A critical point is a point where the gradient is zero. We want our critical points to be *nondegenerate* as well. (e.g. 0 of x^3 is considered degenerate.)

We view our manifold M as a bunch of functions in coordinates, such that at any given altitude, there is at most one critical point. What to do these critical points represent?

Starting at the bottom, each critical point is equivalent to attaching an n-cell.

17.5 Poincaré Conjecture

Theorem 15 (Poincaré Conjecture)**.** *If an n-manifold Σ_n has the same homotopy type of S^n, then Σ_n is homeomorphic to S^n*

$n = 2$ is easy, but $n \geq 5$ was proven in the 60's. The $n = 4$ case is hard and involves "elegant procrastination", but the $n = 3$ case is the hardest.

Gauss-Bonnet Theorem

18 October 24, 2012

18.1 Klein Bottle

One last application of van Kampen:

$$\begin{tikzcd}
\triangle & \triangle \arrow[l, "a"'] \\
\triangle \arrow[u, "b"] \arrow[r, "a"'] & \triangle \arrow[u, "b"']
\end{tikzcd}$$

Etymology: flasche and fiasco are related, but fläche means surface.

Once again, we take punch a hole in the middle of the square. Let V be K minus the hole, and U be a slightly large band-aid. Again, by Seifert van-Kampen, we can get

$$\pi_1(K) = \pi_1 U *_{\pi_1(U\cap V)} \pi_V = 1 *_{\mathbb{Z}} \mathbb{Z} = \left\langle a, b \mid bab^{-1}a \right\rangle.$$

Note: in general, there is no easy way to prove that we've gotten all the relationships, but in our case our $U \cap V$ are all generated by one generator, which makes it easy.

The "word problem" basically states that given a presentation of a group, it is not algorithmically possible to determine whether a word is equal to the identity. It is also not algorithmically possible to determine whether two groups are isomorphic, given their presentations.

18.2 Covering Projections

Now we have diagrams!

$$\begin{tikzcd}
E \arrow[d, "p"'] \\
B
\end{tikzcd}$$

Given two covering projections (E_1, p_1) and (E_2, p_2) onto the same space B, then am orphism between the two is a map φ such that $p_2 \circ \varphi = p_1$ (see last week). Geometrically, φ maps $p_1^{-1}(b_0)$ to $p_2^{-1}(b_0)$.

In other words, the morphisms between covering projections are continuous functions which preserve fibers. Isomorphisms are those which are invertible.

Suppose a "finite" group G acts freely[6] on a space E.

Example. Let $G = (\mathbb{Z}, +)$ and $E = \mathbb{R}$. Let G act on E by $n \cdot x = x + 2\pi n$. This corresponds to the classical covering projection of $\mathbb{R}$ onto S^1, which occurs since $E/G \approx S^1$.

Later we will see that the nicest covering projections are those which can be represented by this group modding.

[6]That is, the stabilizer of each $x \in E$ is trivial. In still other words, $\nexists g \in G \setminus \{1\}, x \in E$ with $g \cdot x = x$.

Again, we have our fundamental lifting correspondence:

$$\begin{tikzcd}
 & \text{path } \gamma & E \arrow[d, "p"] \\
I \arrow[ur] \arrow[r] & {[b_0] \in \pi_1(b)} & B
\end{tikzcd}$$

We will see that with "nice" projections, then either the lifts of a loop $[\gamma] in \pi_1(B)$ are all loops or all non-loops.

18.3 More General Liftings

$$\begin{tikzcd}
 & e_0 \in p^{-1}(b_0) \subseteq E \arrow[d, "p"] \\
(Y, y_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & b_0 \in B
\end{tikzcd}$$

Let $f : Y \to B$ with $b_0 = f(y_0)$. We seek a lifting $\tilde{f}$ of f that takes y_0 to $e_0 \in p^{-1}(b_0)$.

We restrict our attention to connected and locally path connected[7] spaces Y, E, B. (It turns out that spaces which are connected and locally path connected are automatically path-connected).

Theorem 16. *Let $f : (Y, y_0) \to (B, b_0)$ be continuous and consider a covering projection $p : (E, e_0) \to (B, b_0)$. Then a lifting $\tilde{f}$ with $\tilde{f}(y_0) = e_0$ if and only if*

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(E, e_0)).$$

If this lifting exists, it is unique.

Conveniently, $p_*(\pi_1(E, e_0)) \cong \pi_1(E, e_0)$ since we saw earlier that p_* is an injection. In particular, if $Y = I$, then $f_*(\pi_1(Y, y_0))$ is trivial and the inclusion is evident.

Question. If f turns out to be a covering projection, is it the case that $\tilde{f}$ is also a covering projection?

Proof. For the "only if" part, suppose such a lifting $\tilde{f}$ exists. Then $p \circ \tilde{f} = f$. Thus, $f_* = p_* \circ \tilde{f}_*$. Hence,

$$f_* \left(\pi_1(Y, y_0)\right) = p_* \left(f_* \left(\pi_1(Y, y_0)\right)\right) \subseteq p_*(\pi_1(E, e_0))$$

where the last step follows from the obvious $\tilde{f}_* \left(\pi_1(Y, y_0)\right) \subseteq \pi_1(E, e_0)$.

For the other direction, assume $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(E, e_0))$. First, we will show that if $\tilde{f}$ exists, it is unique! (This turns out to be instructive).

Suppose $\tilde{f}_1$ and $\tilde{f}_2$ are both liftings of f that start at e_0. We wish to show $\tilde{f}_1(y_1) = \tilde{f}_2(y_1) \forall y_1 \in Y$. If $y_1 = y_0$ this is obvious. Hence assume $y_1 \neq y_0$ and consider a path α from y_0 to y_1. In that case, $f \circ \alpha$ is a path from $f(y_0) = b_0$ to $f(y_1)$. Lift $f \circ \alpha$ to a path γ starting at e_0; this path is unique. But $p\tilde{\circ}(f_1 \circ \alpha) = f \circ \alpha$, so $\tilde{f}_1 \circ \alpha$ is a lift of $f \circ \alpha$ starting at $\tilde{f}_1(\alpha(0)) = \tilde{f}_1(y_0) = e_0$. Hence, the ending points are certainly the same;

[7] X is locally path-connected if $\forall x \in X$ and U an open neighborhood of x_0, $\exists V$ a path connected open set with $x_0 \in V \subseteq U$. This lets us exclude things like that harmonic fan...

that is, $\gamma(1) = \tilde{f}_1(\alpha(1)) = \tilde{f}_1(y_1)$ (that is, their ending points are the same). Similarly $\tilde{f}_2(y_1) = \gamma(1)$ as well, so we're done.

Anyways, to define $\tilde{f}$ we just use the same thing. That is, if α is a path from y_0 to y_1, then let γ be the lifting of $f \circ \alpha$ and define $\tilde{f}(y_1) = \gamma(1)$. (In practice, it is very hard to clean up this definition of $\tilde{f}$ to something reasonable.)

We need to check that this map is both well-defined (since α was chosen arbitrarily) and that it is continuous. USE THE CONDITION NOW (oops I did not notice that). You need the first condition to get the well-defined part, and the second part is just sort of blah.

To be finished next time... $\square$

The "only if" direction intuitively should be easy: going from topology to algebra tends to lose information, so it is natural to guess that the existence of a lifting is strong, while the algebraic condition is weak.

19 October 29, 2012

19.1 More on Covering Projections

Let B be connected and locally path connected; let E, Y be path connected and Y path-connected.

$$\begin{tikzcd} & e_0 \in P^{-1}(b_0) \subseteq E \arrow[d, "p"] \\ y_0 \in Y \arrow[r, "f"] & b_0 \in B \end{tikzcd}$$

Want to show f lifts to a unique map $\tilde{f}$ with $\tilde{f}(y_0) = e_0$ if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(E, e_0))$.

Idea: connect y_0 to y_1 via α; then $f \circ \alpha$ is a path in B with $f(\alpha(0)) = f(y_0) = b_0$ and $f(\alpha(1)) = f_(y_1) = b_1$. Lift this to a path γ starting at e_0 and define $\tilde{f}(y_1) = \gamma(1)$.

We got up to here last time, and now we need to show this is well-defined and continuous. This is probably a good time to use the condition.

Example. To show what goes wrong without said hypothesis, consider the following:

$$\begin{tikzcd} & E = \mathbb{R} \arrow[d, "p"] & \theta \arrow[d] \\ Y = S^1 \arrow[r, "f = \mathrm{id}"] & B = S^1 & e^{2\pi i\theta} \end{tikzcd}$$

Here, this hypothesis doesn't hold since $f_*(\pi_1(Y, y_0)) \approx \pi_1(Y, y_0) \approx \mathbb{Z}$ and $\pi_1(E, e_0) = \{1\} \Rightarrow p_*(\pi_1(E, e_0)) = \{1\}$. Indeed, this causes the function to not be well-defined: the two paths $\alpha_1, \alpha_2 : [0,1] \to S^1$ from 1 to -1 given by $t \mapsto e^{i\pi t}$ and $t \mapsto e^{i\pi 3t}$ yield different values when lifted.

OK, let's try to prove well-definedness. Consider two paths α and β in Y with the same endpoints, and consider paths $f \circ \alpha$ and $f \circ \beta$ in B from b_0 to b_1. Lift $f \circ \alpha$ and $f \circ \bar{\beta}$ to γ and δ, where $\gamma(1) = \delta(0)$. Also, $\gamma * \delta$ lifts $(f \circ \alpha) * (f \circ \bar{\beta})$, since

$$p(\gamma \circ \delta) = (p \circ \gamma) * (p \circ \delta) = (f \circ \alpha) * (f \circ \bar{\beta}).$$

Now

$$[f \circ (\alpha * \beta)] = f_*([\alpha * \bar{\beta}]) \in f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(E, e_0))$$

so that $f \circ (\alpha * \bar{\beta}) \simeq_{\mathrm{p}} p \circ (\eta)$ where η is some *loop* at e_0.

Hence, η is a lifting of $(f \circ \alpha) * (f \circ \bar{\beta}) = f \circ (\alpha * \bar{\beta})$ which starts at e_0. Now by uniqueness of lifting, then, this is a loop. So $\bar{\delta} = \bar{\bar{\beta}} = \beta$ ends at the same place as γ. Hence, the function is well-defined.

Continuity can be shown from the fact that f is continuous and we have evenly covered neighborhoods (i.e. homeomorphisms on a local scale.)

19.2 Equivalence of Projections

$$
\begin{tikzcd}
\pi_1(E,e_0) \arrow[ddrr, "p_*"] & e_0 \in p^{-1}(b_0) \subseteq E \arrow[rr, "h"] \arrow[dr, "p"] & & e_0' \in (p')^{-1}(b_0) \subseteq E' \arrow[dl, "p'"'] & \pi_1(E',e_0') \arrow[ddll, "(p')_*"'] \\
& & B & & \\
& & \pi_1(B,b_0) & &
\end{tikzcd}
$$

Theorem 17. *There's a unique equivalence h of coverings (E,p) and (E',p') such that $h(e_0)=e_0'$ if and only if*

$$p_*(\pi_1(E,e_0)) = p'_*(\pi_1(E',e_0')).$$

Recall that an equivalence if a homeomorphism h such that $p' \circ h = p$.

Proof. Suppose h exists. Then $p' \circ h = p \Rightarrow (p' \circ h)_* = p_*$; i.e. $p'_* \circ h_* = p_*$. In that case, h is a lifting of p; by our previous theorem we have $p_*(\pi_1(E,e_0)) \subseteq p'_*(\pi_1(E',e_0))$.

$$
\begin{tikzcd}
& E' \arrow[d, "p'"] \\
E \arrow[ur, "h"] \arrow[r, "p"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& E \arrow[d, "p"] \\
E' \arrow[ur, "h^{-1}"] \arrow[r, "p'"] & B
\end{tikzcd}
$$

But now by doing the same thing with h^{-1}, we find $p'_*(\pi_1(E',e_0')) \subseteq p_*(\pi_1(E,e_0))$; this establishes the reverse inclusion and hence the groups in question are equal.

For the reverse direction, we can perform the same trick with double inclusion to find that p lifts to $h : E \to E'$ with $h(e_0)=e_0'$, and p' lifts to $g : E' \to E$ with $g(e_0')=e_0$. We want to show that g and h are inverses, but don't know anything about them yet.

$$
\begin{tikzcd}
& E \arrow[d, "p"] \\
E \arrow[ur, "g \circ h"] \arrow[r, "p"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& E \arrow[d, "p"] \\
E \arrow[ur, "\mathrm{id}_E"] \arrow[r, "p"] & B
\end{tikzcd}
$$

So we look at the map $g \circ h$. Since $(p \circ g \circ h) = (p \circ g) \circ h = p' \circ h = p$, so $g \circ h$ lifts p. But so does id_E. By uniqueness of lifting, we have $g \circ h = \mathrm{id}_E$. Doing the same thing in the other direction gives the conclusion. $\square$

Note that in this theorem, we have specified base points!

Recall:

Fact. If $H_1, H_2 \le G$, then H_2 is called conjugate to H_1 if $\exists g \in G$ such that $H_2 = gH_1g^{-1}$. Since conjugation is an automorphism (the inner automorphism), then $H_1 \cong H_2$ must take place.

Now, consider a single covering projection with two points e_0, e_1.

$$
\begin{tikzcd}
e_0, e_1 \in p^{-1}(b_0) \subseteq E \arrow[d, "p"] \\
b_0 \in B
\end{tikzcd}
$$

Let $H_i = p_*(\pi_1(E, e_i))$, for $i = 0, 1$. If γ is a path from e_0 to e_1 then $\alpha = p \circ \gamma$ is the *loop* from b_0 to itself.

Now, $[\alpha]^{-1} * p_*(\pi_1(E, e_0)) * [\alpha] = p_*(\pi_1(E, e_1))$; indeed,

$$
\begin{aligned}
[\alpha]^{-1} * p_*([\delta]) * [\alpha] &= [\bar{\alpha} * (p \circ \delta) \circ \alpha] \\
&= [(p \circ \bar{\gamma}) * (p \circ \delta) \circ (p \circ \gamma)] \\
&= p_*([\bar{\gamma} \circ \delta * \gamma])
\end{aligned}
$$

In fact, the converse is true: if some subgroup H is conjugate to $p_*(\pi_1(E, e_0))$ in $\pi_1(B, b_0)$, then $\exists e_1 \in E$ and a path α from e_0 to e_1 such that $H = p_*(\pi_1(E, e_1))$.

If (E, p, e_0) and (E', p', e_0') are coverings

$$
\begin{tikzcd}
e_0 \in E \arrow[rr, "h"] \arrow[dr, "p"'] & & e_0' \in E' \arrow[dl, "p'"] \\
& b_0 \in B &
\end{tikzcd}
$$

these coverings are equivalent if and only if $p_*(\pi_1(E, e_0))$ is conjugate to $p'_*(\pi_1(E', e_0'))$. If they are conjugate, then, $\exists e_1' \in E'$ with $p'(e_1') = b_0$ for which $p_*(\pi_1(E, e_0)) = p'_*(\pi_1(E', e_1'))$; actually equal.

Next time: universal covers!

20 Halloween, 2012

Assume all spaces are connected and locally path connected (which implies path-connected and locally path-connected).

Last time, we saw that a lifting of a map $f : Y \to b$ exits if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(Y, y_0))$. Hence, given two coverings of the same space B, there exists an equivalence h with $h(e_0) = e_0'$ iff $p_*(\pi_1(E, e_0)) = p'_*(\pi_1(E', e_0'))$.

$$\begin{tikzcd} & E' \arrow[d, "p'"] \\ E \arrow[ur, "h"] \arrow[r, "p"'] & B \end{tikzcd}$$

Then, there exists a equivalence h, period, if and only if $p_*(\pi_1(E, e_0))$ is conjugate to $p'_*(\pi_1(E_1', e_0'))$ in $\pi_1(B, b_0)$. Here, $e_0 \in E$ and $e_0' \in E$ are both in the fiber of b_0.

20.1 Universal Covers

Definition. Given a projection $E \xrightarrow{p} B$, E is called a universal cover[8] if E is simply connected (1-connected).

In particular, if E_1 and E_2 are both universal covers, then they are equivalent by the above, since $p_*(\pi_1(E_1, e_1)) = p_*(0) = 0$ and similarly for E_2. In particular, E_1 and E_2 are homeomorphic. Hence, up to homeomorphism there is a unique universal covering.

$$\begin{tikzcd} & E_2 \arrow[d, "p_2"] \\ E_1 \arrow[ur] \arrow[r, "p_1"] & B \end{tikzcd}$$

Hence, $\mathbb{R}$ is the universal covering of S^1 since $\mathbb{R}$ is contractible (via $\theta \mapsto e^{2\pi i\theta}$). Similarly, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ covers $S^1 \times S^1 = T$ by $(\theta, \phi) = (e^{2\pi i\theta}, e^{2\pi i\phi})$, so it is also the universal covering. This implies, for example, that S^2 cannot cover the torus because S^2 is also simply connected, but we already have $\mathbb{R}^2$ as the universal covering and $\mathbb{R}^2 \not\approx S^2$.

Why do we call these universal? First, recall the facts:

Fact. Let $r \circ q = p$ be continuous maps by $E_1 \xrightarrow{q} E_2 \xrightarrow{r} B$.

$$\begin{tikzcd} & E_2 \arrow[d, "r"] \\ E_1 \arrow[ur, "q"] \arrow[r, "p"'] & B \end{tikzcd}$$

If p, r are coverings then so is q. If p, q are coverings then so is r.

Claim. If $(\tilde{E}, \tilde{p})$ is a universal covering of B and (E, p) is any covering of B,

[8] Strictly, (E, p).

Proof. Notice that $\tilde{p}_* \pi_1(\tilde{E}, \tilde{e}_0) = \tilde{p}_* 0 = 0 \subseteq p_*(\pi_1(E, e_0))$; this inclusion is strict if E is not a second universal covering (since p_* is trivial). Hence, we must have a lifting ϕ; by the fact above we see that we ϕ is actually a covering projection!

$$\begin{tikzcd}
 & E \arrow[d, "p"] \\
\tilde{E} \arrow[ur, "\phi"] \arrow[r, "\tilde{p}"'] & B
\end{tikzcd}$$

□

20.2 Category Theory

Given a category $\mathcal{C}$, an object $U_r \in \text{ob}(\mathcal{C})$ is called unusually repelling if $\forall A \in \text{ob}(\mathcal{C})$, $\exists \varphi_A \in \text{Hom}(U, A) : U_r \to \varphi_A A$. By uniqueness and such, it is clear that a universally repelling object is unique. Also called an initial object.

Here, our category is the covers (E, p) of B and the morphisms are functions f_{12} such that $p_2 \circ f_{12} = p_1$.

$$\begin{tikzcd}
E_1 \arrow[rr, "f_{12}"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\
 & B &
\end{tikzcd}$$

Now are categories are starting to look more complicated! Here's another example:

Example. Fix groups $\{G_i\}$. The objects of the category are a group and *families of homomorphisms from some external group* H, not fixed. That is, we can view our objects in the form

$$(H, \{H \xrightarrow{\varphi_i} G_i\}).$$

Our morphisms will be homomorphisms Φ_{12} such that $\varphi_{2i} \circ \Phi_{!2} = \varphi_{1i}$ for all i.

$$\begin{tikzcd}
H_1 \arrow[rr, "\Phi_{12}"] \arrow[dr, "\varphi_{1i}"'] & & H_2 \arrow[dl, "\varphi_{2i}"] \\
 & G_i &
\end{tikzcd}$$

The object

$$\left(\prod G_j, \left\{\prod G_j \xrightarrow{\text{pr}_i} G_i\right\}\right),$$

which is the product of all the groups where the homomorphisms are just protections, are universally attractive. Indeed, just take

$$\Phi_{12} : h \to (\varphi_i(h))_i;$$

i.e. just project each thing individually.

20.3 Do Universal Covers Always Exist

Answer: no. But usually.

IF E is a universal cover, then $\forall b \in B \exists U$ a neighborhood such that the inclusion $U \overset{j}{\hookrightarrow} B$ for which $j_* : \pi_1(U, b) \to \pi_1(B, b)$ is trivial. This property is called semi-locally simply connected.

It turns out this is sufficient!

Example (Hawaiian Earring)**.** In $\mathbb{R}^2$ take the subspace topology on the set

$$\bigcup_{n \ge 1} \left\{ (x, y) \mid \left(x - \frac{1}{n} \right)^2 + y^2 = \frac{1}{n^2} \right\}.$$

All these points are tangent at $(0, 0)$. You can show this is continuous and such. Here,

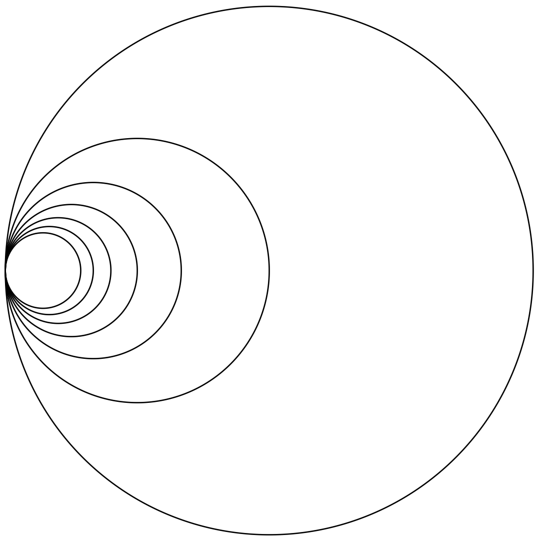

Figure 2: Hawaiian Earing

this is NOT semi-locally simply connected, so there is no universal covering of this space.

You can also look at the higher-dimensional version, where spheres are replaced by circles, then we get some other interesting stuff.

20.4 Subgroups vs Coverings

Suppose a base B is semi-locally simply connected.

$$\begin{tikzcd}
0 = \pi_1(\tilde{E})\ \tilde{E} \arrow[dr, "q"] \arrow[dd, "\tilde{p}"'] & \\
& E \arrow[dl, "p"] \\
B &
\end{tikzcd}$$

$$0 = \tilde{p}_*(\pi_1(\tilde{E})) < p_*(\pi_1(E)) \leq \pi_1(B)$$

Given (E, p) let $\mathcal{C}(E, p, B)$ denote the group of covering transformations of (E, p); that is self-equivalences of the covering (E, p).

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

This is a group under composition; the identity is id_E, et cetera.

Next time: how does $\mathcal{C}(E, p, B)$ act on E?

21 November 5, 2012

21.1 Examples of Group Actions

Definition. Let G be a group. It acts **freely** from the left on X if $\forall x \in X : G_x = \{1\}$. That is, the stabilizer of any $x \in X$ is the trivial subgroup.

Example. $G = (\mathbb{R}, +)$ acts on $X = \mathbb{R}$ by addition.

Definition. Let X/G denote the set of the orbits of elements in X, endowed with the quotient topology.

Our goal is, given a group G and a space X, to get a covering projection $p : X \to X/G$ by $x \mapsto [x]$; that is, we project each element of the space to its image in the group X/G with the quotient topology.

Example. $\mathbb{Z}$ acts on $X = \mathbb{R}$. We get $X/G \approx S^1$. This is really just our standard projection $\theta \mapsto e^{2\pi i\theta}$.

Example. $G = \mathbb{Z}/n\mathbb{Z}$ acts on $X = S^1$ by $m \cdot z = \zeta^m \cdot z$, where $\zeta = e^{\frac{2\pi i}{m}}$. This is a well-defined action but is also a covering projection. Now X/G is just an "interval" of S^1. This corresponds to the covering map $S^1 \to S^1 : z \mapsto z^n$.

Why do we need the group to act freely? Consider the following example with $X = B^2$, where G does not act freely be the stabilizer of 0 is all of G.

Example. Let $X = B^2 = \{z \mid |z| \le 1\}$. Let $G = \mathbb{Z}/n\mathbb{Z}$ act on B^2 by $m \cdot w = \zeta^m \cdot w$. Now B^2/G cannot be identified with some covering projection, because the cardinality of the fiber must be constant; this is not possible here.

This isn't enough to guarantee a covering projection: for example, if $X = \mathbb{R}$ and $G = \mathbb{R}$ acts on X by addition, then X/G is trivial. It is not possible to cover using $\mathbb{R}$ the singleton space.

21.2 Stronger Conditions

Definition. G acts on a space X *properly discontinuously* if $\forall x \in X$, there exists a neighborhood U such that $U \cap gU = \emptyset \forall g \neq 1$.

Intuitively, this means that G acts in such a way that if $1 \neq g \in G$, then elements of X are moved "sufficiently far away". The example $X = G = \mathbb{R}$ does not work, but...

Fact. If G acts on X properly discontinuously, then $X \to X/G$ is a covering projection.

Note. Some book use *freely and properly discontinuously.* However, if an action is properly discontinuous, then it is automatically free (for obvious reasons).

We can start viewing X as topological groups; a group endowed with a topology. For example,

$$\langle \zeta_n \rangle = \mathbb{Z}_n \le \mathbb{R} \cong S^1$$

Other important groups: $\mathrm{SO}(n)$, $\mathrm{GL}_n(\mathbb{R})$. Anyways, on a side note

$$\mathrm{O}(n) \subseteq \mathrm{GL}_n(\mathbb{R}) = \mathrm{O}(n) \times \mathbb{R}^?.$$

21.3 Covering Transformations

Let $G = \mathcal{C}(E, p, B)$ be the group of covering transformations of (E, p, B); that is, it is the set of all homeomorphisms

$$\{h : E \to E \mid p \circ h = p\}.$$

This is a group with identity is id_E.

Claim. G acts on E from the left by $h \cdot e := h(e)$.

Proof. Check that this is an action. To see that it is a free action, consider an $h(e) = e$, so that h is the stabilizer of $e \in E$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ E \arrow[ur, "h"] \arrow[r, "p"'] & B \end{tikzcd}$$

h is a lifting of p, where $h(e) = e$. But by uniqueness of lifting, we immediately obtain $h = \mathrm{id}_E$.

It turns out that this is properly discontinuous as well. Consider some $e \in E$, let $b_0 = p(e)$, take some neighborhood U_{b_0} of b_0. Now $p^{-1}(U_{b_0})$ is a collection of disjoint neighborhoods around $p^{-1}(b_0)$, per even covering. Let U_e be the one around V.

Then, $p(U_e) = U_{b_0}$. If h is some homeomorphism which preserves projection, then $h(U_e)$ must be U_{b_0}. Unless h is the identity, then $h(U_e) \in p^{-1}(U_{b_0})$, and this is a bunch of disjoint sets. $\square$

21.4 Coverings and Such

Again, G denotes $\mathcal{C}(E, p, B)$.

$$\begin{tikzcd} E \arrow[d, "\pi"'] \arrow[dr, "p"] & \\ E/G & B \end{tikzcd}$$

Let $\pi : E \to E/G$ be a group quotient map, and $p : E \to B$ be the standard covering projection. Can $E/G = B$?

Let $\sim_\pi$ be the relation that equates elements of E in the same orbit of π. Let $\sim_p$ be the relation that equates elements in the same fiber of p.

Fact. $e \sim_\pi e' \Rightarrow e \sim_p e'$. The converse is not necessarily true.

Proof. Compute

$$\begin{aligned} e \sim_\pi e' &\Rightarrow e' = ge = g(e) \\ \Rightarrow pp(e') &= p(g(e)) \\ &= p(e) \\ &\Rightarrow e' \sim_p e \\ &\Rightarrow e \sim_p e' \end{aligned}$$

$\square$

22 November 7, 2012

22.1 Groups

We are given a covering projection $p : E \to B$, with $b_0 \in B$ having fiber $F = p^{-1}(b_0)$. Let $G = \pi_1(B, b_0)$ from now on.

1. G acts from the right on F. The action $F \times \pi_1(B, b_0) \to F$ sends $(e, [\gamma])$ to $\tilde{\gamma}(1)$ where $\tilde{\gamma}$ is a path starting at e.[9]

2. This G-action is transitive. (This is obvious.) Therefore, $eG = F$; i.e. the orbit of E is all of F. Furthermore, F is equivalent to the cosets of G_e, the stabilizer of E.

3. $p^{-1}(b_0)$ is essentially equivalent to $p_*(\pi_1(E, e)) \backslash \pi_1(B, b_0)$.

Consider $\mathcal{C}(E, p, B)$ as defined in the previous section. Then, define $\text{Aut}(F)$ to be the set of G-automorphisms of F; that is

$$\text{Aut}(F) = \{\Psi : F \to F \text{ bijective} \mid \Psi(eg) = \Psi(e)g \ \forall g \in G\}.$$

These are both groups under composition. Now we claim these groups are isomorphic! (This implies that $\mathcal{C}(E, p, B)$ is completely determined by what p does to a fiber.)

Claim. $\mathcal{C}(E, p, B) \cong \text{Aut}(F)$.

Proof. Consider $\phi \in \mathcal{C}(E, p, B)$. Now consider some $e \in F = p^{-1}(b_0)$.

$$p(\varphi(e)) = (p \circ \varphi)(e) = p(e) = b_0 \Rightarrow \varphi(e) \in p^{-1}(b_0).$$

Hence, $\varphi|_F$ is a function from F to F.

Therefore we are motivated to construct the isomorphism

$$\Phi : \mathcal{C}(E, p, B) \to \text{Aut}(F) : \varphi \mapsto \varphi|_F.$$

Lots of things to check:

- **Check that** $\varphi|_F \in \text{Aut}(F)$. Since φ is a homeomorphism, it is globally bijective, so immediately $\varphi|_F$, and it is surjective since $\varphi^{-1}|_F$ is also injective (as φ^{-1} is also bijective). Hence $\varphi \mid_F$ is bijective.

  To show that $\varphi|_F$ preserves the right action by $G = \pi_1(B, b_0)$, take some $e \in F$. Then we want to show that $\varphi|_F(e[\gamma]) = (\varphi|_F(e))[\gamma]$. But

  $$\varphi|_F(e[\gamma]) = \varphi(e[\gamma]) = \varphi(\tilde{\gamma}(1)).$$

  But then

  $$p \circ (\tilde{\varphi} \circ \tilde{\gamma}) = p \circ \tilde{\gamma} = \gamma.$$

  So $\varphi \circ \tilde{\gamma}$ is a lifting of γ starting at $\varphi(\tilde{\gamma}(0)) = \varphi(e)$. So, $\varphi(\tilde{\gamma}(1)) = \varphi(e)[\gamma]$ by definition. This establishes the desired equality. Hence, we finally get $\varphi|_F \in \text{Aut}(F)$.

- **Check that** $\varphi|_F$ **is a homomorphism**. This is trivial since the operations are composition in both groups.

[9] Since $p(\tilde{\gamma}(1)) = \gamma(1) = b_0$, we see that this is well-defined in the sense that $\tilde{\gamma}(1) \in F$.

- **Check that Φ is injective**. We can just check that the kernel is trivial. Suppose $\varphi|_F$ is the identity; i.e. in the kernel. Then $\forall e \in F : \varphi|_F(e) = e \Rightarrow \varphi(e) = e$. Then φ is a lifting of p that takes e to e, so it must be the identity.

- **Check that Φ is surjective**. We prove an algebraic lemma: Suppose F is a transitive right G-set. We wish to show for all $e_1, e_2 \in F$, we have the equality $G_{e_1} = G_{e_2}$ if and only if $\exists \phi \in \mathrm{Aut}(F)$ such that $\phi(e_1) = e_2$. (This is purely algebraic).

  First suppose such a ϕ exists. Take some $g \in G_{e_1}$. Then $e_2 = \phi(e_1) = \phi(e_1 g) = \phi(e_1)g = e_2 g$. Therefore, $g \in G_{e_2}$. Hence, $G_{e_1} \subseteq G_{e_2}$. Doing the same thing with ϕ^{-1} yields the reverse inclusion.

  For the other direction, suppose that $G_{e_1} = G_{e_2}$. We wish to construct $\phi \in \mathrm{Aut}(F)$. Select some $e \in F$; by transitivity, $e = e_1\tilde{g}$ for some $\tilde{g}$ in G by transitivity. We select the map

  $$\phi : e_1\tilde{g} = e \mapsto e_2\tilde{g}.$$

  This satisfies the condition $\phi(e_1) = \phi(e_1 \cdot 1) = e_2 \cdot 1 = e_2$. To see that ϕ is well-defined suppose that $e = e_1\hat{g} = e_1\tilde{g}$. Then $\hat{g}\tilde{g}^{-1} \in G_{e_1} = G_{e_2}$. Thusly $e_2\hat{g}\tilde{g}^{-1} = e_2$ which implies $e_2\hat{g} = e_2\tilde{g}$ and hence ϕ is well defined.

  Let us now check injectivity; suppose $\phi(e) = \phi(e')$ and $e = e_1\tilde{g}$, $e' = e_1\tilde{g}'$. Evidently $e_2\tilde{g} = e_2\tilde{g}'$. Then $e_2\tilde{g}(\tilde{g}')^{-1} = e_2$, so $\tilde{g}(\tilde{g}')^{-1} \in G_{e_2} = G_{e_1}$ which forces $e = e'$.

  glosses over remaining details

  We can now use the lemma to show Φ is onto. Let $\phi \in \mathrm{Aut}(F)$, take $e \in F$, and let $e' = \phi(e)$. Now $G_e = G_{e'}$. Recalling that $G = \pi_1(B, b_0)$ we see that

  $$G_e = p_*\pi_1(E, e) \qquad G_{e'} = p * \pi_1(E, e')$$

  So $p_*\pi_1(E, e) = p_*\pi_1(E, e')$.

  $$\begin{tikzcd} & E \arrow[d, "p"] \\ E \arrow[ur, "\varphi"] \arrow[r, "p"'] & B \end{tikzcd}$$

  By our general lifting criterion, we see that $\exists\varphi : (E, e) \to (E, e')$ an equivalence.

  $\square$

It turns out later that $\mathrm{Aut}(F) = N_{G_e}(G)/G_e$, where $N_{G_e}(G)$ is the normalizer of G_e in G. In the case of covering projections, we know that $G_e = p_*\pi_1(E, e_e)$ and $G = \pi_1(B, b_0)$ and we can compute stuff. In particular, if $\pi_1(E) = 1$ then $\mathrm{Aut}(F) = \pi_1(B, b_0)$.

23 November 14, 2012

23.1 Groups of Covering Transformations

We discussed a group of covering transformations $\mathcal{C}(E, p, B) \cong \text{Aut}(F)$, where F is the fiber of a $b_0 \in B$ with a covering projection $E \xrightarrow{p} B$. Here $\text{Aut}(F)$ consists of the permutations of F that preserve the right action of $\pi_1(B, b_0)$.

If F is a transitive right G-set (i.e. E is connected) and $e_0 \in F$, then F is equivalent (as a G-set) to the set of right cosets $G_{e_0} \backslash G$, where G_x denotes the stabilizer of x.

Now, the new claim is:

Claim. If $G = \pi_1(B, b_0)$ we have

$$\text{Aut}(F) \cong N_G(G_{e_0})/G_{e_0}.$$

Here $G_{e_0} = p_* \pi_1(E, e_0)$ is the stabilizer of the element e_0 and $N_G(G_{e_0}) = \{g \in G \mid gG_{e_0}g^{-1} = G_{e_0}\}$ is the normalizer of G_{e_0} in G.

Note that in general

$$N_G(G_{e_0}) \neq \{g \in G \mid gG_{e_0}g^{-1} \subseteq G_{e_0}\}$$

although it is certainly true for finite groups. Also, recall the facts (i) $H \le N_G(H) \le G$ for any $H \le G$, (ii) $H \trianglelefteq G \Leftrightarrow N_G(H) = G$, and (iii) $H \trianglelefteq N_G(H)$

Proof. We will do this completely in the context of group theory; let $F = G_e \backslash G$ where G_{e_0} is the stabilizer of some $e_0 \in F$ and suppose G acts transitively on F. Consider some $\phi \in \text{Aut}(F) = \text{Aut}(G_e \backslash G)$ mapping $Hg \mapsto \phi(Hg) = \phi(H)g$. Notice that Φ is completely determined by any H. So, for any automorphism ϕ, we let g_ϕ denote a representative of the coset $\phi(H1_G) = \phi(H)$.

Hence, we are motivated to construct the automorphism $\Gamma : \text{Aut}(F) \to N_G(G_{e_0})/G_{e_0}$ by

$$\phi \mapsto Hg_\phi.$$

Okay, details...

- First we claim $g_\phi \in N_G(G_{e_0})$. We have

  $$Hg_\phi = \phi(H) = \phi(Hh) = \phi(H)h \; \forall h \in H.$$

  So, $H = Hg_\phi h g_\phi^{-1}$ which implies $g_\phi h g_\phi^{-1} \in H \; \forall h \in H$. Thus $g_\phi H g_\phi^{-1} \subseteq H$... which is not enough. To establish the reverse inclusion observe that because $H = \phi^{-1}(\phi(H)) = \phi^{-1}(H)g_\phi = (H_{g_{\phi^{-1}}})g_\phi \Rightarrow g_\phi^{-1} g_\phi \in H$. Do some blah.

- Next, we claim that $\Gamma(\phi_1\phi_2) = \phi_1\phi_2(H)$, so that Γ is a homomorphism. Compute

  $$\phi_1\phi_2(H) = \phi_1(Hg_{\phi_2}) = \phi_1(H)g_{\phi_2} = (Hg_{\phi_1})g_{\phi_2} = H(g_{\phi_1}g_{\phi_2}) = H_{g_{\phi_1}}H_{g_{\phi_2}} = \Gamma(\phi_1)\Gamma(\phi_2).$$

- To get that Γ is injective, notice that $\Gamma(\phi) = \phi(H) = H$. Then $\phi(Hg) = \phi(H)g = Hg$ for every g, so ϕ is the identity, and hence $\ker \Gamma = \{1\}$ and we win.

- To demonstrate that Γ is onto, consider $H\hat{g} \in N_G(G_{e_0})/G_{e_0}$ and define ϕ by $\phi(Hg) = \phi(H)g = H\hat{g}g$. Then trivially $\Gamma(\phi) = H\hat{g}$. Do blah.

$\square$

Returning to the context of topology we have

$$\mathcal{C}(E,p,B) \cong N_{\pi_1(B,b_0)}(p_*\pi_1(E,e_0))/p_*\pi_1(E,e_0).$$

In particular, $p_*\pi_1(E,e_0) \trianglelefteq \pi_1(B,b_0)$ if and only if

$$\mathcal{C}(E,p,B) \cong \pi_1(B,b_0)/p_*\pi_1(B,b_0).$$

In this case, we call the covering *regular*.

23.2 A Diagram

We have that $\mathcal{C}(E,p,B)$ acts properly discontinuously on E.

$$\begin{tikzcd} E \arrow[d, "\pi"'] \arrow[dr, "p"] & \\ E/\mathcal{C}(E,p,B) \arrow[r, dotted] & B \end{tikzcd}$$

Last time we showed that $\sim_1$ and $\sim_2$ defined by $e \sim_1 e' \Leftrightarrow \exists \phi \in \mathcal{C}(E,p,B) : \phi(e) = e'$ and $e \sim_2 e' \Leftrightarrow p(e) = p(e'0)$ satisfy $e \sim_1 e' \Rightarrow e \sim_2 e'$. Hence we get an induced map $E/\mathcal{C}(E,p,B) \to B$.

When are $\sim_1$ and $\sim_2$? This occurs iff $e \sim_2 e' \Rightarrow e \sim_1 e'$; that is $p(e) = p(e') \Rightarrow \exists \phi : \phi(e) = e'$. But the former condition is equivalent to e, e' being in the same fiber. In English (?), that means that for every e, e' in the same fiber one can find a covering transformation taking one to the other.

Such a ϕ is a lifting of p.

$$\begin{tikzcd} E & E \arrow[d, "p"] \arrow[dl] \\ E \arrow[r, "p"] \arrow[ur, "\phi"] & B \end{tikzcd}$$

Hence, we can find a ϕ if and only if $p_*\pi_1(E,e) = p_*\pi_1(E,e')$; yet $p_*\pi_1(E,e') = [\gamma] * p_*\pi_1(E,e)[\bar{\gamma}]$ for every $[\gamma] \in \pi_1(B,b)$.

Yet this is normality.

In conclusion, $p : E \to B$ is effectively an orbit projection if and only if it is a regular covering.

23.3 Universal Coverings

Suppose E is simply connected; that is E is the universal cover of B. Then

$$\mathcal{C}(E,p,B) \cong \pi_1(B,b_0)$$

since $p_*\pi_1(E,e)$ is trivial and hence normal in everything. Thus, universal coverings are regular.

$$\begin{tikzcd} E \arrow[d] \arrow[dr, "p"] & \\ E/H \arrow[r, dotted] & B = E/G \end{tikzcd}$$

Furthermore, suppose that $H \le \pi_1(B, b_0) = \mathcal{C}(E, p, B)$. Now $\pi_1(E/H) = H$!

We will also get an induced map from E/H to B that turns out to be a covering projection, although not necessarily regular.

23.4 Example

Let $E = \mathbb{R}^2$ and $G = \mathbb{Z}\times\mathbb{Z}$, where G acts on E in the usual manner giving $E/G = S^1 \times S^1$. Now $\pi_1(S^1 \times S^1) = \pi_1(S^1) \times \imath_1(S^1) = \mathbb{Z} \times \mathbb{Z}$.

Consider $\mathbb{Z} \times 2\mathbb{Z} \le \pi_1(S^1 \times S^1)$.

Then $E/(\mathbb{Z} \times 2\mathbb{Z})$ we get another torus which is "not as tightly wrapped". Now our induced map is a covering projection from a torus to a torus.

24 November 19, 2012

24.1 Manifolds

Manifolds are spaces which locally look like Euclidean space. For example, S^1 is a manifold since sufficiently small neighborhoods are homeomorphic to $\mathbb{R}^1$.

A n-manifold with boundary is a space such that every point has either a neighborhood homeomorphic to $\mathbb{R}^n$ or has a neighborhood homeomorphic to

$$\mathbb{R}^n_+ := \{(x_1, x_2, \cdots, x_n) \mid x_n \geq 0\}.$$

It turns out that the boundary of a manifold with boundary is an $n-1$ manifold with boundary.

24.2 Groups acting dis-continuously

Here, we consider left actions. Recall

Definition. A group G acts *properly discontinuously* on a space X if for every $x \in X$ there exists an open neighborhood U of x such that $U \cap gU = \emptyset$ for every $g \neq 1_G$ in G.

If G acts properly discontinuously on E, then the projection E onto its orbit space is a covering projection.[10] The proof was discussed in an earlier section which I am too lazy to look up; the basic idea is that one takes a small neighborhood around each of the elements in the fiber, which is possible because the action is properly discontinuous.

Claim. The group of covering transformations of this projection is precisely G; that is,

$$\mathcal{C}(E, p, E/G) = G.$$

Proof. Define the map $\lambda_g : E \to E$ by $e \mapsto ge$ for every $g \in G$, and then construct the map $\tau : G \to \mathcal{C}(E, p, E/G)$ by $g \mapsto \lambda_g$. We claim this is an isomorphism. Clearly $\lambda_g \in \mathcal{C}(E, p, E/G)$ by construction. □

24.3 An Example of a Semidirect product

Let $G = \mathbb{Z} \rtimes_\phi \mathbb{Z}$, where $\phi : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z})$ is given by $1 \mapsto (n \mapsto -n)$.

Then, our group operation is simply

$$(h_1, k_1) \cdot (h_2, k_2) = \left(h_1 + (-1)^{k_1} h_2, k_1 + k_2\right).$$

We observe that $(m, n) = (m, 0)(0, n) = (1, 0)^m (0, 1)^n$. Now define $a = (1, 0)$ and $b = (0, 1)$; these generate the group. Now, check that $bab^{-1} = a^{-1}$.

Hence, we may write

$$G = \left\langle a, b \mid bab^{-1} = a^{-1} \right\rangle.$$

24.4 The Action

Now let G act on $E = \mathbb{R}^2$ by

$$(n, m)(x, y) = \begin{cases} (x + n, y + m) & m \text{ even} \\ (1 - x + n, y + m) & m \text{ odd} \end{cases}$$

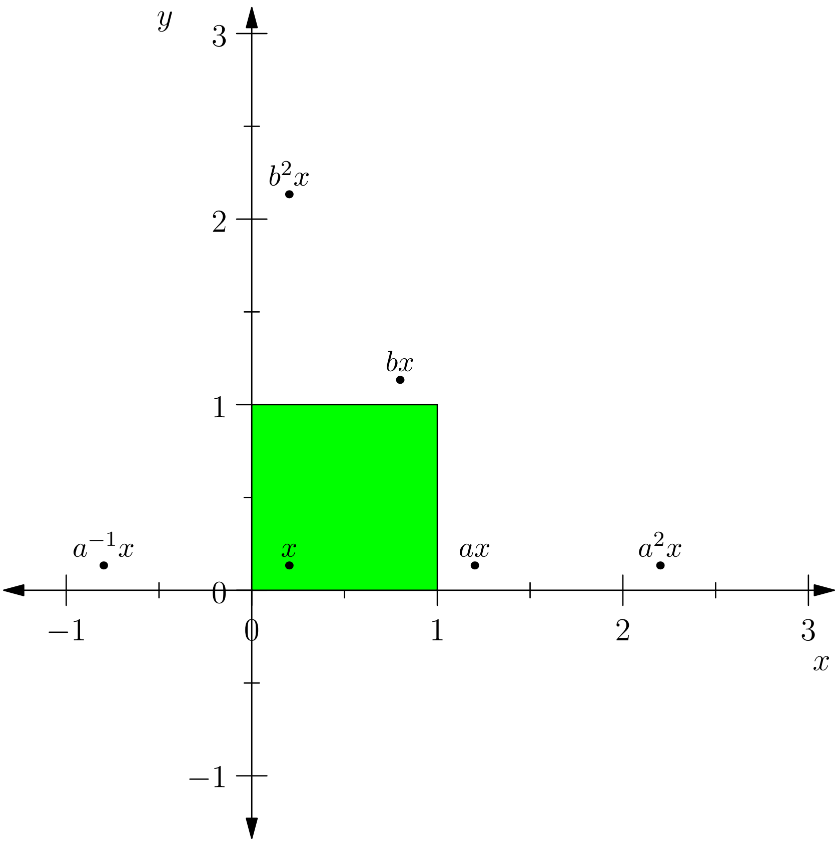

Figure 3: Diagram of group action

It is easy to see that

$$\begin{aligned} a(x,y) &= (x+1,y) \\ b(x,y) &= (1-x,y+1) \end{aligned}$$

Now this action is properly discontinuous. Hence, we get a regular covering $p : E \to E/G$. Then $G = \mathbb{Z} \rtimes \mathbb{Z} = \mathcal{C}(E,p,E/G) = \pi_1(E/G)/p_*\pi_1(E)$. But $\pi_1(E)$ is trivial and we obtain $\pi_1(E/G) = G$.

Anyways, everything is equivalent to something in the green square, so that:

Definition. A *fundamental domain* is a set containing exactly one representative of each equivalence class.

Now a fundamental domain of our group is our unit square with the left and right edges identified nicely, and the top/bottom twisted. This is a **Klein Bottle**.

$$\begin{tikzcd} \square & \square \arrow[l] \\ \square \arrow[u] \arrow[r] & \square \arrow[u] \end{tikzcd}$$

Therefore, the fundamental group of a Klein bottle is $\mathbb{Z} \rtimes \mathbb{Z}$. We've seen this earlier with the van Kampen Theorem, when the condition $\beta\alpha\beta^{-1}\alpha = 1$ was added by the patching.

[10] This is also a regular covering projection; it's equivalent to $p_*\pi_1(E) \trianglelefteq E/G$.

25 November 21, 2012

25.1 Subgroups and Snowmen

Check that groups are nonempty. That is all.

Also, snowmen are topologically equivalent to B^3, but it does not have an optimal volume-per-surface-area ratio, and hence melts.

25.2 Unoriented Cobordism and Manifolds

Let M_1 and M_2 be closed n-manifolds. We write $M_1 \sim M_2$ if the disjoint union $M_1 \sqcup M_2 = \partial W$ for some space W; in English, the two manifolds can be joined in such a way that they are the boundaries of an $n+1$ manifold.

This equivalence relation is called *bordism.*

If we make the convention that $\emptyset$ is an n-manifold for every n, then we can define $\mathcal{N}_k$ to be the equivalence classes of k-manifolds.

We now have

$$\mathcal{N}_0 = \{\emptyset, \{1\}\}.$$

because a 0-manifold with k points is cobordant to a 0-manifold with $k+2$ points, but $\emptyset$ and $\{1\}$ are not cobordant.

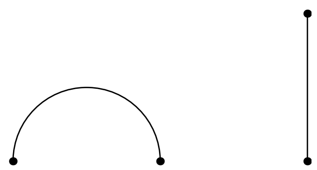

Figure 4: The three-element set is cobordant to the one-element set.

Now the only closed manifold of dimension one is S^1; hence $\mathcal{N}_1 = \{\emptyset\}$.

Now we can define an addition by $[M_1] + [M_2] = [M_1 \sqcup M_2]$ which will give us a group action with identity $[\emptyset]$. The inverses are themselves: check that $[M] + [M] = [\emptyset]$.

Fun fact: since the Cartesian product of an m-manifold and n-manifold is an $m+n$-manifold, with some work we can get our structure to become a *graded ring.*

25.3 Bottles Again

Last time, we had a group

$$G = \mathbb{Z} \rtimes \mathbb{Z} = \left\langle a, b \mid bab^{-1} = a^{-1} \right\rangle$$

defined by

$$(m,n)(k,\ell) = (m + (-1)^n k, n + \ell).$$

Our group action gave a map $\mathbb{R}^2 \to \mathbb{R}^2/G$ from $\mathbb{R}^2$ to its orbit space, which gave a regular covering projection from $\mathbb{R}^2$ to the Klein Bottle K.

Again, as we saw last time, since this is a regular covering we get $\pi_1(K) = G = \mathbb{Z} \rtimes \mathbb{Z}$.

Now, suppose $H \le G$. Then H acts on $\mathbb{R}^2$ as well, and we can get mod out again:

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[d] \arrow[dr] & \\
\mathbb{R}^2/H \arrow[r, dotted] & \mathbb{R}^2/G = K
\end{tikzcd}$$

Obviously, two things equivalent under the H action are equivalent under the G action, but not vice versa. Hence, the equivalence classes of H are "finer" than those of G. In still other words, there is less collapsing.

Now we get a covering projection of $\mathbb{R}^2/H$ onto $\mathbb{R}^2/G$. Note, however, that if H is nontrivial then this cannot be regular.

25.4 Example

Let

$$H_{12} = \mathbb{Z} \rtimes 2\mathbb{Z} = \{(n, 2m) \mid n, m \in \mathbb{Z}\} \leq G.$$

What does our action look like?

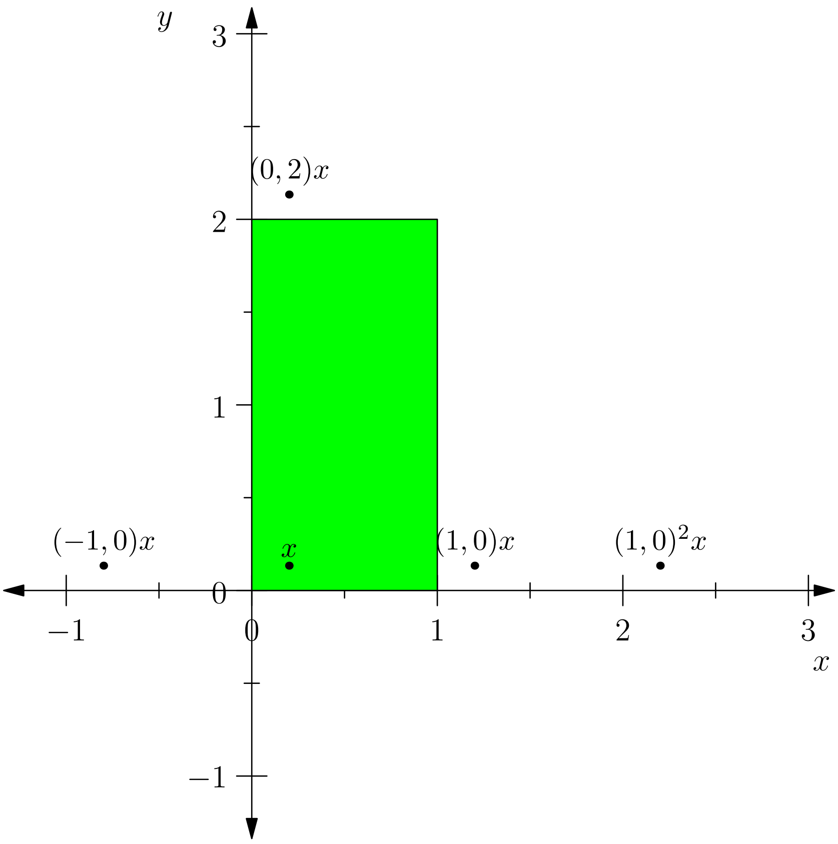

Figure 5: H_{12} acts on $\mathbb{R}^2$.

This time, there's no "twist": so the result is that $\mathbb{R}^2/H_{12} \approx T$, the torus. Now we can rewrite our diagram earlier:

$$\begin{tikzcd}
 & \text{Torus} \arrow[d] \\
\mathbb{R}^2 \arrow[ur] \arrow[r] & \text{Klein Bottle}
\end{tikzcd}$$

Actually, $H_{12} \cong \mathbb{Z} \times 2\mathbb{Z}$ if we just check that the group multiplication is "normal". This is consistent with the fact that the fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$.

Now, consider the covering we have obtained, $q : T \to K$. Then

$$\mathcal{C}(T, q, K) = N_{\pi_1(K} q_* \pi_1(T) / q_* \pi_1(T)$$

But $q_*\pi_1(T)$ is precisely $\mathbb{Z} \rtimes 2\mathbb{Z}$ with index 2 in $\mathbb{Z} \times \mathbb{Z}$. Since index 2 subgroups are always normal, and with some easy computations we get that the group of covering transformations in $\mathbb{Z}/2\mathbb{Z}$.

25.5 The other way around

Let $H_{21} = 2\mathbb{Z} \rtimes \mathbb{Z} \le G$ now. What is the new picture?

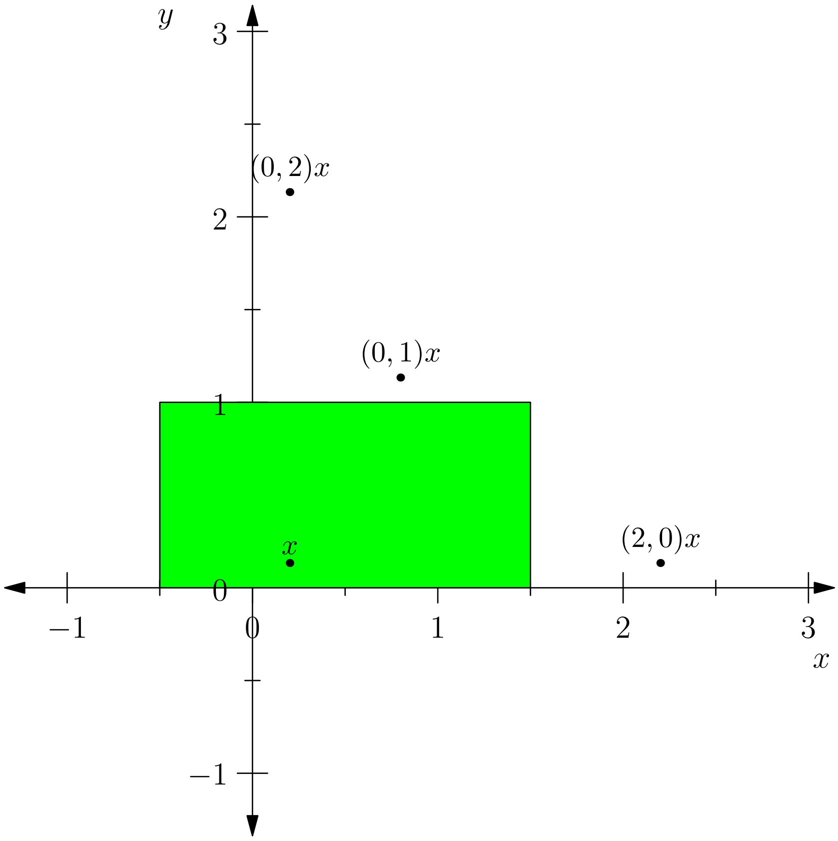

Figure 6: H_{21} this time

Note that the $[0,2] \times [0,1]$ doesn't work here, since $(2,0) \sim (-1,1)$ which doesn't get mapped to anything on the line $y = 1$ within that rectangle.

And... we get a Klein bottle. Hence the Klein bottle covers itself. In fact, if we use $H_{33} = 3\mathbb{Z} \rtimes 3\mathbb{Z}$ we get the Klein bottle once again. Unfortunately, H_{33} is not normal in $\mathbb{Z} \rtimes \mathbb{Z}$.

25.6 GAP

Just something to Google.

26 November 26, 2012

Once again we begin with $G = \mathbb{Z} \rtimes \mathbb{Z}$ as before acting on $X = \mathbb{R}^2$. If $H \le G$, then H acts on $\mathbb{R}^2$. Again, we get a covering from $\mathbb{R}^2/H$ to $\mathbb{R}^2/G$.

Question. When is such a covering regular?

Example. If $H = 3\mathbb{Z} \rtimes 3\mathbb{Z}$, do we have $H \trianglelefteq G$?

Solution. This is equivalent to $ghg^{-1} \in H \forall g \in G$. It suffices to check that this is true when g is a generator of G. Compute

$$(1,0)(3s,3t)(1,0)^{-1} = (3s+1+(-1)^{3t}, 3t) = (3s+1+(-1)^t, 3t).$$

Unfortunately, this is false for t odd. Hence it is not the case that $H \trianglelefteq G$. □

Returning to our example, suppose $H \trianglelefteq N \le G$. Now

$$[G : H] = [G : N][N : H].$$

We have in fact seen that $[G : N] \neq 1$. Hence, we have two possibilities:

- $[G : N] = [N : H] = 3$, in which case G/N and N/H are both $\mathbb{Z}_3$.
- $[G : N] = 9$ and $[N : H] = 1$ in which case N/H is trivial.

What we care about is the group of covering transformations, which is N/H is the stuff above. We also know that the covering $\mathbb{R}^2/H \xrightarrow{\hat{p}} \mathbb{R}^2/G$ is not regular because it is not the case that H is normal in G. (Recall that $E \to B$ is regular iff $p_*\pi_1(E) \trianglelefteq \pi_1(B)$, but $\pi_1(X/G) = G$.)

OK, so let us try to compute $N_G(H)$. If $(m,n) \in N$ then $g(3s,3t)g^{-1}$. Computing,

$$(m,n)(3s,3t)(m,n)^{-1} = \left((-1)^n \cdot 3s + m(1-(-1)^{2n+3t}), 3t\right).$$

This lies in $3\mathbb{Z} \rtimes 3\mathbb{Z}$ if and only if $m(1-(-1)^t)$ is divisible by 3. Since this must hold for all t, this is true when $m \in 3\mathbb{Z}$.

We then conclude that $N_G(H)$ is $3\mathbb{Z} \rtimes \mathbb{Z}$, by our index argument.

26.1 Pictures

Let $H = 3\mathbb{Z} \times 3\mathbb{Z}$.

The fundamental domain is $[-1,2) \times [0,3)$. This is obvious. Both yield Klein bottles.

26.2 A Characterization of Regular Coverings

Suppose $E \xrightarrow{p} B$. We are not assuming that E is simply connected. This covering p is regular if and only if for each individually $[\gamma] \in \pi_1(B)$, either all lifts of γ to E are open[11], or they are all closed.

For example, in our example above, we can take a "loop" α from $(0,0)$ to $(0,1)$ in $\mathbb{R}^2/G$ with fundamental domain $[0,1)^2$.

Then, a lifting of α^3 can be viewed as a path from $(-1,0)$ to $(-1,3)$ and also one from $(0,0) = (1,3)$ to $(0,3)$. The latter is closed but the former is open.

[11] i.e. $\tilde{\gamma}(0) \neq \tilde{\gamma}(1)$.

We can also show generally that

$$\mathbb{Z} \rtimes \ell\mathbb{Z} \trianglelefteq \mathbb{Z} \rtimes \mathbb{Z}$$

by showing that the covering of $G/(\mathbb{Z} \rtimes \ell\mathbb{Z})$ onto the Klein bottle is regular.

How de prove this claim? One direction is straightforward. Suppose $p_*\pi_1(E) \trianglelefteq \pi_1(B)$ so that $\mathcal{C}(E, p, B) = \pi_1(B)/p_*\pi_1(E)$.

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

Take $[\varphi] \in \pi_1(B)$. Lift it to $\tilde{\gamma}$ starting at e_0. Then every $\varphi \circ \tilde{\gamma}$ is a lift of γ if $\varphi \in \mathcal{C}(E, p, B)$. Now every $\varphi \circ \tilde{\gamma}$ is also a lift of γ if $\varphi \in \mathcal{C}(\mathbb{Z}, p, B)$. This turns out to be all the possible liftings; some technical details omitted here.

Conversely, consider the case where $\forall[\gamma]$ all lifts are open or all are closed. We wish to show

$$p_*\pi_1(E, e_0) \trianglelefteq \pi_1(B, b_0). \Leftrightarrow [\alpha] * p_*\pi_1(E, e_0) * [\alpha]^{-1} = p_*\pi_1(E, e_0).$$

But this is equivalent to $p_*\pi_1(E, e_1) = p_*\pi_1(E, e_0)$.

27 November 28, 2012

Definition. The order of a covering is the number of elements above the fiber of any given element. This is well-defined.

If G is a group then the covering $E \to E/G = B$ is the order of G.

Given a fundamental domain D, we can act on it by the elements of G, which will construct a "tessellation" of our space X.

27.1 Fractional Linear Transformations

Consider a function $F(z) = \frac{az+b}{cz+d}$. This is either called a *fractional linear transformation* or *linear fractional transformation*. We assume $ad - bc \neq 0$.

Then, we can define a group $\mathcal{G}$ functions, and this is a group under transformation. Remarkably, the "product" corresponds to exactly the matrix multiplication; that is, the canonical bijection between $\mathcal{G}$ and $M_2(\mathbb{C})$ respects multiplication.

However, $\mathcal{G} \neq GL_2(\mathbb{C})$ since a function can have multiple representations (indeed, scale the coefficients). This ends up being:

$$\mathcal{G} \cong GL_2(\mathbb{C})/\{I_2, -I_2\}.$$

where I_2 is the 2×2 identity.

Now it is not hard to show that $GL_2(\mathbb{C}) = \det^{-1}(\mathbb{C} \setminus \{0\})$ is a topological group. Now, $GL_2(\mathbb{Z})$ is a discrete subgroup of $GL_2(\mathbb{C})$.

The modular group is the subgroup of $\mathcal{G}$ with the addition stipulation that $ad - bc = 1$ and $a, b, c, d \in \mathbb{Z}$. Now look at its fundamental domain is given at `http://en.wikipedia.org/wiki/Fundamental_domain#Fundamental_domain_for_the_modular_group`.

28 December 3, 2012

Recall the definition of a semidirect product.

$$\begin{tikzcd} \mathbb{R}^2 \arrow[d] \arrow[dr] & \\ K = \mathbb{R}^2/G & \mathbb{R}^2/H \arrow[l] \end{tikzcd}$$

Same setup as before, where $H \le G$.

28.1 An Aside

Actually, as a curiosity

$$\mathbb{Z} \rtimes \mathbb{Z} = \left\langle a, b \mid bab^{-1} = a^{-1} \right\rangle \cong \left\langle \alpha, \beta \mid \alpha^2 = \beta^2 \right\rangle.$$

The bijection is $\alpha = ab$ and $\beta = b$.

It thus turns out that $K = \mathbb{R}P^2 \# \mathbb{R}P^2$, where we are taking the connected sum. Now we can also use the Seifert van-Kampen theorem to get

$$\pi_1(\mathbb{R}P^2 \# \mathbb{R}P^2) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) = \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}.$$

Here $U = V = \mathbb{R}P^2$. This is where we get the relation $\alpha^2\beta^{-2} = 1$.

Sidenote : isomorphism problem in combinatorial group theory. Word problem also.

28.2 Galois and Intermediate Covers

Let B be a "nice" space with a universal cover $\tilde{E}$.

$$\begin{tikzcd} \tilde{E} \arrow[dd, "\tilde{p}"'] \arrow[dr, "r"] & \\ & \tilde{E}/H \arrow[dl, "\nu"] \\ \tilde{E}/G = B & \end{tikzcd}$$

We have $\pi_1(B) \cong G = \mathcal{C}(\tilde{E}, \tilde{p}, B)$. Now G acts on $\tilde{E}$ to yield B, et cetera.

Let $H \le G$. Then H acts on $\tilde{E}$, and $\tilde{E} \xrightarrow{r} \tilde{E}/H$ is a covering transformation as well.

So, we find a get a map from the subgroups of G to the quotients of $\tilde{E}$ by

$$H \mapsto (\tilde{E}, r, \tilde{E}/H).$$

At the extremes $\{1\} \mapsto E$ and $G \mapsto E/G = B$. On the other hand, any covering $Y \to B$ can be recovered[12] as $\tilde{E}/H$ for some H.

[12] No pun intended.

Now we get

$$\begin{tikzcd} \tilde{E} \arrow[r, "r"] \arrow[d] & Y \arrow[dl, "q"] \\ B & \end{tikzcd}$$

We claim that $\mathcal{C}(\tilde{E}, r, Y) \le \mathcal{C}(\tilde{E}, \tilde{p}, B)$. Suppose that

$$\varphi \in \mathcal{C}(\tilde{E}, r, Y) \Rightarrow \varphi : \tilde{E} \to \tilde{E} \text{ has } r \circ \varphi = r.$$

We wish to show that $\tilde{p} \circ \varphi = \tilde{p}$. But this follows by $p = q \circ r$.

$$\begin{tikzcd} \tilde{E} \arrow[rr, "\varphi \in \mathcal{C}(\tilde{E}, r, Y)"] \arrow[dr, "r"'] & & \tilde{E} \arrow[dl, "r"] \\ & Y \arrow[d, "q"] & \\ & B & \end{tikzcd}$$

Hence, with a few more details we obtain

Fact. For "nice" B, the spaces which cover B correspond precisely to the subgroups of $G = \mathcal{C}(\tilde{E}, \tilde{p}, B)$.

How does this correspond to Galois theory? Let k be a subfield of K and suppose

$$G = \mathrm{Aut}_k(K) = \{\varphi : K \to K \mid \varphi \in \mathrm{Aut}(K), \varphi \mid_k = \mathrm{id}_k\}.$$

For every subgroup $\{1\} \le H \le G$ define K^H to be the fixed field of H, defined by

$$K^H = \{k \in K \mid \sigma(x) = x \ \forall \sigma \in H\}.$$

Clearly, $k \subseteq K^G$ and $G_L = \{\sigma \in G : \sigma \mid_L = \mathrm{id}_L\} \le G$

28.3 Pullback Diagrams

$$\begin{tikzcd} f^! E \arrow[r, dotted, "pr_2"] \arrow[d, dotted, "pr_1"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

Given a covering $p : E \to B$ and a map $f : B' \to B$, we define

$$f^! E = \{(x, e) \in B' \times E \mid f(x) = p(e)\}.$$

Then we get a canonical projection

$$pr_1(x, e) = x \text{ and } pr_2(x, e) = e.$$

It turns out that pr_1 is a covering. For an $x_0 \in B'$, the fibre of x in $f^! E$ is

$$pr^{-1}(x_0) = \{(\tilde{x}_0, e) \mid pr_1(\tilde{x}, e) = x; p(e) = f(x)\} = \{(x_0, e) \mid p(e) = f(x_0)\} = \{x_0\} \times p^{-1}(f(x)).$$

so the fibers of pr correspond canonically to the fibers of p.

Now we can show that if f_1 and f_2 are homotopic, then the corresponding pr_1 are "equivalent". And more to come...

29 December 5, 2012

If $n < m$ then any map from $S^n \to S^m$ if $n < m$ since it is not surjective, and then one can take the missed point and expand it to something else. However, $S^n \to S^m$ is often not nulhomotopic if $n \geq m > 1$.

29.1 Fiber Bundles

Definition. A map from $p : E \to B$ is called a *fibre bundle* with structure group G if for every $b \in B$ there exits a neighborhood U_b of b such that $\exists h : p^{-1}(U_b) \to U \times F$ such that (i) h is a homeomorphism (ii) $h(e) = (p(e), h_2(E))$

Example. Locally the Möbius Strip looks like $U \times [-1, 1]$

$$\begin{tikzcd} E' \arrow[r, "q"] \arrow[d, "p'"'] & E \\ B & \end{tikzcd}$$

q is called a bundle map if $p \circ q = p'$; that is q "preserves" fibers.

If $b_0 \in B$ then
$$q\left(\left(p'\right)^{-1}(b_0)\right) \subseteq p^{-1}(b_0).$$

29.1.1 Vector Bundles

Tangent bundles. Some very small subset of the tangent bundles have trivial tangent bundles.

Also disk bundles.

29.1.2 Covering Projections

A covering projection is a fiber bundle where the bundle projection is actually a local homeomorphism. In particular, the fiber is discrete.

Consider $b_0 \in U, V$ where U, V are some subsets of B. Suppose we have $p : E \to B$.

$$\begin{tikzcd} (V \cap U) \times F \arrow[dr, "pr_1^{(V)}"'] & p^{-1}(V \cap U) \arrow[l, "h_V"'] \arrow[r, "h_U"] \arrow[d] & (V \cap U) \times F \arrow[dl, "pr_1^{(U)}"] \\ & V \cap U & \end{tikzcd}$$

We have h_U and h_V are homeomorphisms. Then $(b, f) \overset{h_U \circ h_V^{-1}}{\mapsto} (b, L(b, f))$ for some arbitrary function L.

29.2 Ending Notes

If $E \to B$ is a fiber bundle[13] with fiber F then there exists a magical space B_G and E_G and a map $f : B \to B_G$ such that the pullback $f^! E_G$ is equivalent to E.

$$\begin{tikzcd} E = f^! E_G \arrow[r] \arrow[d] & E_G \arrow[d] \\ B \arrow[r, "f"] & B_G \end{tikzcd}$$

These are called "classifying spaces".

[13] e.g. $F = \mathbb{R}^n$ and $G = GL_n(\mathbb{R})$ or $F = \{a_1, \cdots, a_n\}$ and $G = S_n$ the symmetric group

30 December 10, 2012

Last class!

30.1 Fiber Bundles

Fix F and a group G which acts on F from the left.

Example. Let $F = \mathbb{R}^n$ and $G = O(n)$ the group of orthogonal $n \times n$ matrices. Then if $A \in G$, $\vec{x} \in F$ we could make a group action $A\vec{x} \in F$.

Example. Let $F = \{1, 2, \cdots, n\}$ and $G = S_n$ the symmetric group. Then the action is just $\sigma \cdot i = \sigma(i)$.

Definition. Suppose $p : E \to B$. This is a fibre bundle with fibre F and structure group G if $\forall b \in B$ there exists a neighborhood $U_b \subseteq B$ such that $p^{-1}(U_b)$ is essentially a product; that is, $\exists \varphi_U : U_b \times F \to p^{-1}(U_b)$ a homeomorphism such that $p \circ \varphi_U$ is projection. In other words, the following diagram should commute:

$$\begin{tikzcd} U_b \times F \arrow[r, "\varphi_U", "\approx"'] \arrow[dr, "pr_1"'] & p^{-1}(U_b) \arrow[d, "p"] \\ & U_b \end{tikzcd}$$

Furthermore, if $b \in U_b \cap V_b$, then $p^{-1}(U_b \cap V_b)$ we get a diagram

$$\begin{tikzcd} (U_b \cap V_b) \times F \arrow[r, "\varphi_U", "\approx"'] \arrow[dr, "pr_1"'] & p^{-1}(U_b \cap V_b) \arrow[d, "p"] & (U_b \cap V_b) \times F \arrow[l, "\varphi_V"', "\approx"] \arrow[dl, "pr_1"] \\ & U_b \cap V_b & \end{tikzcd}$$

Then, we require that there is a "transition" function $\Gamma_{VU} : U_b \cap V_b \to G$ for which $x \mapsto \Gamma_{VU} x$ for which

$$(\varphi_V^{-1} \circ \varphi_U)(x, f) = (x, \Gamma_{VU}(x) f)$$

for all $f \in F$.

Finally, if $b \in U_b \cap V_b \cap W_b$ we require $\Gamma_{WV} \circ \Gamma_{VU} = \Gamma_{WU}$.

Finally, we can define morphisms as such:

$$\begin{tikzcd} E' \arrow[r, "u"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

A morphism between fiber bundles with the same fiber F and structure group G is a pair (u, f) such that $p \circ u = f \circ p'$.

Thus, we have a category of fiber bundles with fiber F and group G.

30.2 Categories

Now, let H be the category of topological spaces and homotopy classes of maps. The objects consists of category of topological spaces X, and arrows are the homotopy classes of maps $[X, Y]$.

Also, Set is the category whose objects are sets and arrows are arbitrary set functions.

Now we consider a map $k_G : X \mapsto k_G(X)$, the set of all fiber bundles with structure group G and fiber F over X. This is a set, of course. Then k_G is a *contravariant*[14] *functor*. We want to send $[f] \in [X, Y]$ to some map $f^* : k_G(Y) \to k_G(X)$.

$$\begin{tikzcd} f^! E_y \arrow[r] \arrow[d] & E_y \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

So, we just use the pullback. The claim is that $f^! E_y \to X$ is a fiber bundle with the same fiber and such.

Now, we say that k_G is *representable* in the sense that there is a unique (up to homotopy) space B_G and an element of $k_G(B_G)$ (i.e. a fiber bundle $E_G \to B_G$) such that $\forall X : k_G(X) = [X, B_G]$.

That is, if $E \to X$ is a fiber bundle, there is a unique $[f] \in [X, B_G]$ such that this commutes:

$$\begin{tikzcd} E \approx f^! E_G \arrow[r] \arrow[d] & E_G \arrow[d] \\ X \arrow[r, "f"'] & B_G \end{tikzcd}$$

Hence, for every group G we can get a "master" $E_G \to B_G$ which describes all fiber bundles with structure group G.

As an example, we have for $F = \{1, 2\}$ and $G = \mathbb{Z}_2$

$$\begin{tikzcd} S^1 \arrow[d] & \subseteq S^2 \arrow[d] & \subseteq S^3 \arrow[d] & \subseteq \cdots \\ {} & {} & {} & \cdots \mathbb{R}P^1 \subseteq \mathbb{R}P^2 \subseteq \mathbb{R}P^3 \subseteq \cdots \end{tikzcd}$$

So we can define an $S^\infty = \bigcup_{n \ge 1} S^n$ and $\mathbb{R}P^\infty = \bigcup_{n \ge 1} \mathbb{R}P^n$.

[14]Meaning the direction is flipped.