Sparse Recovery and Fourier Sampling

Eric Price

MIT

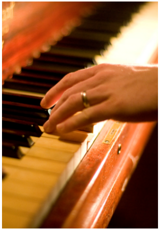
The Fourier Transform

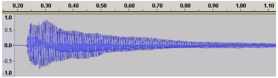
Conversion between time and frequency domains

Time Domain

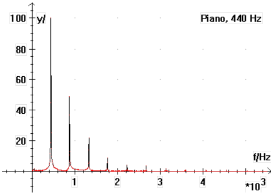
Frequency Domain

Fourier Transform ⟷





Displacement of Air

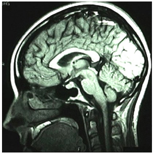


Concert A

The Fourier Transform is Ubiquitous

Audio

Video



Medical Imaging

Radar

GPS

Oil Exploration

Computing the Discrete Fourier Transform

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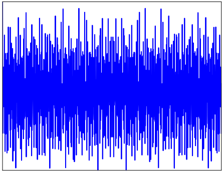
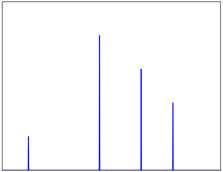
When can we compute the Fourier Transform in *sublinear* time?

Idea: Leverage *Sparsity*

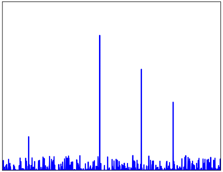
Often the Fourier transform is dominated by a small number of peaks:



Time Signal



Frequency
(Exactly sparse)



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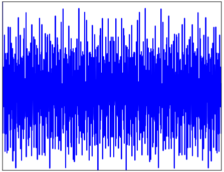

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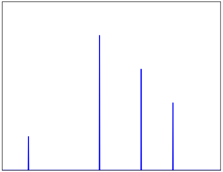

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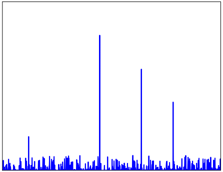

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Sparsity is common:

Audio

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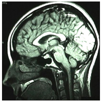

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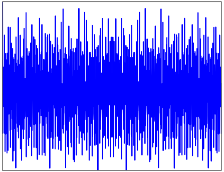

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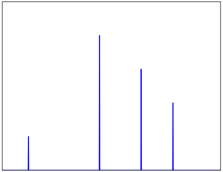

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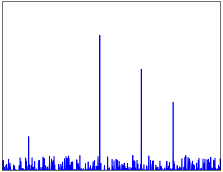

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Sparsity is common:

Goal of this work: a *sparse* Fourier transform

Faster Fourier Transform on sparse data.

Talk Outline

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   - Overview
   - Technical Details

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My Contributions

Goal: Compute the Fourier transform $\widehat{x} = Fx$ when $\widehat{x}$ is k-sparse.

- Theory:
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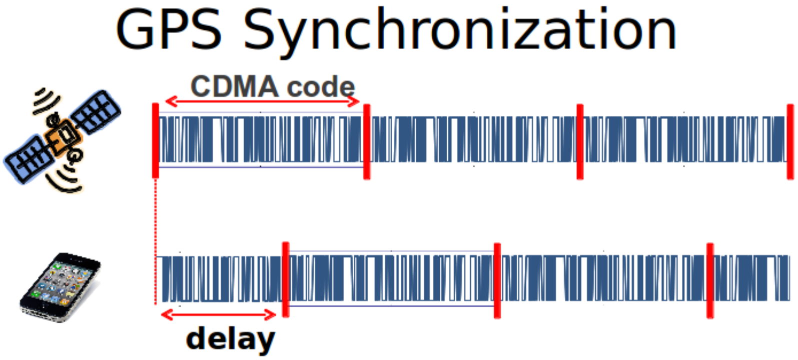
- Theory:
  - The fastest algorithm for Fourier transforms of sparse data.
  - The only algorithms faster than FFT for all $k = o(n)$.
- Practice:
  - Implementation is faster than FFTW for a wide range of inputs.
  - Orders of magnitude faster than previous sparse Fourier transforms.
  - Useful in multiple applications.

Applications of ideas

http://groups.csail.mit.edu/netmit/sFFT/workshop.html



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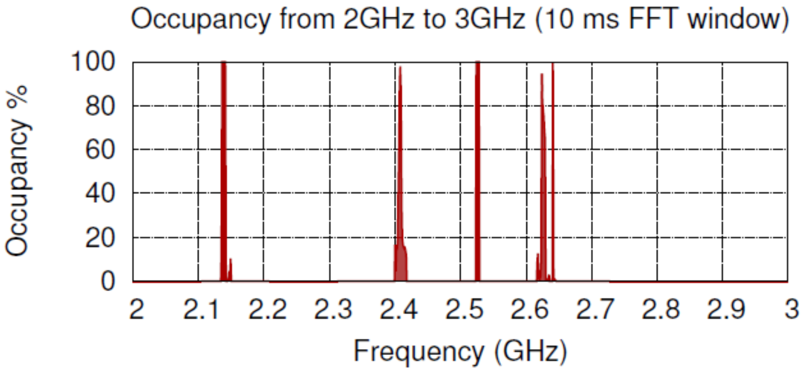

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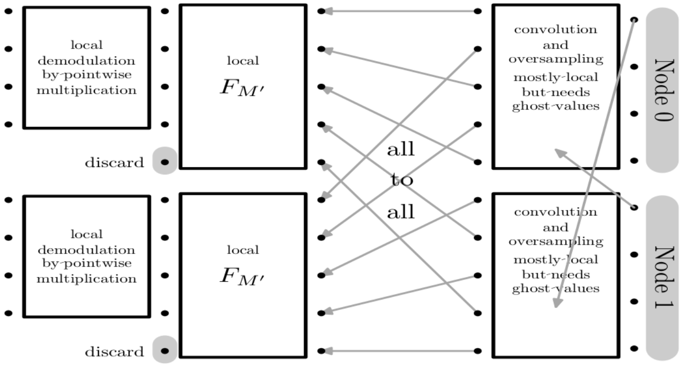

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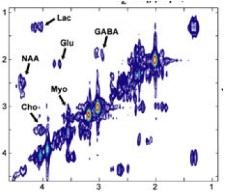

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For a signal of size *n* with *k* large frequencies

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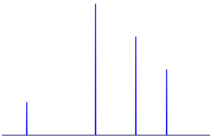

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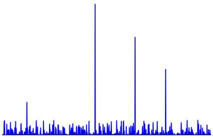

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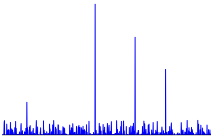

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- Given $x \in \mathbb{C}^n$, compute Fourier transform $\widehat{x}$:

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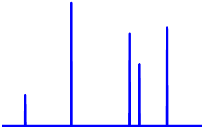

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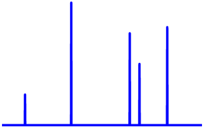

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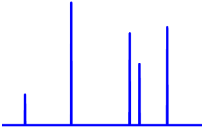

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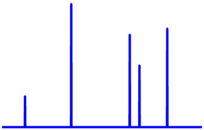

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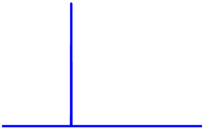

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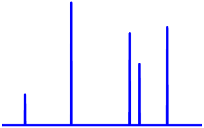

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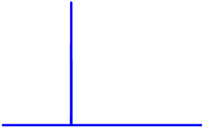

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Algorithm for $k = 1$

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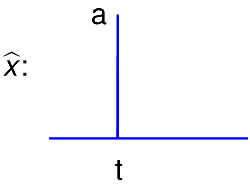

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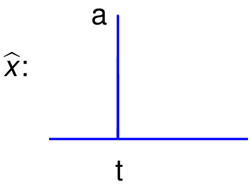

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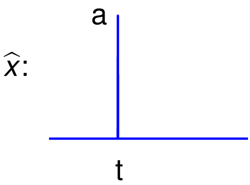

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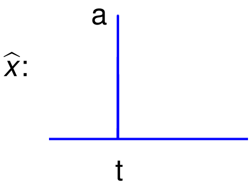

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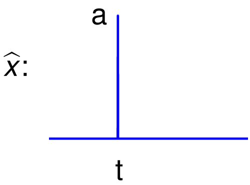

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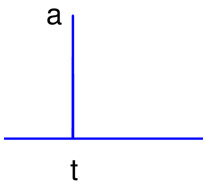
$\widehat{x}$: 

- Then $x = (a, a\omega^t, a\omega^{2t}, a\omega^{3t}, \ldots, a\omega^{(n-1)t})$.

$$x_0 = a \qquad\qquad x_1 = a\omega^t$$

- $x_1/x_0 = \omega^t \implies t$. $\blacksquare$

Algorithm for $k = 1$

Lemma

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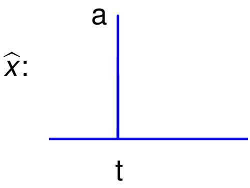

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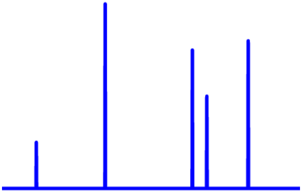
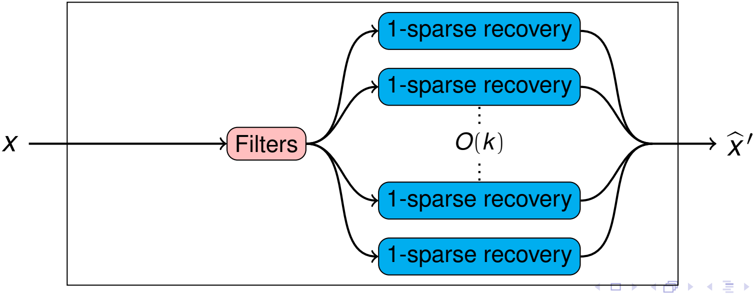
- (Related to OFDM, Prony's method, matrix pencil.)

Algorithm for general k

- Reduce general k to $k = 1$.





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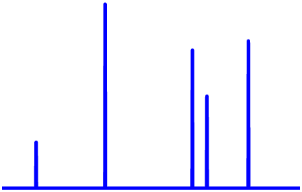

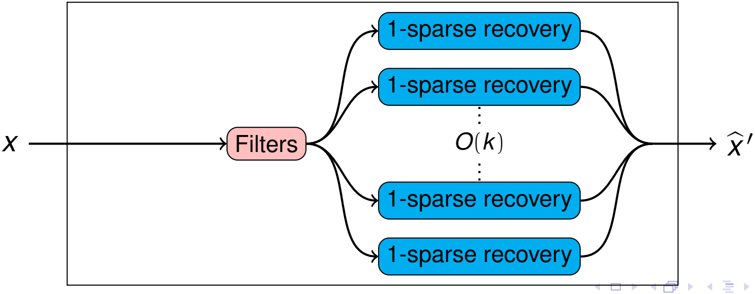

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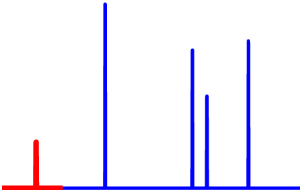

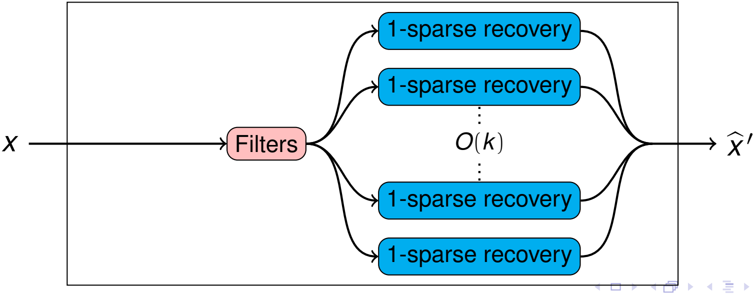

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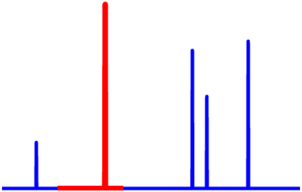

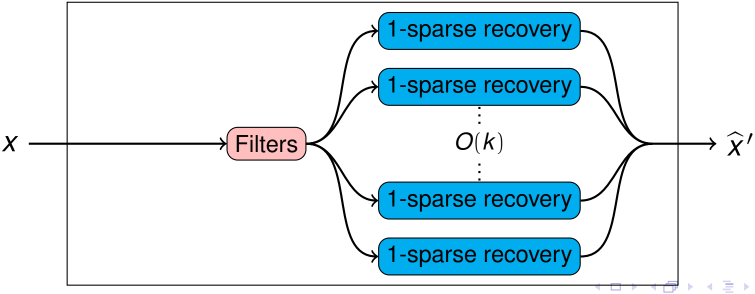

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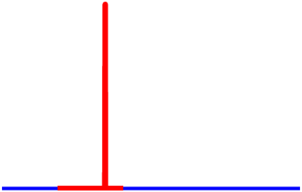

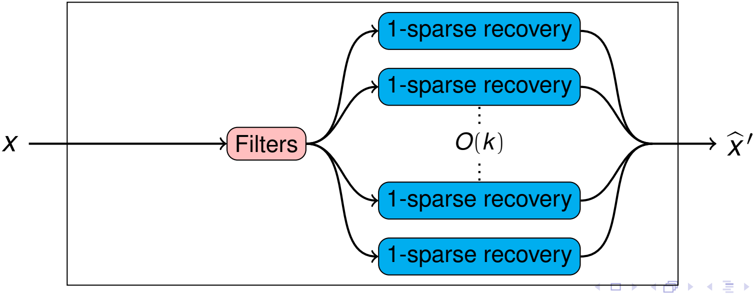

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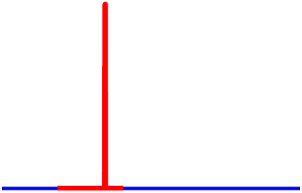

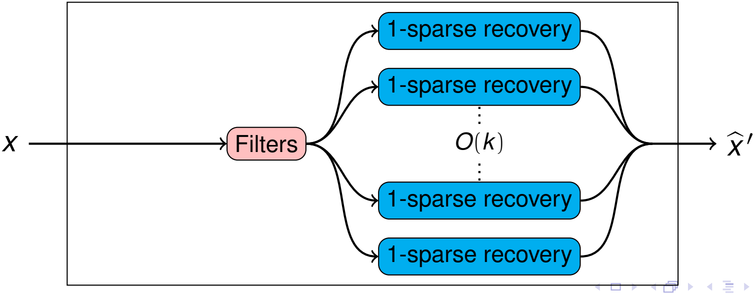

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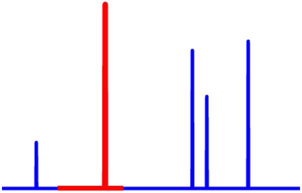

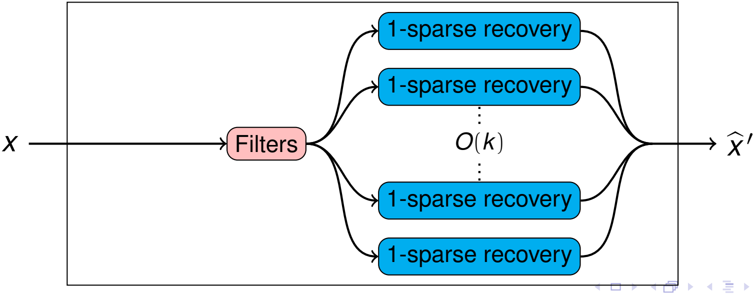

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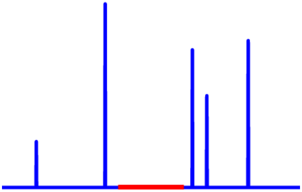

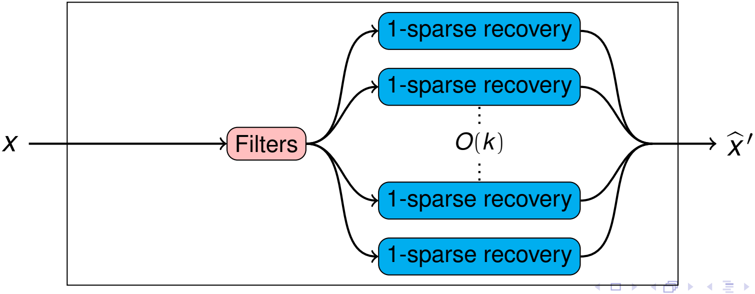

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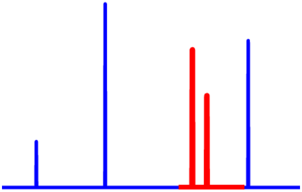

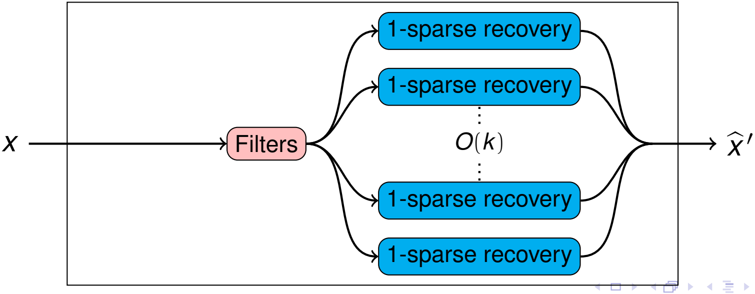

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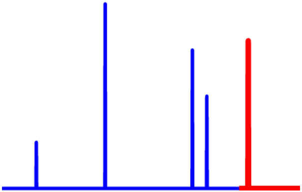

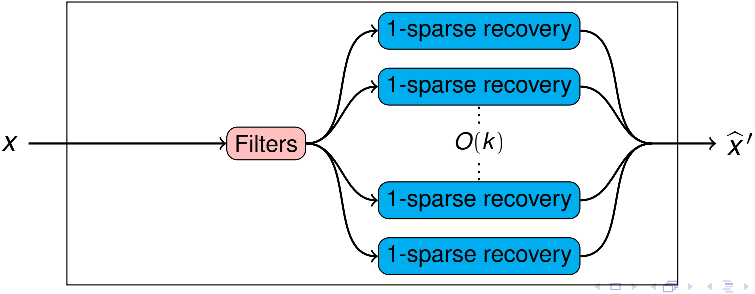

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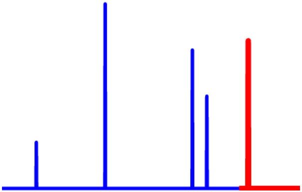

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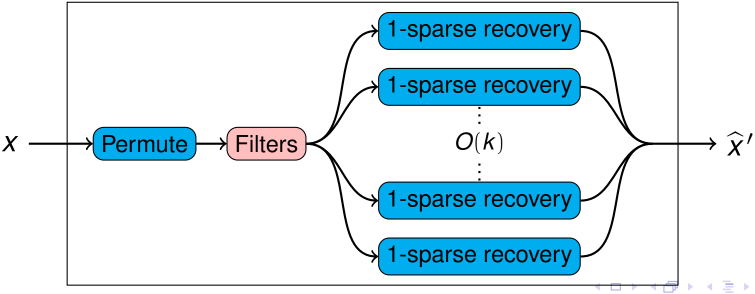

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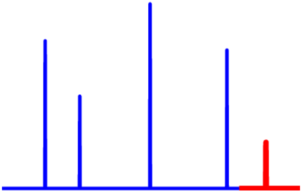

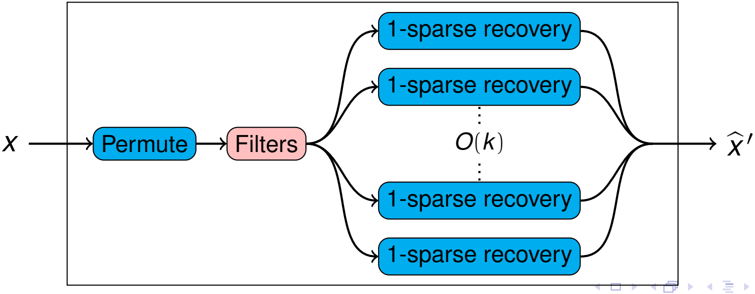

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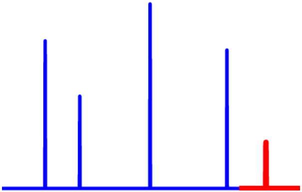

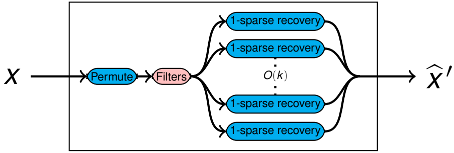

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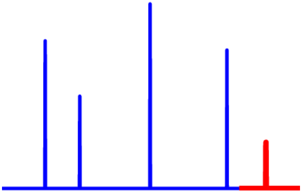
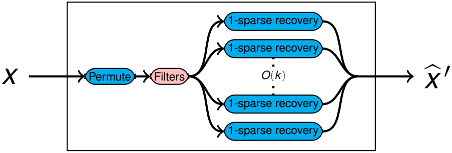


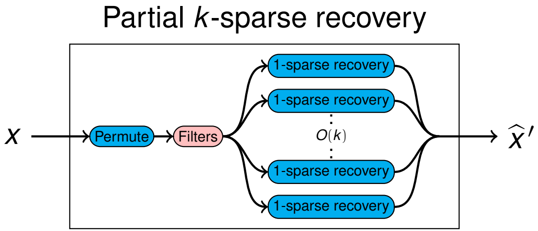


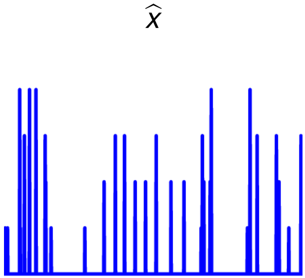
Recovers *most* of $\widehat{x}$:

Lemma (Partial sparse recovery)

In $O(k \log n)$ expected time, we can compute an estimate $\widehat{x}'$ such that $\widehat{x} - \widehat{x}'$ is $k/2$-sparse.

Overall outline





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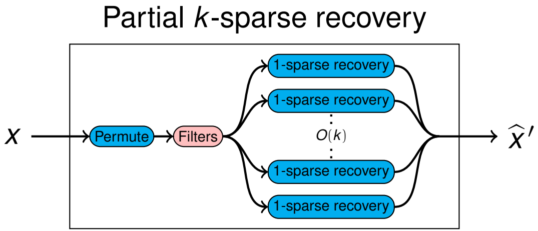

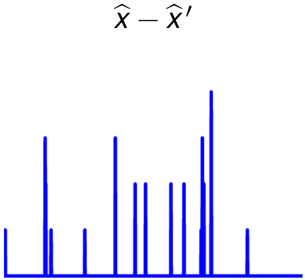

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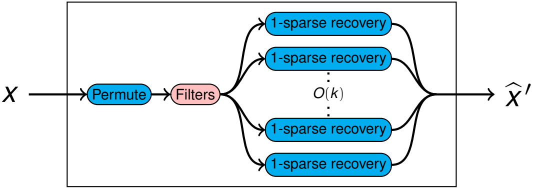

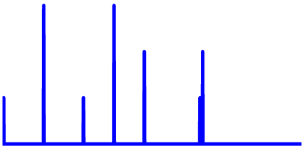

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Repeat, $k \to k/2 \to k/4 \to \cdots$

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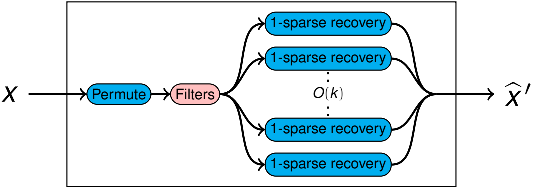

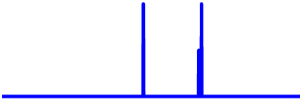

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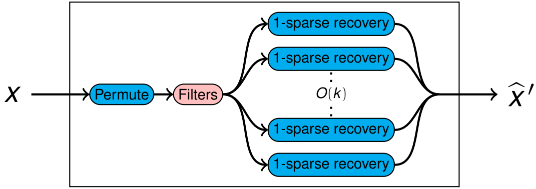

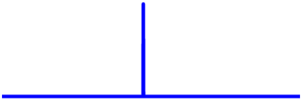

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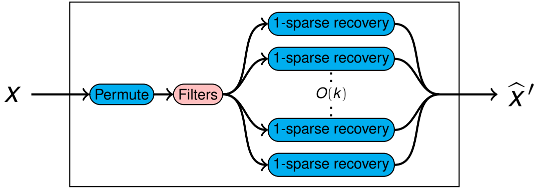

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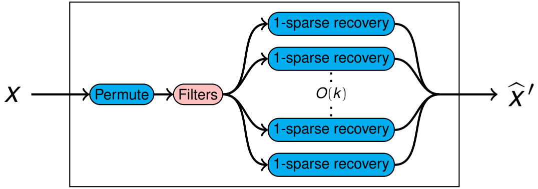

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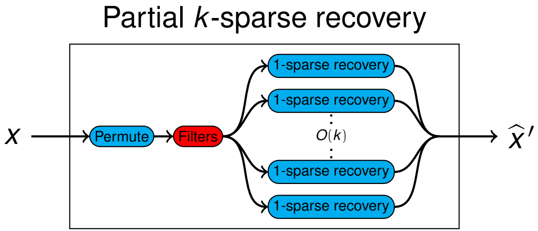

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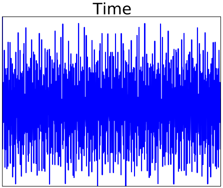

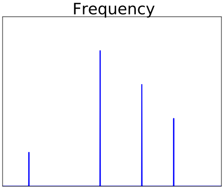

n-dimensional DFT:
$O(n \log n)$
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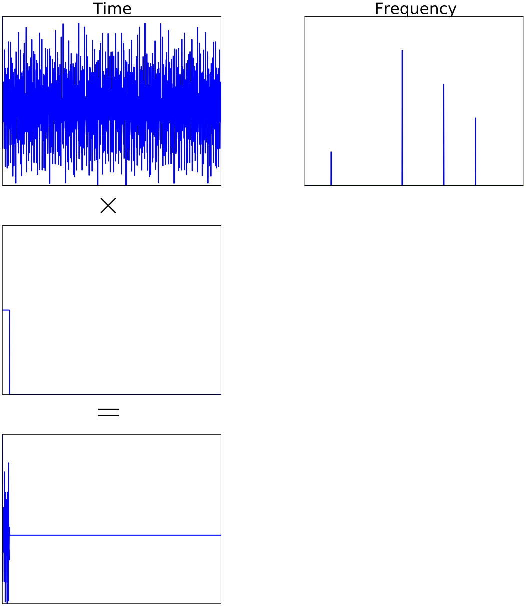

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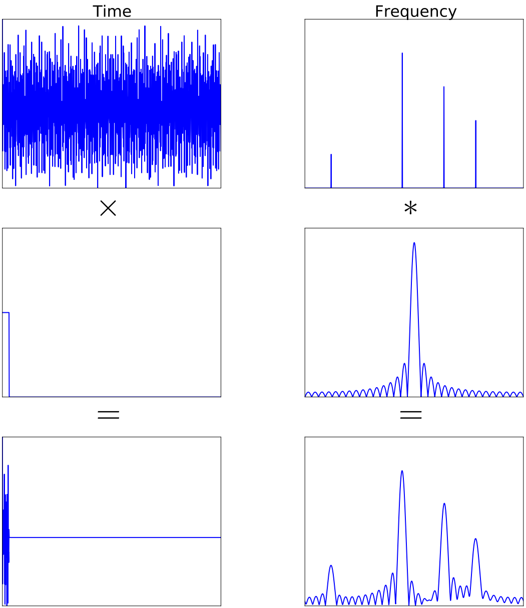

n-dimensional DFT: $O(n \log n)$
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$x \cdot \text{rect} \to \widehat{x} * \text{sinc}$.

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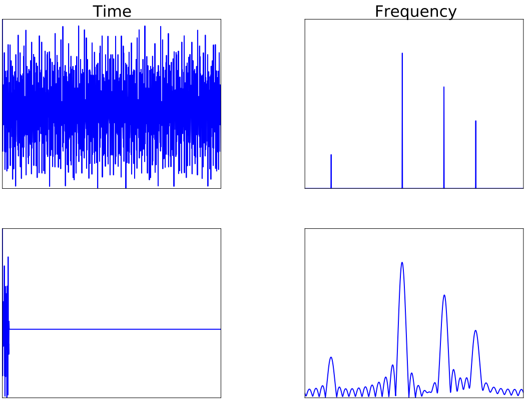

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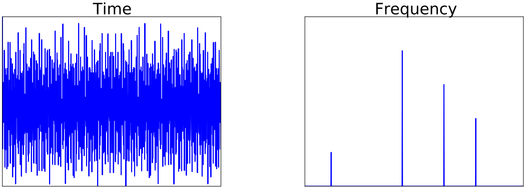

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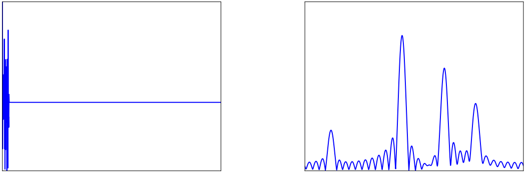

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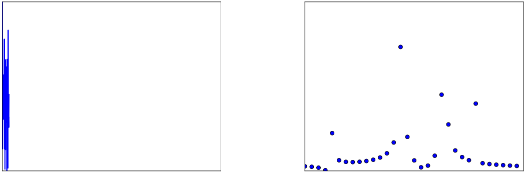

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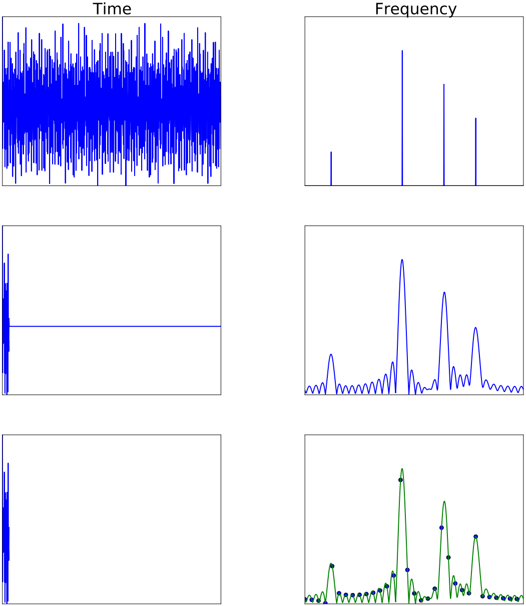

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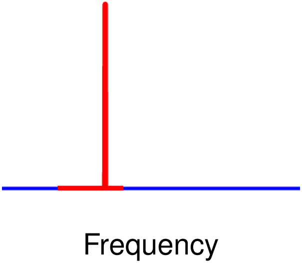

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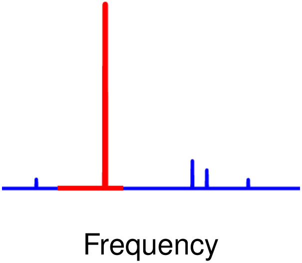
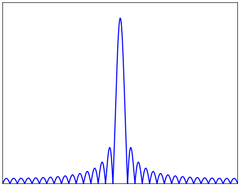
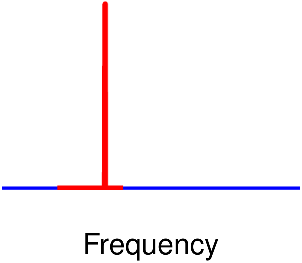


The sinc filter “leaks”.
Contamination from other buckets.



The issue

We want to isolate frequencies.



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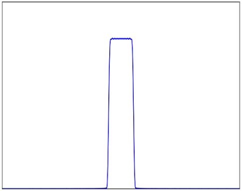
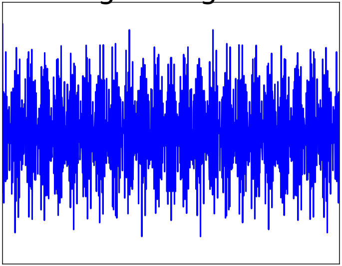


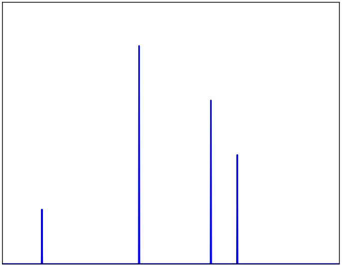
We introduce a better filter:
(Gaussian / prolate spheroidal sequence) convolved with rectangle.

Algorithm for *exactly sparse* signals

Original signal x



Goal $\hat{x}$



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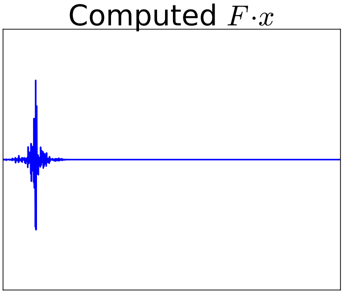
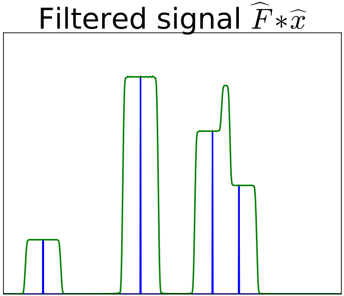
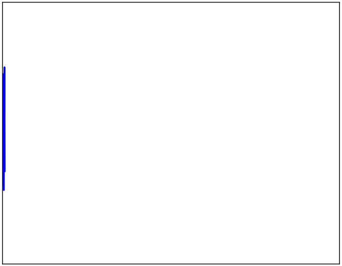


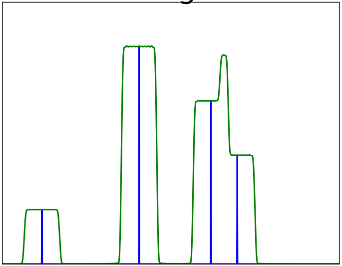


Algorithm for *exactly sparse* signals

$F \cdot x$ aliased to k terms



Filtered signal $\widehat{F} * \widehat{x}$



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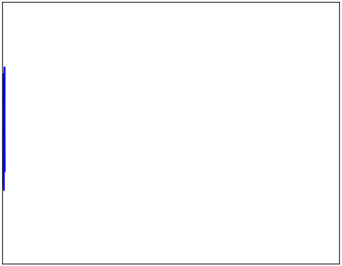
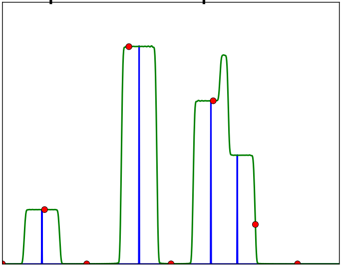


Computed samples of $\widehat{F} * \widehat{x}$



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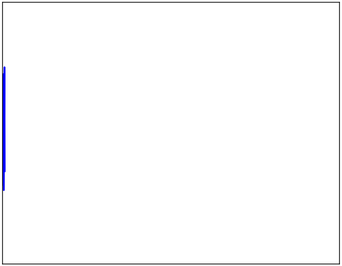

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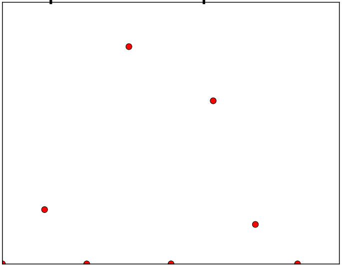

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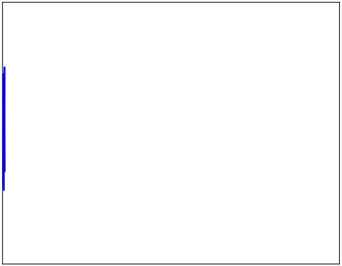
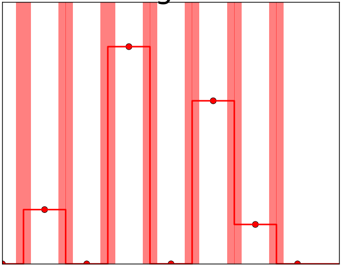


Knowledge about $\hat{x}$



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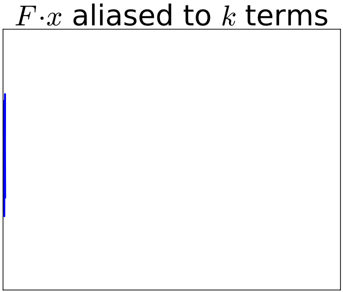

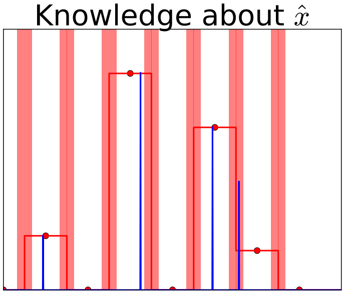

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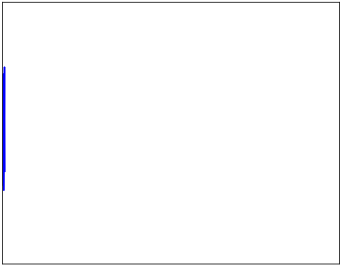

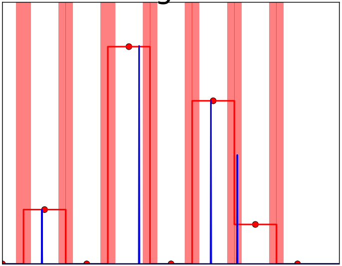

Lemma

If t is isolated in its bucket and in the "super-pass" region, the value b we compute for its bucket satisfies

$$b = \widehat{x}_t.$$

Computing the b for all $O(k)$ buckets takes $O(k \log n)$ time.

Algorithm for *exactly sparse* signals

Lemma

For most t, the value b we compute for its bucket satisfies

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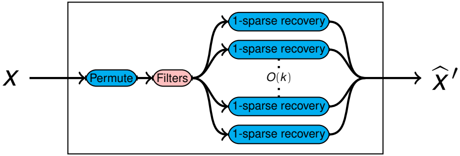

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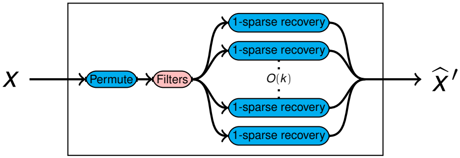

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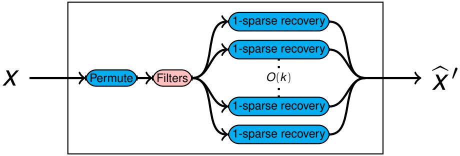
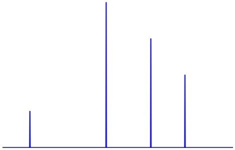


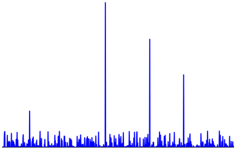
- Repeat $k \to k/2 \to k/4 \to \cdots$
- $O(k \log n)$ time sparse Fourier transform.

Algorithm for *approximately sparse* signals



Algorithm for *approximately sparse* signals

- What changes with noise?



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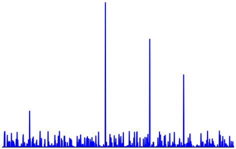

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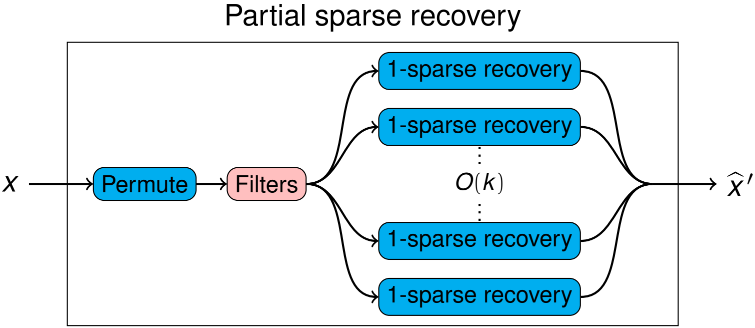
- Identical architecture:



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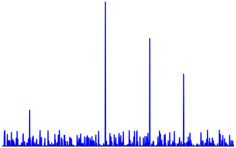

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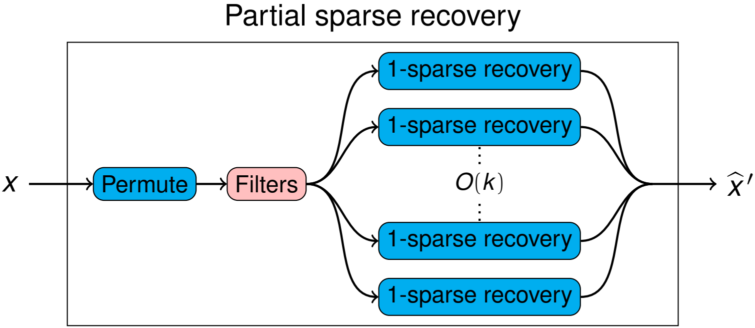

- Just requires robust 1-sparse recovery.

Algorithm for *approximately sparse* signals: $k = 1$

Lemma

Suppose $\widehat{x}$ is approximately 1-sparse:

$$|\widehat{x}_t| / \|\widehat{x}\|_2 \geqslant 90\%.$$

Then we can recover it with $O(\log n)$ samples and $O(\log^2 n)$ time.

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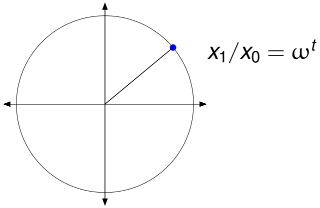

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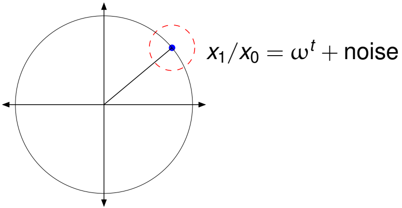

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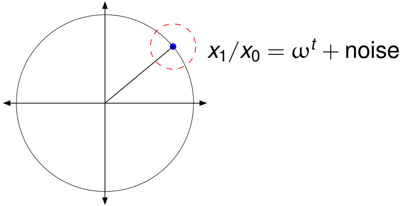

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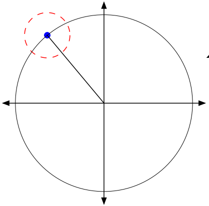

$$x_{c_2}/x_0 = \omega^{c_2 t} + \text{noise}$$

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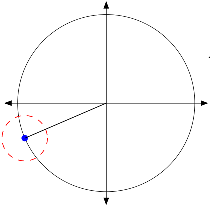

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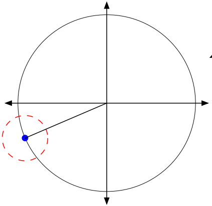

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- With exact sparsity: $\log n$ bits in a single measurement.
- With noise: only constant number of useful bits.
- Choose $\Theta(\log n)$ time shifts c to recover i.
- Error correcting code with efficient recovery $\implies$ Lemma. ∎

Algorithm for *approximately sparse* signals: general k

Lemma

If $\widehat{x}$ is approximately 1*-sparse, we can recover it with* $O(\log n)$ *samples and* $O(\log^2 n)$ *time.*

Algorithm for *approximately sparse* signals: general k

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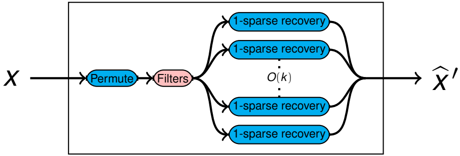
If $\widehat{x}$ is approximately 1-sparse, we can recover it with $O(\log n)$ samples and $O(\log^2 n)$ time.



Reduce k-sparse to 1-sparse on buckets of size n/k, with $\log n$ overhead per sample.

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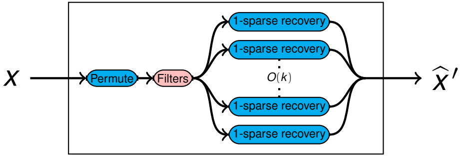

Reduce k-sparse to 1-sparse on buckets of size n/k, with $\log n$ overhead per sample.

Theorem

If $\widehat{x}$ is approximately k-sparse, we can recover it in $O(k \log(n/k) \log n)$ time.

Empirical performance

- Compare to
  - FFTW, the "Fastest Fourier Transform in the West"
  - AAFFT, the [GMS05] sparse Fourier transform.

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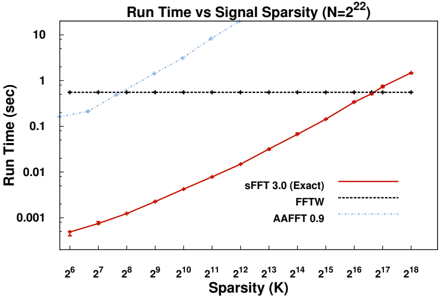

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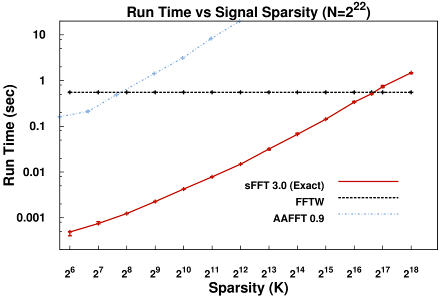

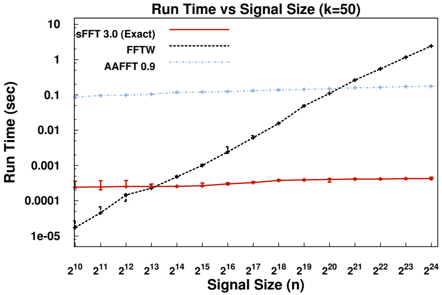

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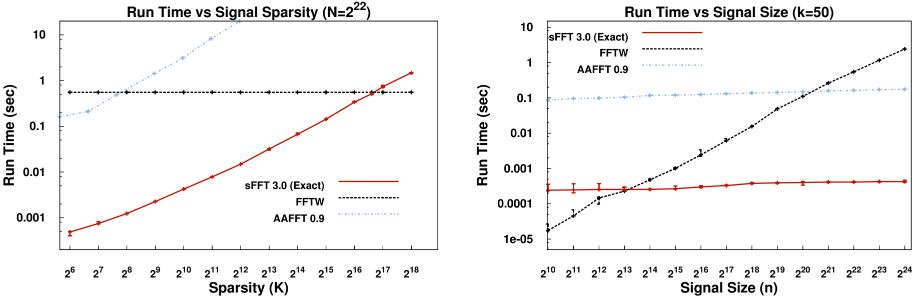

- Faster than FFTW for wide range of values.

Recap of Sparse Fourier Transform

- Theory:
  - The fastest algorithm for Fourier transforms of sparse data.
  - The only algorithms faster than FFT for all $k = o(n)$.

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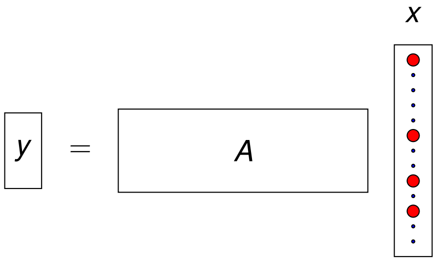
- Theory:
  - The fastest algorithm for Fourier transforms of sparse data.
  - The only algorithms faster than FFT for all $k = o(n)$.
- Practice:
  - Implementation is faster than FFTW for a wide range of inputs.
  - Orders of magnitude faster than previous sparse Fourier transforms.
  - Useful in multiple applications.

Talk Outline

1. Sparse Fourier Transform
   - Overview
   - Technical Details
2. Beyond: Sparse Recovery / Compressive Sensing
   - Overview
   - Adaptivity
   - Conclusion

Sparse Recovery / Compressive Sensing

Robustly recover sparse x from linear measurements $y = Ax$.



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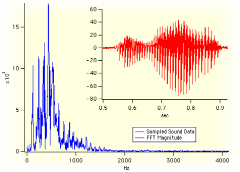
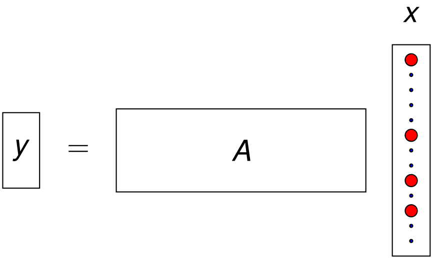


Sparse Fourier



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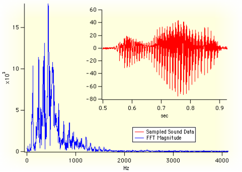

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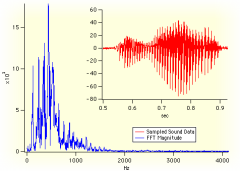

Sparse Fourier

MRI

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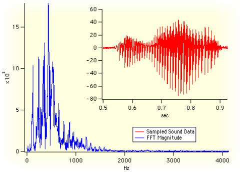
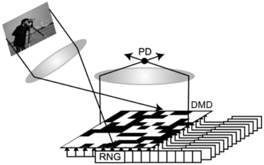


Sparse Fourier

MRI



Single-Pixel Camera

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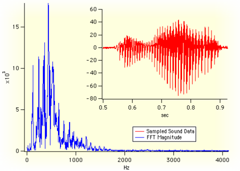

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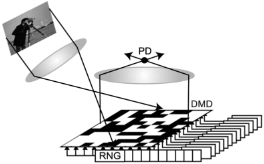

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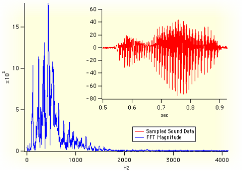

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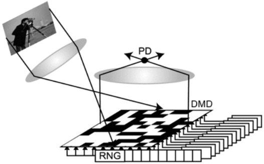

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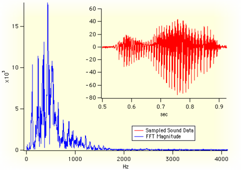

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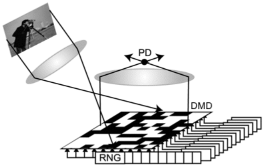

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Applications of Adaptivity

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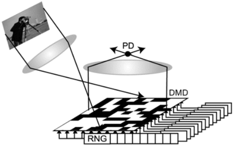

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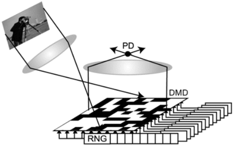

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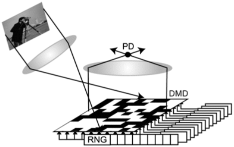

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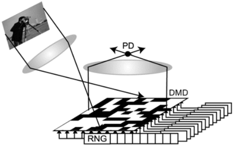

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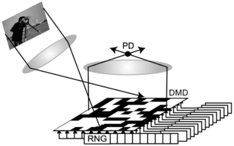

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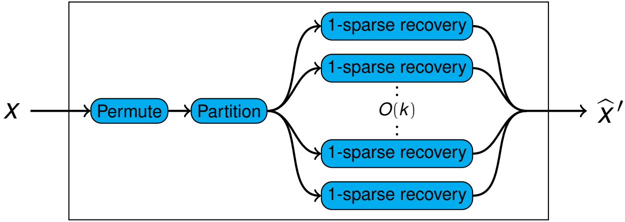

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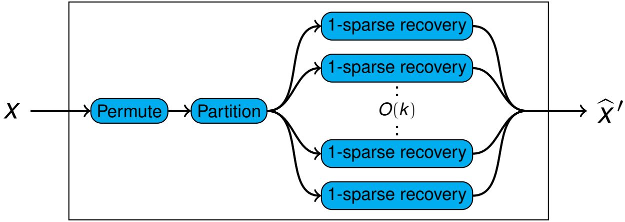

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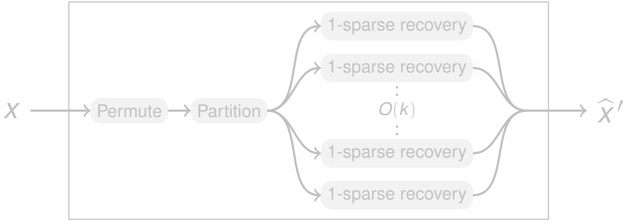

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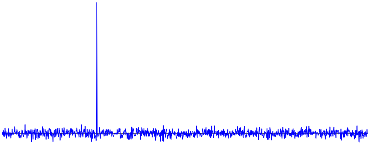

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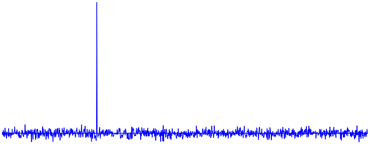

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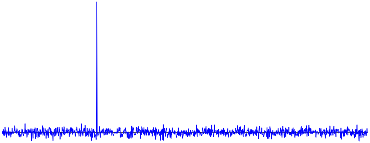

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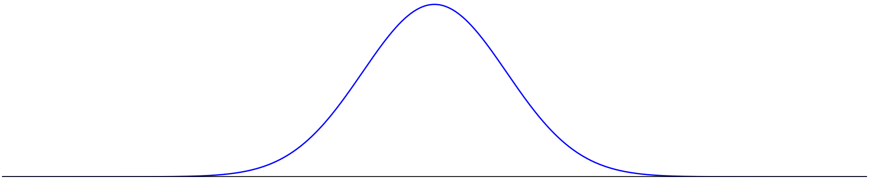

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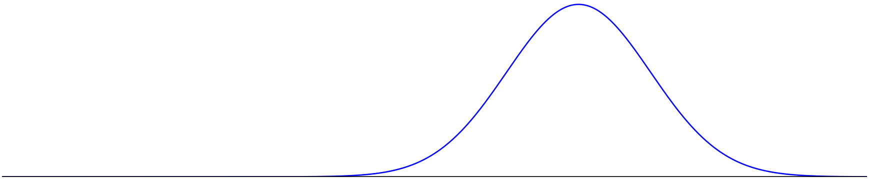

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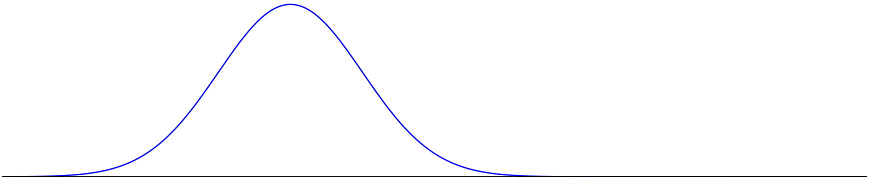

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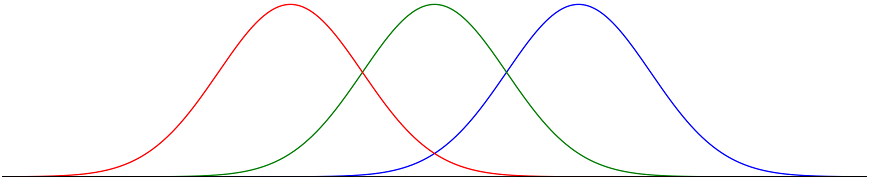

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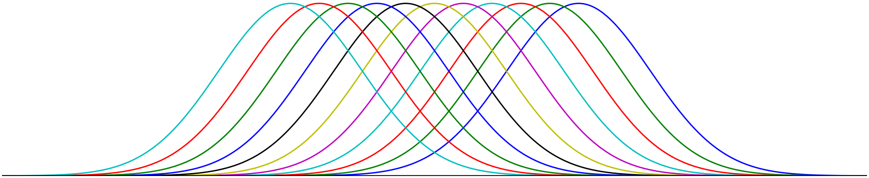

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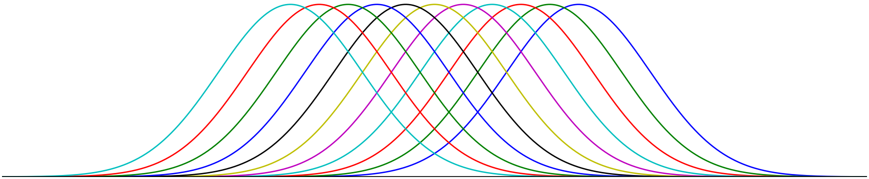

- Shannon 1948: information capacity

$$I(i, \langle v, x\rangle) \leqslant \frac{1}{2}\log(1 + \text{SNR})$$

where *SNR* denotes the "signal-to-noise ratio,"

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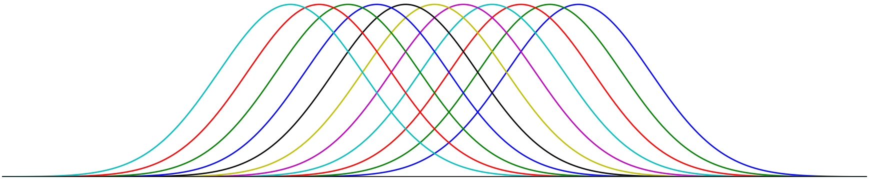

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- Finding i needs $\Omega(\log n)$ non-adaptive measurements.

1-sparse recovery: changes in adaptive setting

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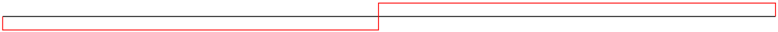
- As we learn about i, we can increase the SNR.

1-sparse recovery: idea

$x = e_i + w$

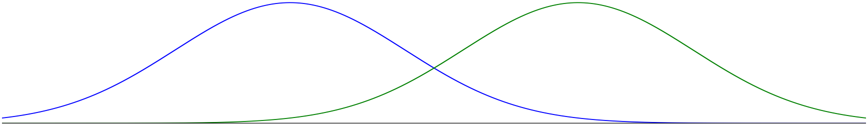
v



$SNR = 2$

$I(i, \langle v, x \rangle) \leqslant \log SNR = 1$

$\langle v, x \rangle = v_i + \langle v, w \rangle$



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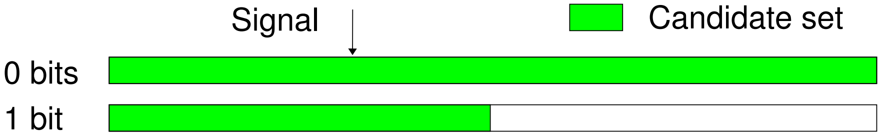

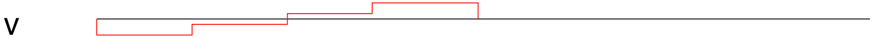

$SNR = 2^2$ $\qquad$ $I(i, \langle v, x \rangle) \leqslant \log SNR = 2$

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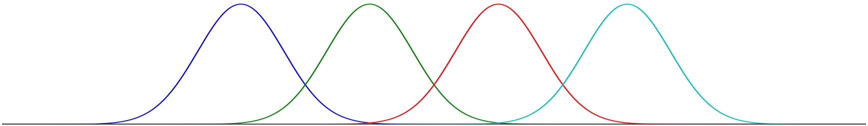

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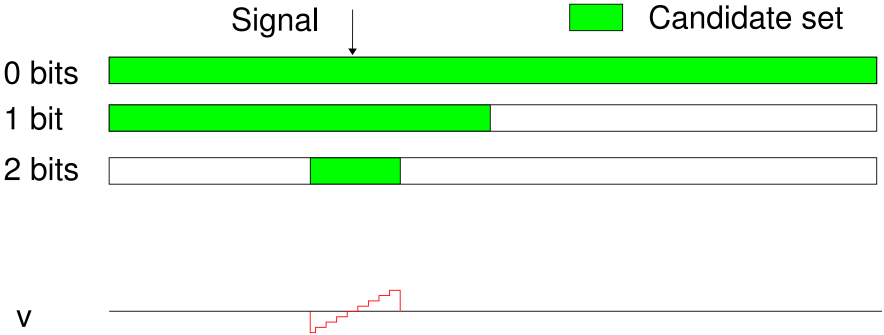

$SNR = 2^4$ $\qquad$ $I(i, \langle v, x \rangle) \leqslant \log SNR = 4$

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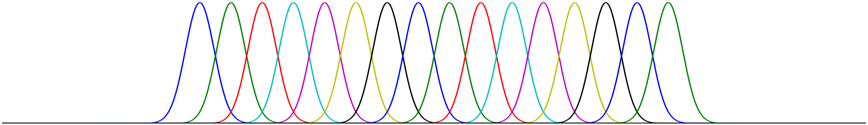

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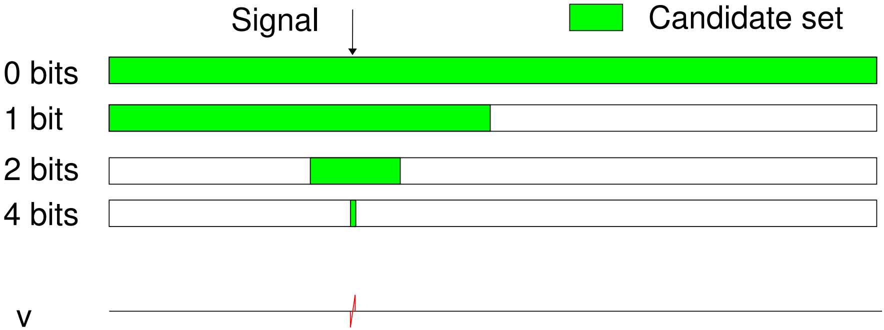

$SNR = 2^8$ $\qquad I(i, \langle v, x \rangle) \leqslant \log SNR = 8$

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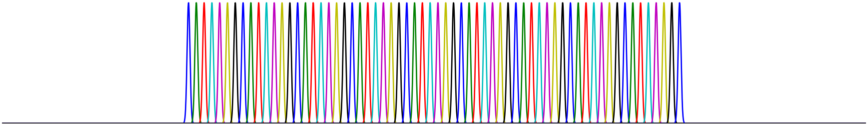

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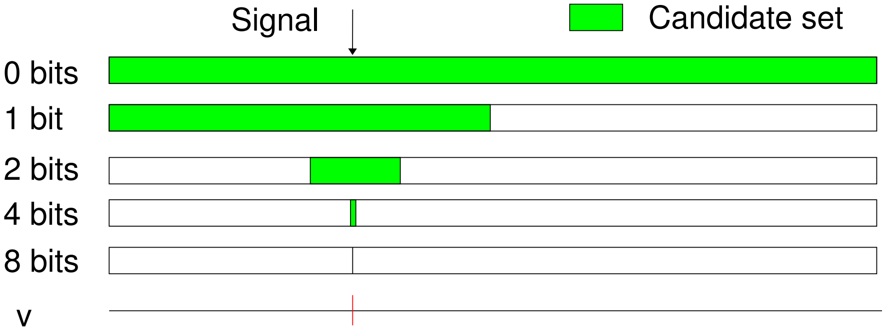

$SNR = 2^{16}$

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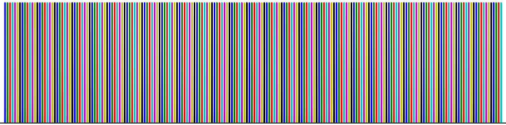

1-sparse recovery

Lemma (IPW11)

Adaptive 1-sparse recovery takes $O(\log \log n)$ measurements.

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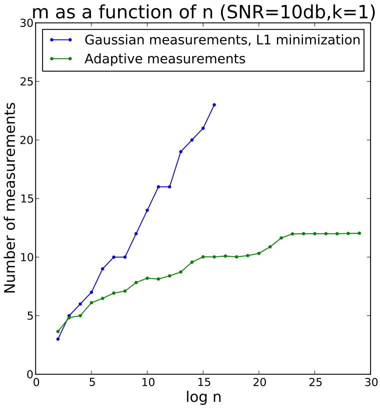

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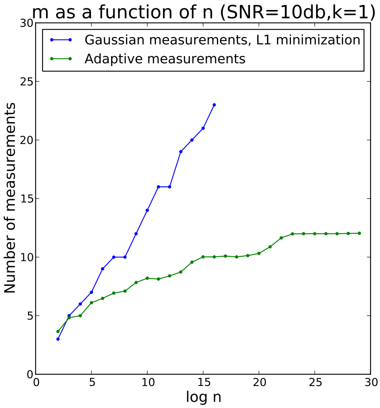

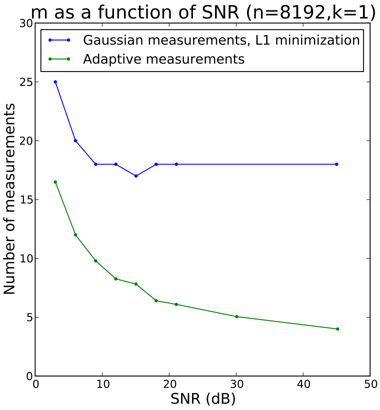

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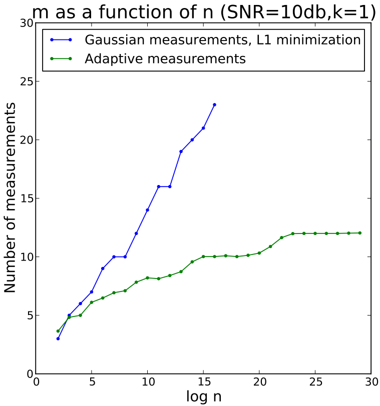

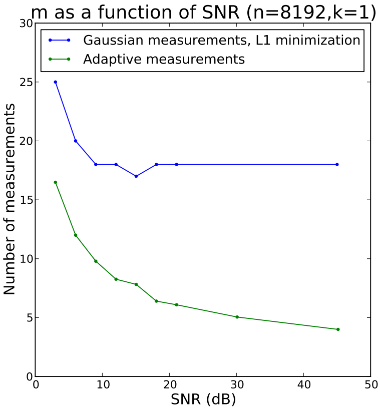

Gives $\Theta(k \log\log(n/k))$ k-sparse recovery via general framework.

Summary

- Sparse Fourier transform
  - Fastest algorithm for Fourier transforms on sparse data
  - Already has applications with substantial improvements

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  - Fastest algorithm for Fourier transforms on sparse data
  - Already has applications with substantial improvements
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Thank You

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