On the Sufficiency of Power Control for a Class of Channels with Feedback

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Abstract — **We show that, for a particular class of channels that we believe applies to many physical problems of interest, the utility of feedback, insofar as channel capacity is concerned, is simply for allowing the transmitter to perform power control. This class of channels, which assumes noiseless feedback but allows for the feedback to be of arbitrary rate, includes channels that model slow, flat fading channels with variable input power. Thus our result gives some guidance on the design of effective transmission schemes for slow, flat fading channels with feedback.**

Suppose we have a feedforward finite-state channel (FSC) with finite input and output alphabets, $\mathcal{X}$ and $\mathcal{Y}$, respectively, and a noiseless feedback link with finite input and output alphabet $\mathcal{U}$. Messages are mapped to a sequence of random *code-functions* F^N, $F_n : \mathcal{U}^{n-1} \to \mathcal{X}$, so $X_n = F_n(U^{n-1})$. At the receiver end, there is a sequence of deterministic functions g^N, $g_n : \mathcal{Y}^n \to \mathcal{U}$, so $U_n = g_n(Y^n)$. The symbols that are fed back are received at the transmitter after a delay of one time unit. Such a channel has noiseless feedback of arbitrary rate.

We extend [1] and establish the following.

Theorem 1. *The capacity of channels with noiseless feedback of arbitrary rate is given by*[2]

$$C = \sup_{\mathbf{P}} \text{p-liminf} \frac{1}{N} \sum_{n=1}^{N} i_n(Y_n; X^n | Y^{n-1}), \tag{1}$$

where

$$i_n(Y_n; X^n|Y^{n-1}) := \frac{P_{Y_n|Y^{n-1},X^n}(Y_n|Y^{n-1}, X^n)}{P_{Y_n|Y^{n-1}}(Y_n|Y^{n-1})}, \tag{2}$$

and $\mathbf{P}$ *is the sequence of conditional probability mass functions* $P_{X_1}, P_{X_2|X_1,U_1}, P_{X_3|X^2,U^2}, \ldots$.

Proposition 1. *If the sequence of pairs* $\{(X_n, Y_n)\}$ *is ergodic, then*

$$\text{p-liminf} \frac{1}{N} \sum_{n=1}^{N} i_n(Y_n; X^n|Y^{n-1}) = \lim_{n\to\infty} \frac{1}{n} I(X^n; Y_n|Y^{n-1}). \tag{3}$$

We expand definitions from [2] to allow for the concept of power control at the transmitter.

Definition. *We say that an FSC is* variable-power *with* L *energy levels if it is associated with* L *disjoint, non-empty sets* $\mathcal{X}_1, \ldots, \mathcal{X}_L$ *such that* $\mathcal{X} = \bigcup_{l=1}^{L} \mathcal{X}_l$*; and a sequence of* input energies $\{E_n\}$ *such that* $E_n = l$ *if the input symbol at time* n *is from alphabet* $\mathcal{X}_l$*. The FSC that results when* $E_n = l$ *for all* n *is called the FSC with input energy* l*.*

We assume that $\mathcal{X}_1, \ldots, \mathcal{X}_L$ *is indexed in such a way that the capacity of the FSC with input energy* l *and perfect CSI is greater than or equal to that with input energy* l' *if* $l > l'$*.*

Definition. *A FSC is* uniformly-symmetric *if, for every state* $s \in \mathcal{S}$*, the discrete memoryless channel with input/output probabilities given by* $p_{Y|X,S}(\cdot|\cdot, s)$ *is output-symmetric.*

A variable-power FSC with L *energy levels is uniformly-symmetric if the FSC with input energy* l *is uniformly-symmetric for all* $l = 1, \ldots, L$*.*

Definition. *Let* X_n *and* Y_n *denote the input and output, respectively, of an FSC. We call an FSC* variable-noise *if there exists a function* φ *such that* $Z_n = \varphi(X_n, Y_n)$*,* Z^n *is independent of* X^n*, and* Z^n *is a sufficient statistic for* S^n *(i.e.* $(X^n, Y^n) \to Z^n \to S^n$ *forms a Markov chain).*

A variable-power FSC is variable-noise if Z^n *is independent of* X^n *when conditioned on* E^n *and* (Z^n, E^n) *is a sufficient statistic for* S^n*.*

The following theorem is the main result.

Theorem 2. *Suppose we have a uniformly-symmetric variable-noise variable-power FSC with noiseless feedback of arbitrary rate. Let*

$$\begin{aligned} \mathbf{P}_e := \{&(P_{X_1}, P_{X_2|X_1,U_1}, P_{X_3|X^2,U^2}, \ldots) \mid \\ &\exists (P_{E_1}, P_{E_2|X_1,U_1}, P_{E_3|X^2,U^2}, \ldots) \text{ s.t. for all } n \\ &P_{X_n|X^{n-1},U^{n-1}}(x_n|x^{n-1}, u^{n-1}) \\ &= \frac{P_{E_n|X^{n-1},U^{n-1}}(\lambda(x_n)|x^{n-1}, u^{n-1})}{|\mathcal{X}_{\lambda(x_n)}|} \\ &\text{for all } x^n \in \mathcal{X}^n, u^{n-1} \in \mathcal{U}^{n-1}\}, \end{aligned} \tag{4}$$

where $\lambda : \mathcal{X} \to \{1, \ldots, L\}$*,* $\lambda(x) = l$ *if* $x \in \mathcal{X}_l$*. Suppose* P^* *achieves the supremum over* $\mathbf{P}_e$ *of* $\lim_{n\to\infty}(1/n)I(X^n; Y_n|Y^{n-1})$ *and that* $\{(X_n, Y_n)\}$ *is ergodic under the input distribution* P^**. Then the capacity of the channel is given by*

$$C = \sup_{P \in \mathbf{P}_e} \lim_{n\to\infty} \frac{1}{n} I(X^n; Y_n|Y^{n-1}). \tag{5}$$

Acknowledgments

The authors would like to thank Ibrahim Abou-Faycal for helpful discussions and suggestions.

References

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[2] A. J. Goldsmith and P. P. Varaiya, "Capacity, mutual information, and coding for finite-state Markov channels," *IEEE Trans. Inform. Theory*, vol. 42, pp. 868–886, May 1996.

[1]This work was supported by the National Science Foundation under grants CCR-00993349 and ANI-0335256, and by Hewlett-Packard under grant 008542-008.

[2]We use p-liminfA_n to denote the liminf in probability of a sequence of random variables $\{A_n\}$, i.e. the supremum of all real numbers α such that $\Pr(A_n \leq \alpha) \to 0$ as $n \to \infty$.