Integral Operators and Exploding Reflectors: Geometric Semantics for Migration/Inversion

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Schlumberger-Doll Research

MIT/Schlumberger Workshop on Geophysical Inversion

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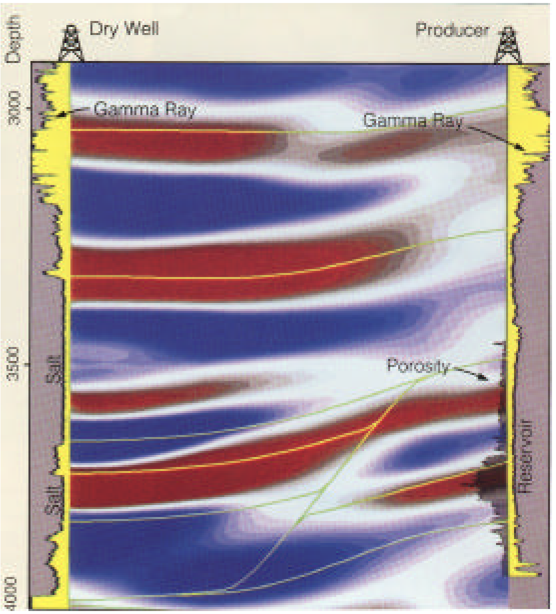
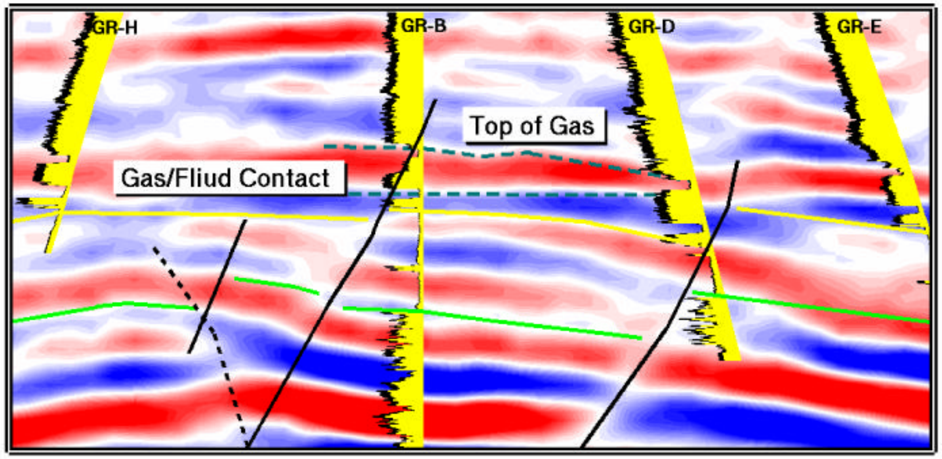
Collaborators

- **Mike Oristaglio, Greg Beylkin**
  - Miller, D., Oristaglio, M., and Beylkin, G., *A new slant on seismic imaging: classical migration and integral geometry*. Geophysics, vol. 52(1987), pp. 943-964.
- **Bob Burridge**
  - Miller, D., and Burridge, R., *Multiparameter inversion, dip-moveout, and the generalized Radon transform*, Proceedings of SIAM Workshop on Geophysical Inversion, Houston, 1989.
- **Martijn deHoop, Carl Spencer, Scott Leaney, Bill Borland**
  - R. Burridge, M. V. de Hoop, D. Miller and C. Spencer, *Multi-parameter inversion in anisotropic media using the generalised radon transform*. Geophysics J. Int., Vol. 134, pp. 757-777,

General Theme

- Geometric analysis can provide a *precise* guide to the development of multidimensional inversion theory





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Mathematics

The radon transform / Sigurdur Helgason.—Boston, Basel, Stuttgart : Birkhäuser, 1980.
(Progress in mathematics : 5)
ISBN 3-7643-3006-6

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It was proved by J. Radon in 1917 that a differentiable function on $\mathbb{R}^3$ can be determined explicitly by means of its integrals over the planes in $\mathbb{R}^3$. Let $J(\omega,p)$ denote the integral of f over the hyperplane $(x,\omega) = p$, ω denoting a unit vector and $(\,,)$ the inner product. Then

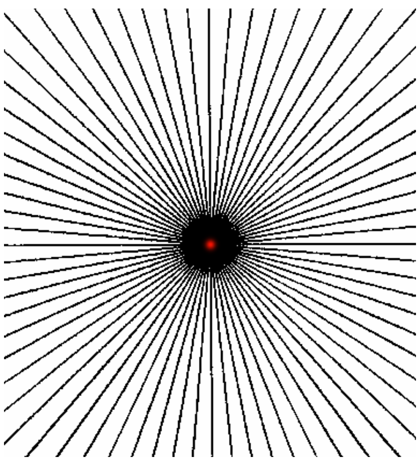
$$f(x) = -\frac{1}{8\pi^2} L_x \left(\int_{\mathbb{S}^2} J(\omega,(\omega,x))\, d\omega \right),$$

where L is the Laplacian on $\mathbb{R}^3$ and $d\omega$ the area element on the sphere $\mathbb{S}^2$ (cf. Theorem 3.1).

We observe that the formula above contains two integrations dual to each other: first one integrates over the set of points in a hyperplane, then one integrates over the set of hyperplanes passing through a given point. This suggests considering the transform $f \to \hat{f}$, $\phi \to \check{\phi}$ defined below.

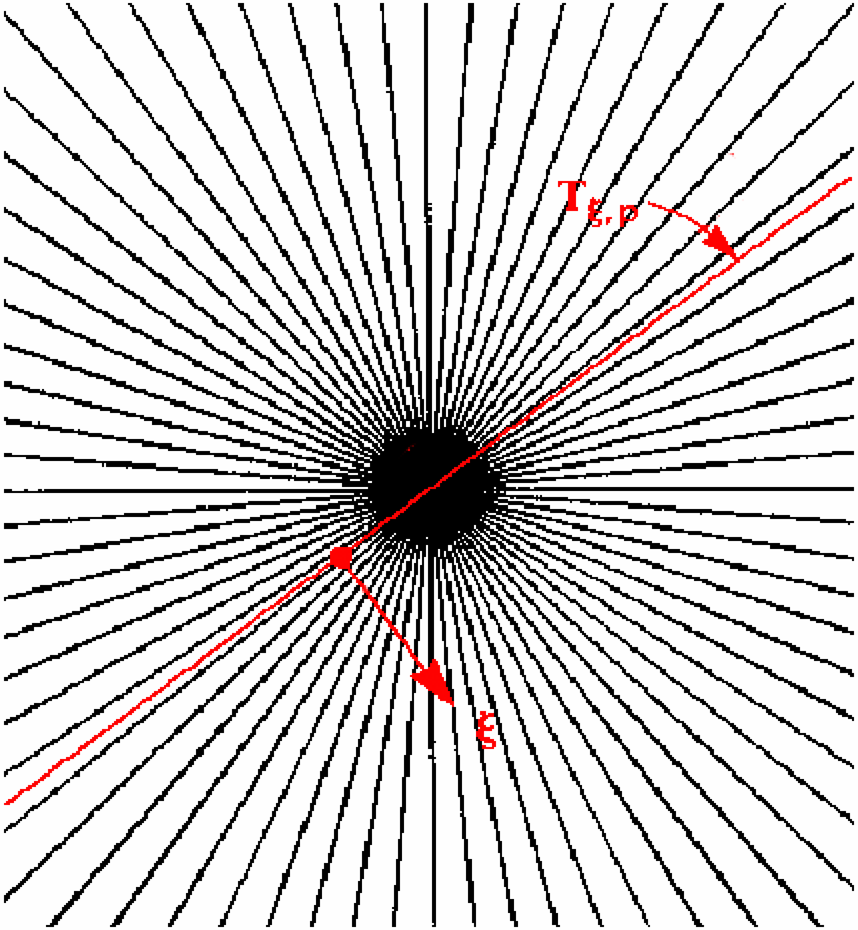
The formula has another interesting feature. For a fixed ω the integrand $x \to J(\omega,(\omega,x))$ is a plane wave, that is a function constant on each plane perpendicular to ω. Ignoring the Laplacian the formula gives a continuous decomposition of f into plane waves. Since a plane wave amounts to a function of just one variable (along the normal to the planes) this decomposition can sometimes reduce a problem for $\mathbb{R}^3$ to a similar problem for $\mathbb{R}$. This principle has been particularly useful in the theory of partial differential equations.

One Picture Summary :



Consider the density of ink

Forward Radon Transform



$$f^{\triangle}(\xi, p) = \int d^3\mathbf{x}\, f(\mathbf{x})\, \delta(p - \xi \cdot \mathbf{x})$$

$f^{\triangle}(\xi, p)$ is the integral of f over the plane perpendicular to ξ and distance p from the origin.

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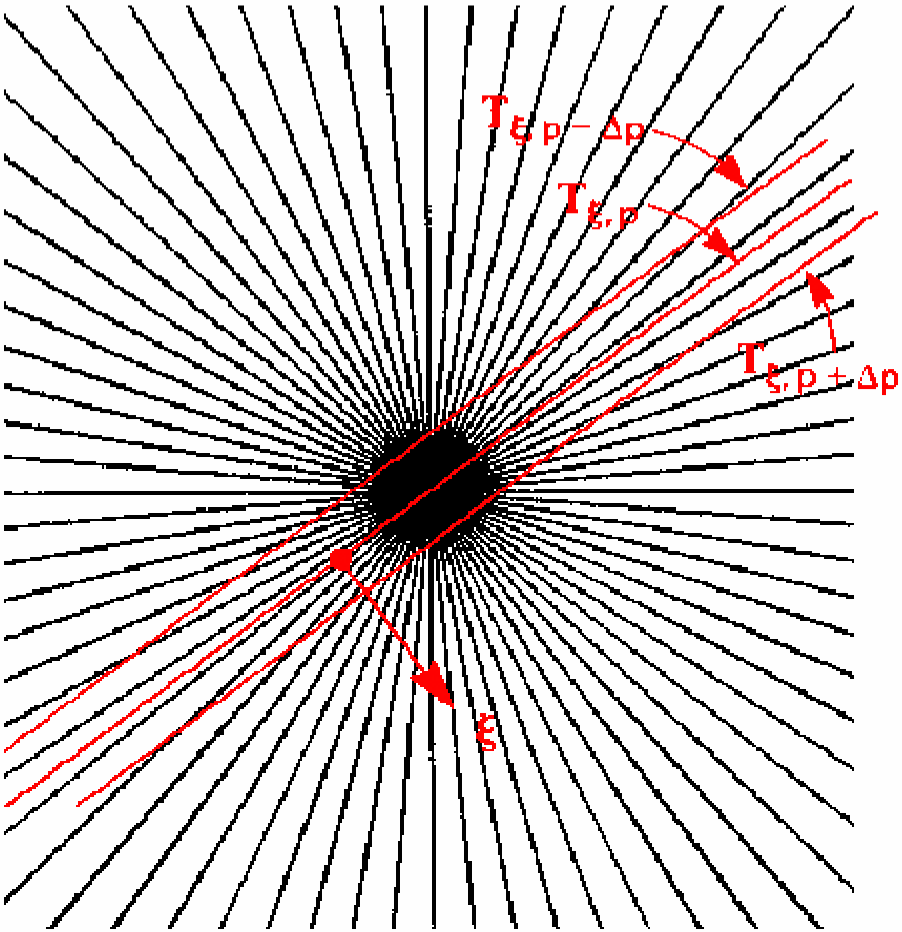
Inverse Radon Transform



$$f(\mathbf{x}_o) = -\frac{1}{8\pi^2}\int d^2\xi \, \frac{\partial^2}{\partial p^2} f^{\triangle}(\xi, p = \xi \cdot \mathbf{x}_o)$$

$f(\mathbf{x}_o)$ is the filtered average of $f^{\triangle}$ over all planes passing through $\mathbf{x}_o$.

The filter is a second derivative with respect to parallel planes.

Bandwidth is needed in spatial frequency and in **angle**.

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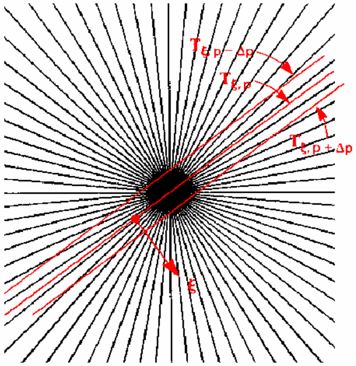


2 Radon Inversion

2.1 Forward Transform

- Given a scalar potential $f(\mathbf{x})$ define the Radon Transform:

$$f^{\triangle}(\xi, p) = \int d^3\mathbf{x}\, f(\mathbf{x})\, \delta(p - \xi \cdot \mathbf{x}). \tag{1}$$

- ξ is a unit orientation vector normal to the plane of integration. p is a scalar parameterizing parallel planes.

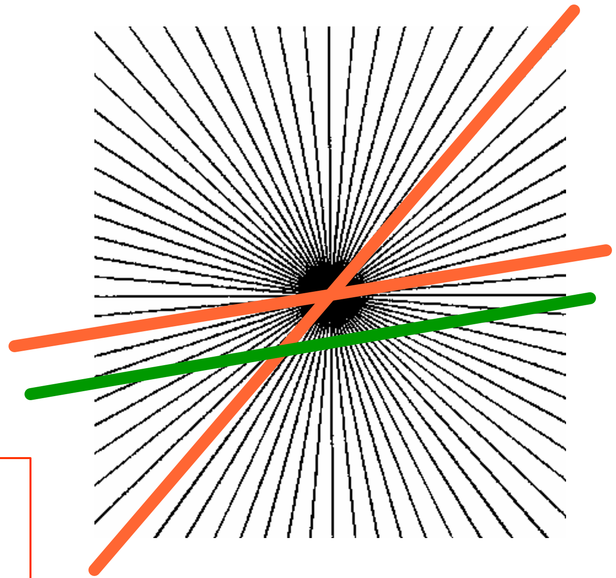
2.2 Inverse Transform

- f can be recovered from $f^{\triangle}$ by the Radon Inversion Formula (filtered backprojection):

$$f(\mathbf{x}_o) = -\frac{1}{8\pi^2} \int d^2\xi\, \frac{\partial^2}{\partial p^2} f^{\triangle}(\xi, p = \xi \cdot \mathbf{x}_o). \tag{2}$$

Physics

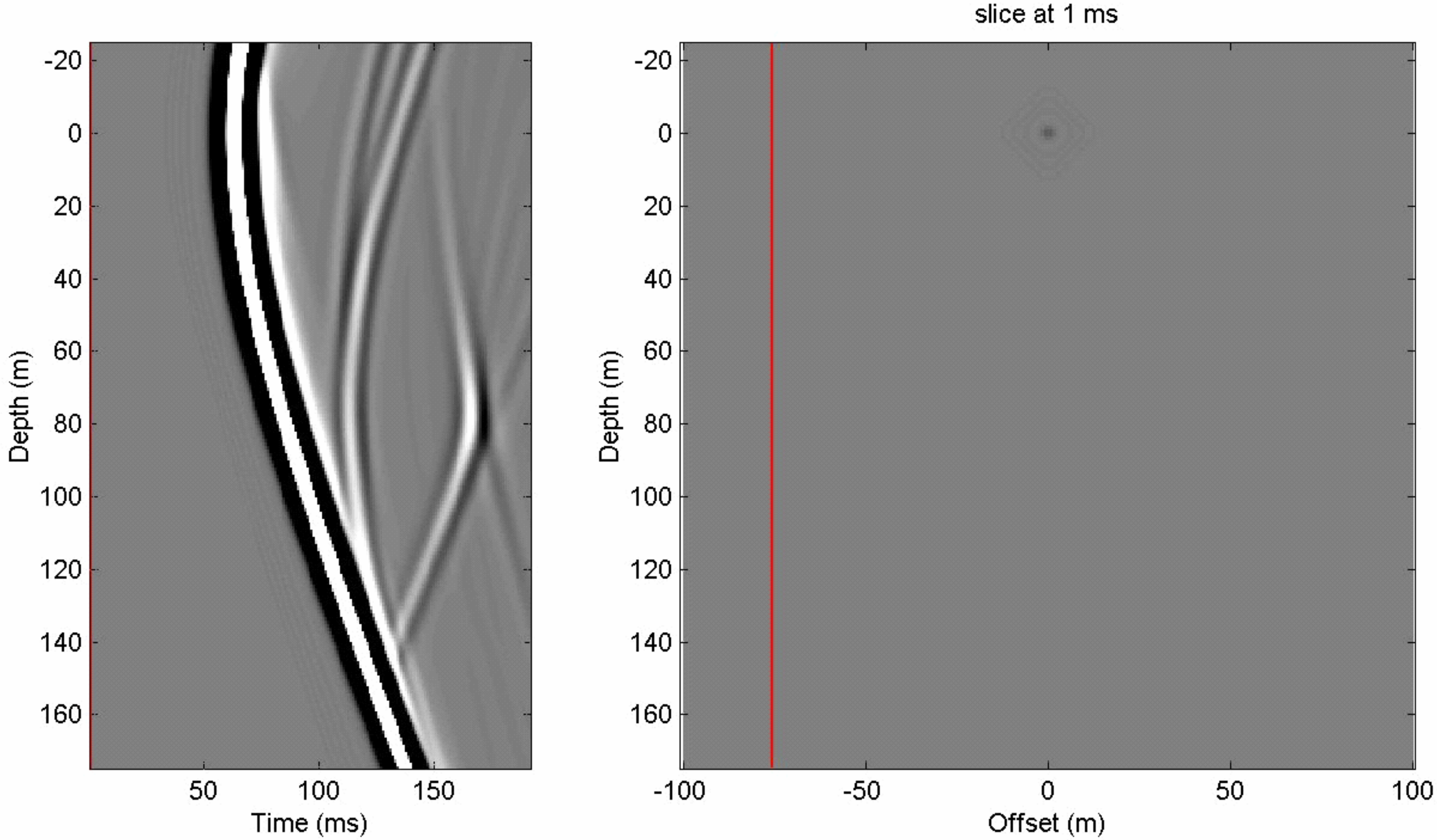
- **Data** consists of integrals
  - X-Ray tomography:
    - X-Ray beams
  - MRI:
    - Surfaces of constant magnetic field strength
  - Slowness tomography:
    - raypaths
  - Seismic Migration/Inversion:
    - Surfaces of constant total traveltime (a.k.a. isochrons)



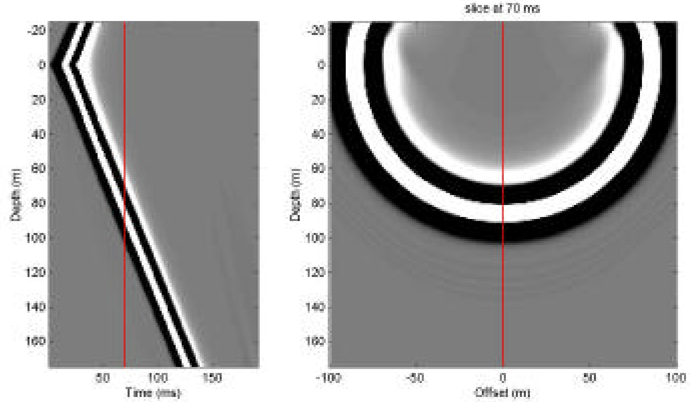
- Appropriate scattering theories relate physical quantities to surface or line integrals.

- **Inversion** by backprojection of filtered data.

A Short Movie



Background



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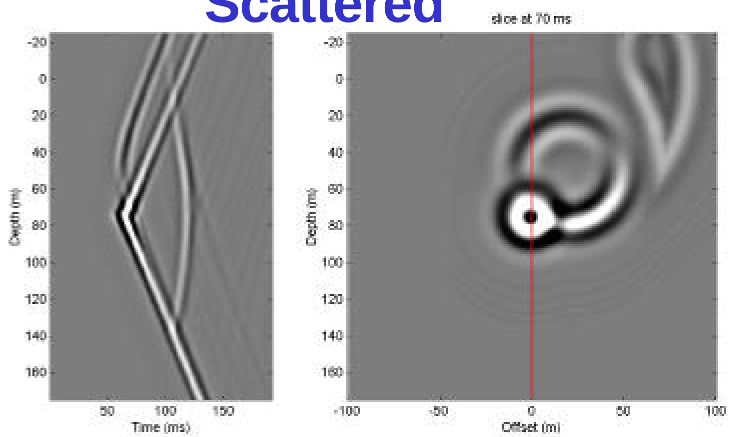
Scattered



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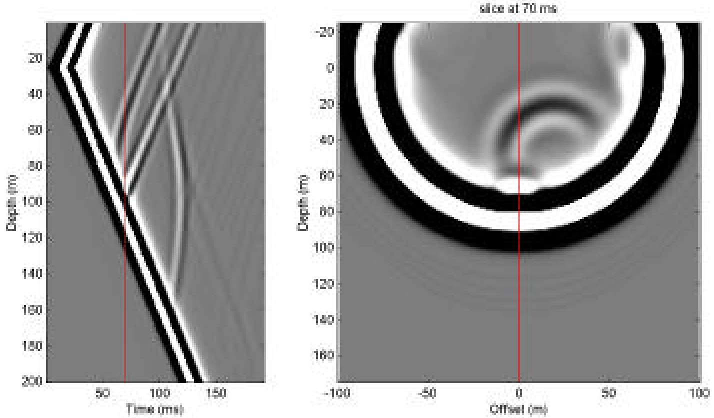
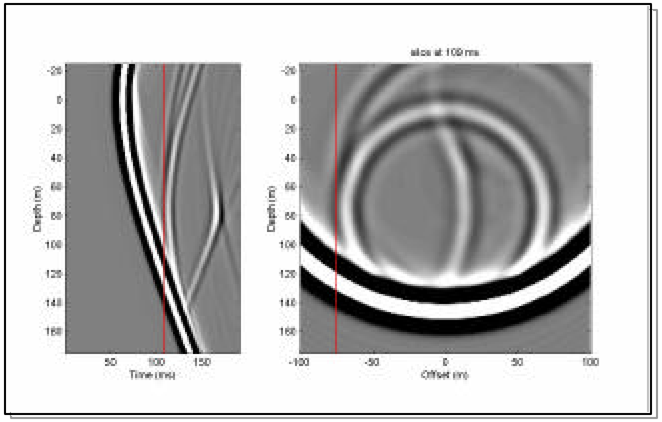
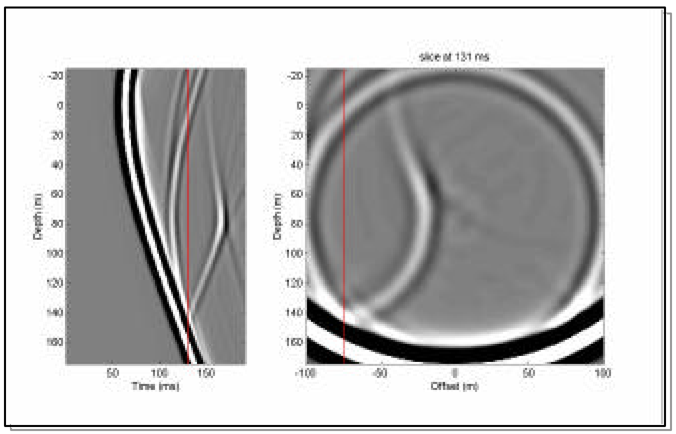
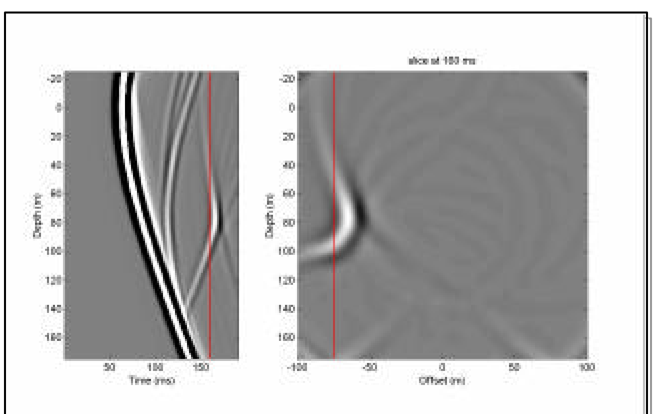
Total

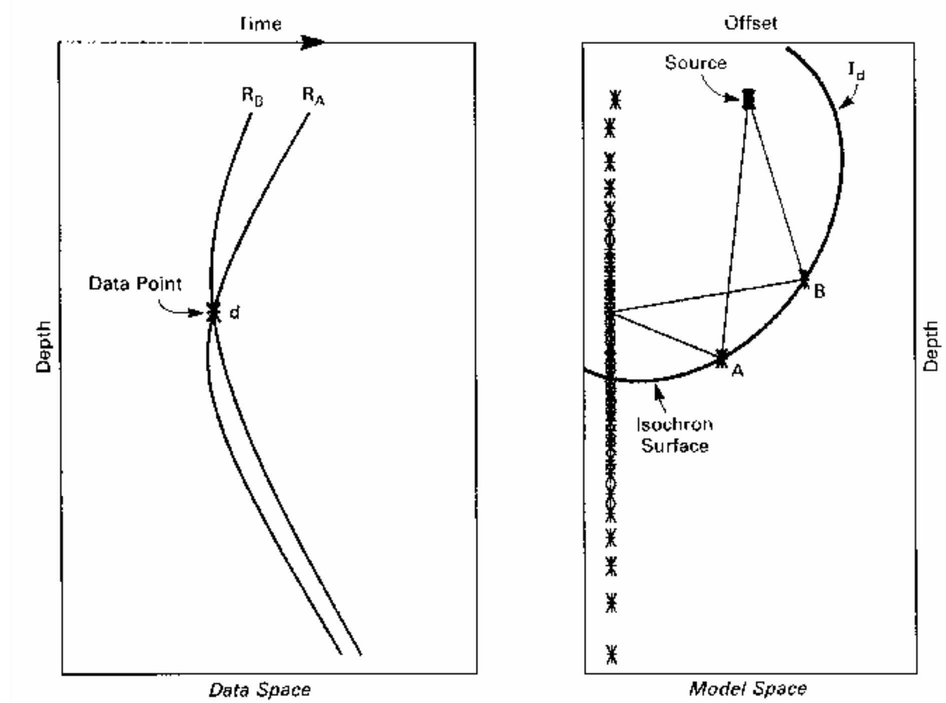


Dual Surfaces









Points in earth correspond to **Reflection Time Surfaces in data**.

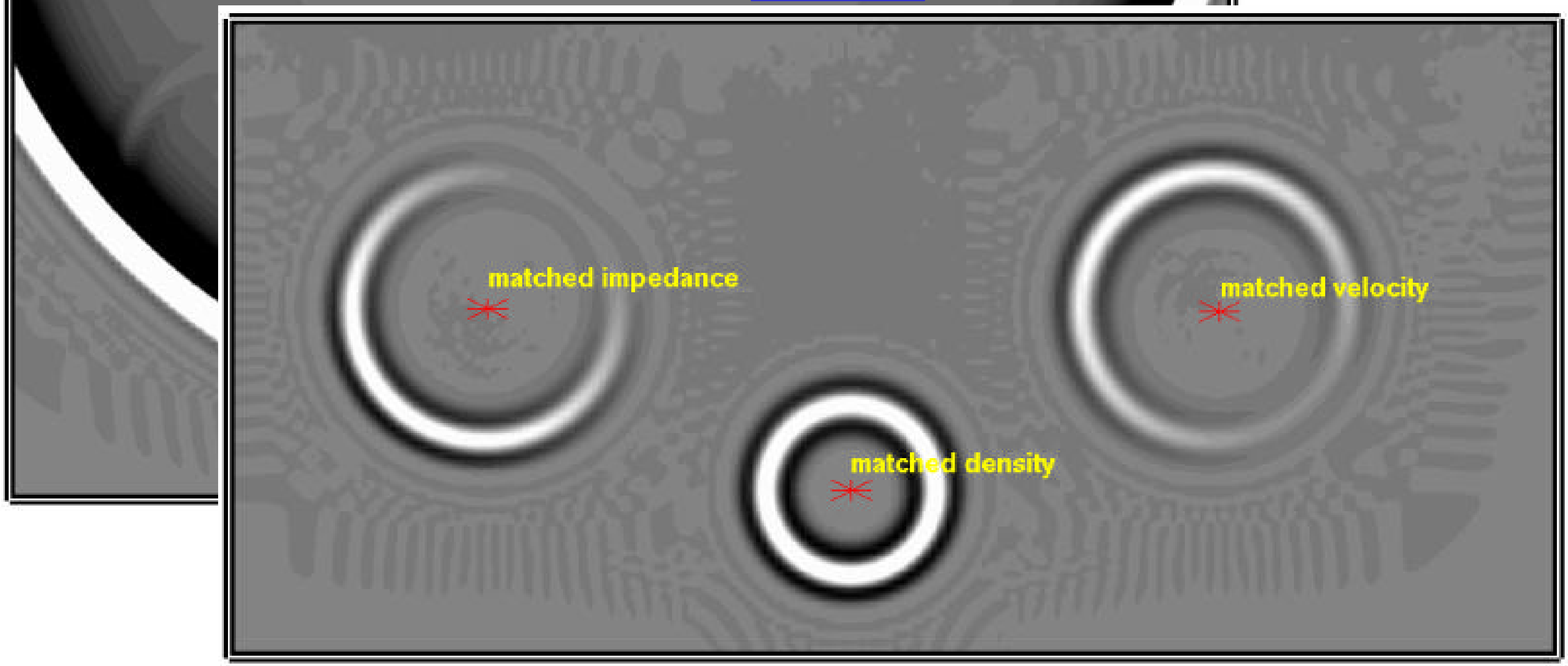
Points in data correspond to **Isochron surfaces in earth**

$$u_{sc}(\mathbf{r},\mathbf{s},\omega) = \int d^3\mathbf{x}\, G(\mathbf{r},\mathbf{x},\omega)\left[\omega^2\kappa + \nabla\sigma\nabla\right] u(\mathbf{x},\mathbf{s},\omega)$$

Propagation

Scattering

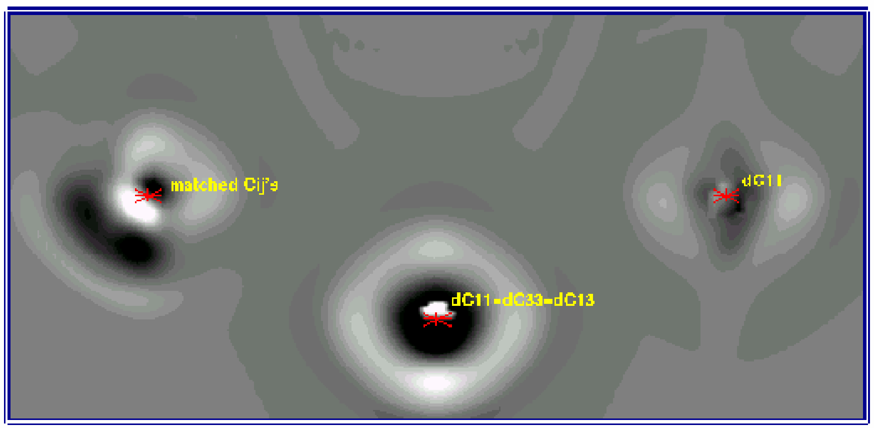
Incident field



The Full Catastrophe

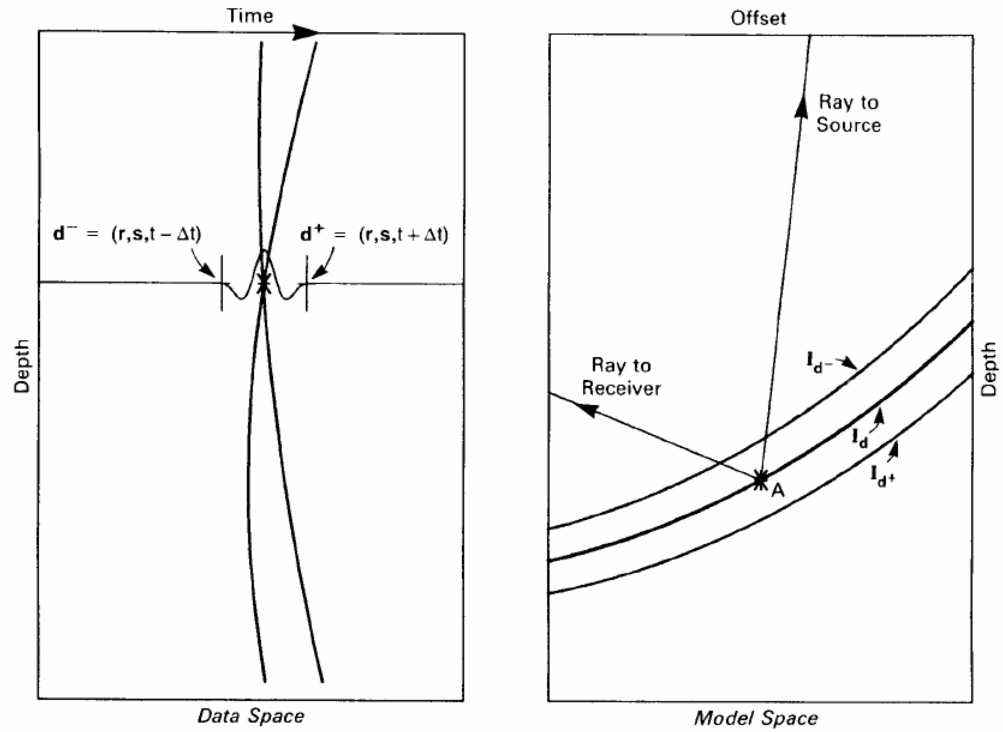
$$u_{pq}^{(1)(NM)}(\boldsymbol{r},\boldsymbol{s},t) = -\int_{\mathcal{D}} \grave{A}(\boldsymbol{x})\acute{A}(\boldsymbol{x})\grave{\xi}_k(\boldsymbol{x})\grave{\xi}_q(\boldsymbol{s})\acute{\xi}_p(\boldsymbol{r})\acute{\xi}_i(\boldsymbol{x})$$

$$\times\, \delta''(t-\grave{\tau}(\boldsymbol{x})-\acute{\tau}(\boldsymbol{x}))\,[\rho^{(1)}(\boldsymbol{x})\delta_{ki} + c_{ijk\ell}^{(1)}(\boldsymbol{x})\grave{\gamma}_\ell(\boldsymbol{x})\acute{\gamma}_j(\boldsymbol{x})]\,\mathrm{d}\boldsymbol{x}.$$



$$\frac{1}{A^{(N)}}\frac{\mathrm{d}A^{(N)}}{\mathrm{d}\tau^{(N)}} = -\tfrac{1}{2}\boldsymbol{\nabla}\cdot(\rho^{(0)}\boldsymbol{v}^{(N)}).$$

$$u_{sc}(\mathbf{r}, \mathbf{s}, t) = -\frac{\partial^2}{\partial t^2} \int d^3\mathbf{x} \, [\kappa + \sigma \cos\theta] \, A(\mathbf{r}, \mathbf{x}, \mathbf{s}) \, \delta[t - \tau(\mathbf{r}, \mathbf{x}, \mathbf{s})]$$



- $A = A(\mathbf{r})A(\mathbf{s})$ is a product of geometrical spreading terms.

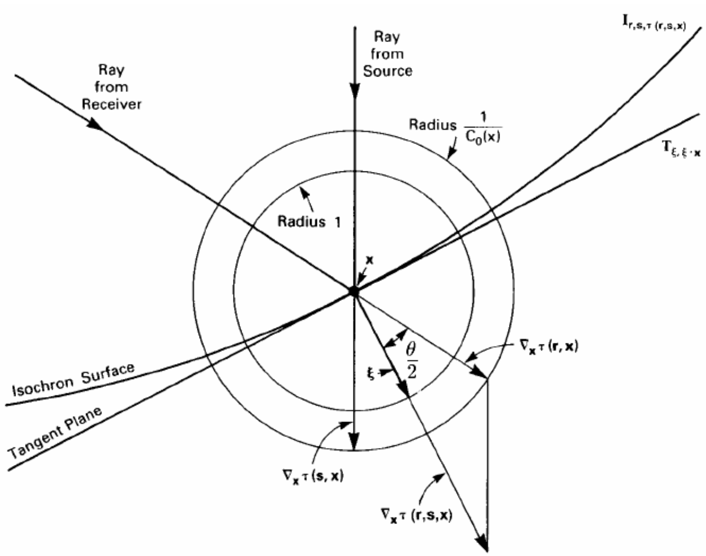
- $\tau(\mathbf{r}, \mathbf{x}, \mathbf{s}) = \tau(\mathbf{r}, \mathbf{x}) + \tau(\mathbf{x}, \mathbf{s})$ is a sum of source and receiver traveltimes.

- u_{sc} represents a (filtered) Generalized Radon Transform over an isochron surface $I_{\mathbf{r},\mathbf{s},t}$ of a "scattering potential" $[\kappa + \sigma \cos\theta]$.

Inverse Acoustic GRTransform

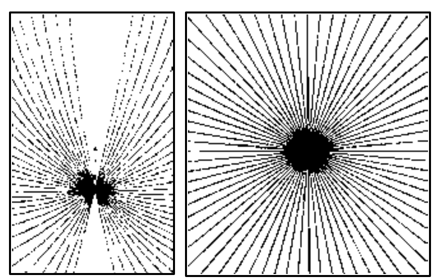
At each image point we match isochron surfaces with their tangent planes and change variables in the Radon inversion formula.

$$f(\mathbf{x}_o) = -\frac{1}{8\pi^2}\int d^2\xi \; \frac{\partial^2}{\partial p^2} f^{\triangle}(\xi, p = \xi \cdot \mathbf{x}_o)$$



- Radon's variables are ξ (= dip) and p (= distance normal to the plane). Experimental variables are $\mathbf{r}$, $\mathbf{s}$, and t.

- t maps directly to p: $\quad \frac{\Delta t}{\Delta p} = 2\cos(\frac{\theta}{2})$
- $\frac{\partial^2}{\partial p^2}$ maps to $\cos^3(\frac{\theta}{2})\,\frac{\partial^2}{\partial t^2}$
- $\mathbf{s}, \mathbf{r}$ map to ξ (and θ).



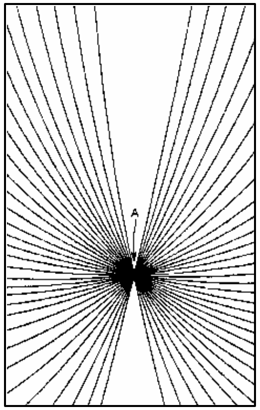
Acoustic GRT Inversion

$$f(\mathbf{x}_o) = -\frac{1}{8\pi^2}\int d^2\xi\, \frac{\partial^2}{\partial p^2} f^{\triangle}(\xi, p = \xi\cdot\mathbf{x}_o)$$

$$u_{sc}(\mathbf{r},\mathbf{s},t) = -\frac{\partial^2}{\partial t^2}\int d^3\mathbf{x}\,[\kappa + \sigma\cos\theta]\, A(\mathbf{r},\mathbf{x},\mathbf{s})\,\delta\,[t - \tau(\mathbf{r},\mathbf{x},\mathbf{s})]$$

- **Constant Density** ($\sigma = 0$). The scattering potential is κ, the inversion equation:

$$\langle\kappa(\mathbf{x}_o)\rangle = \int d^2\xi\, \frac{\cos^3(\frac{\theta}{2})}{A}\, u_{sc}(\mathbf{r},\mathbf{s},\tau(\mathbf{r},\mathbf{x},\mathbf{s})).$$



- **Constant Velocity** ($\sigma = \kappa$). The scattering potential is $\sigma + \sigma\cos\theta$ $(= 2\sigma\cos^2(\frac{\theta}{2}))$, the inversion equation:

$$\langle\sigma(\mathbf{x}_o)\rangle = \int d^2\xi\, \frac{\cos(\frac{\theta}{2})}{A}\, u_{sc}(\mathbf{r},\mathbf{s},\tau(\mathbf{r},\mathbf{x},\mathbf{s})).$$

- $d^2\xi$ represents a Jacobian term that relates dip at the image point to trace index.

- t maps directly to p: $\frac{\Delta t}{\Delta p} = 2\cos(\frac{\theta}{2})$
- $\frac{\partial^2}{\partial p^2}$ maps to $\cos^3(\frac{\theta}{2})\,\frac{\partial^2}{\partial t^2}$
- $\mathbf{s},\mathbf{r}$ map to ξ (and θ).

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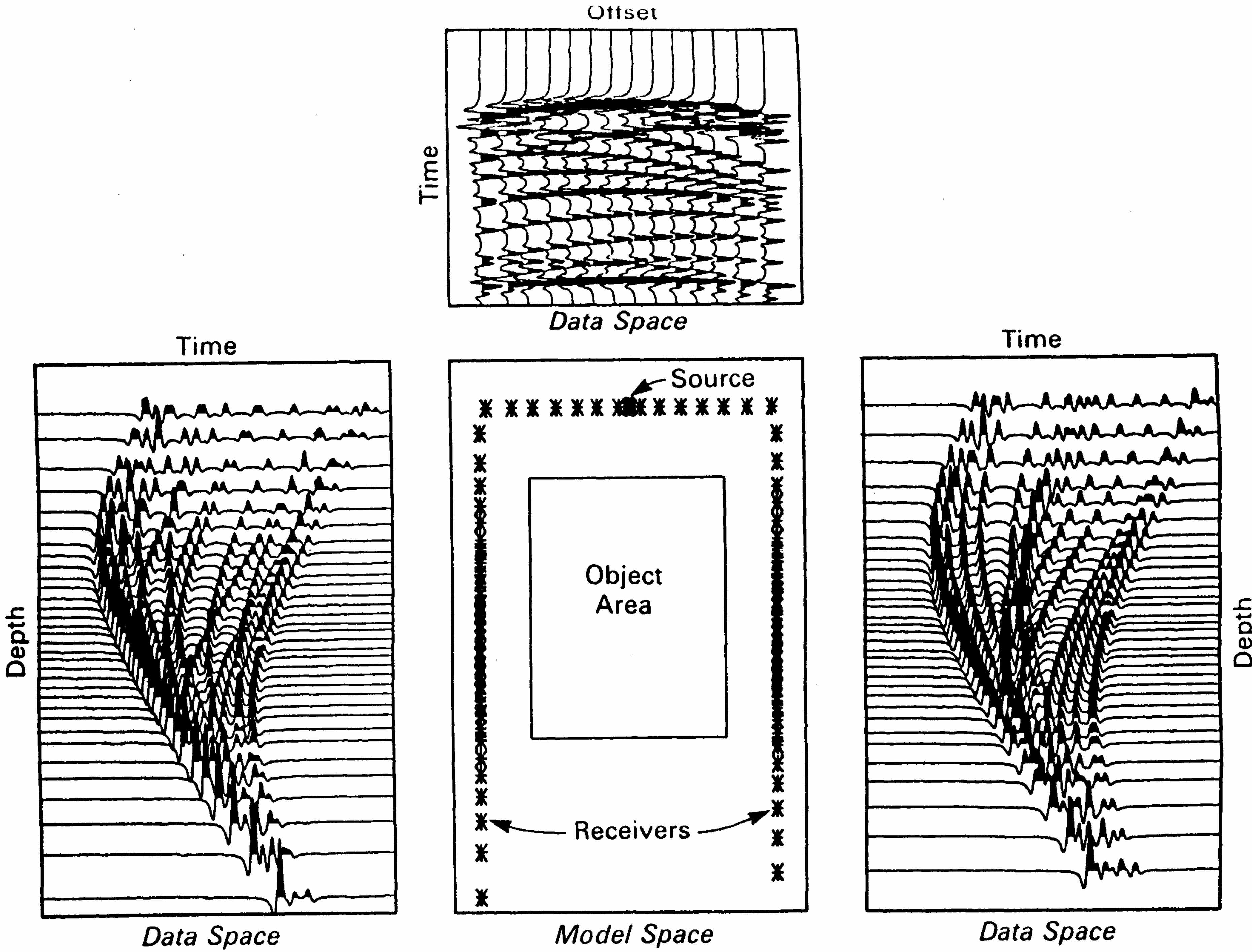
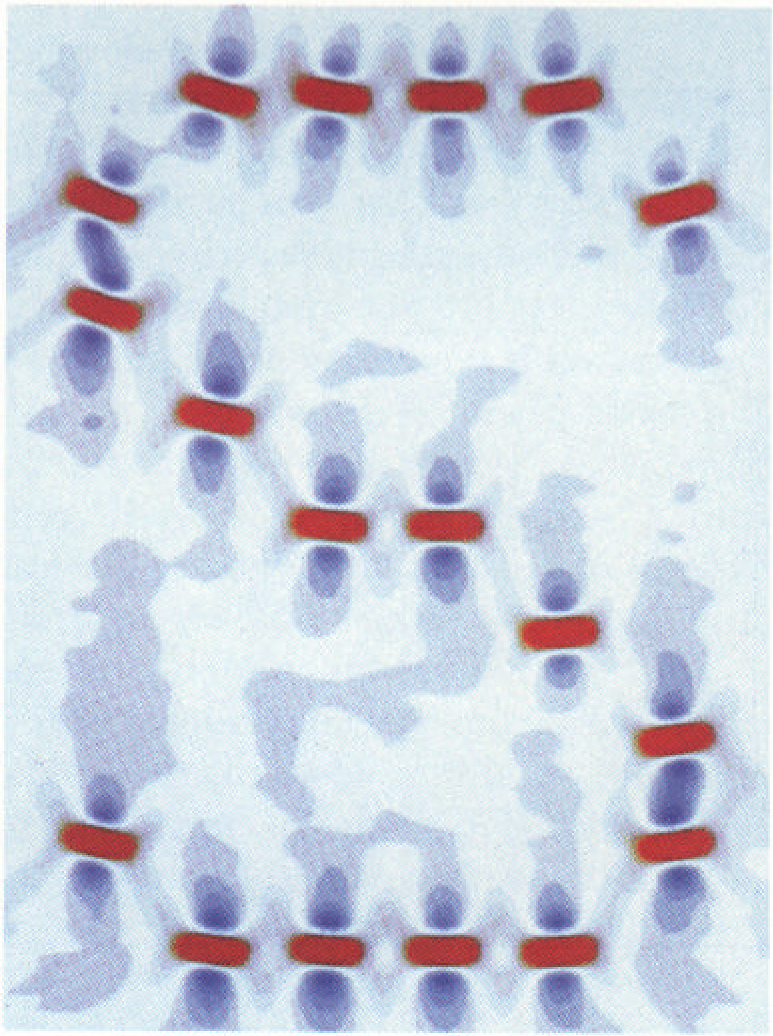
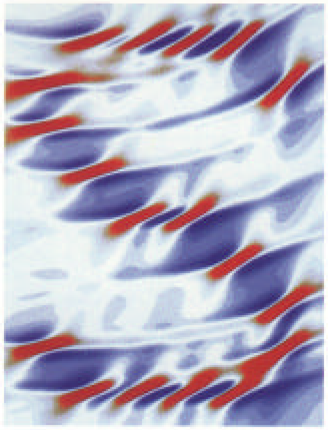
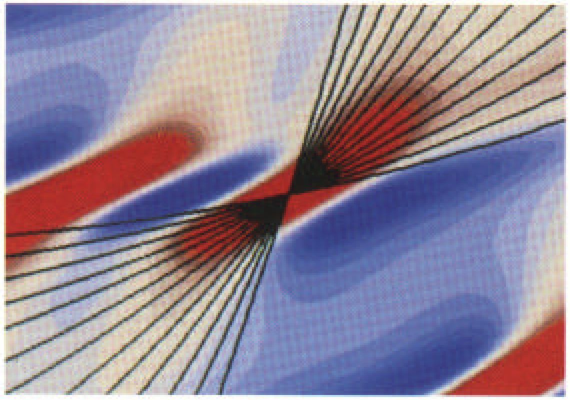

Offset
Time
Data Space
Time
Depth
Data Space
Source
Object Area
Receivers
Model Space
Time
Depth
Data Space

Each data trace contributes resolution perpendicular to the corresponding isochron







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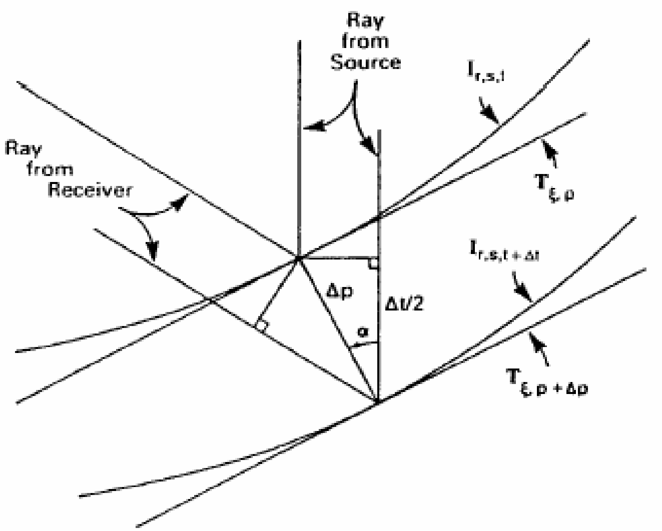
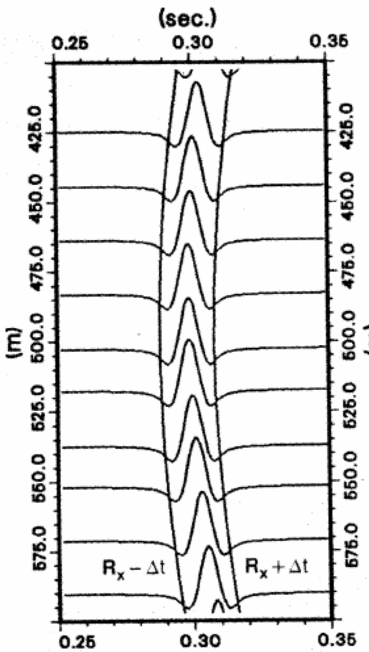
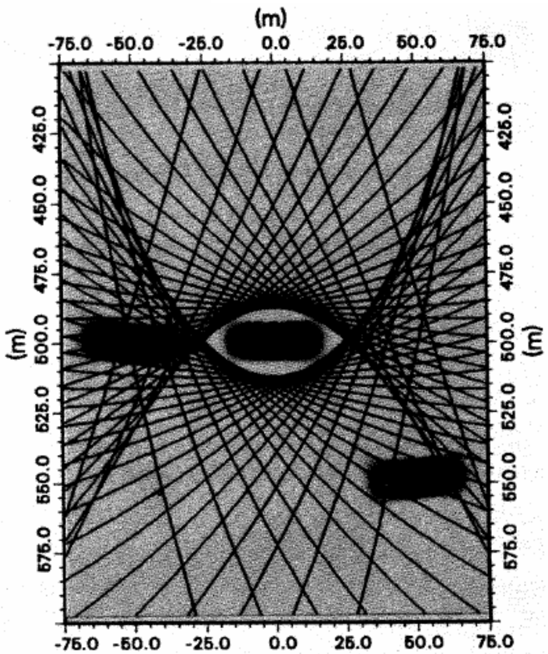
Large Scattering Angles map to low Spatial frequencies

$$\frac{\Delta t}{\Delta p} = 2\cos\left(\frac{\theta}{2}\right)$$

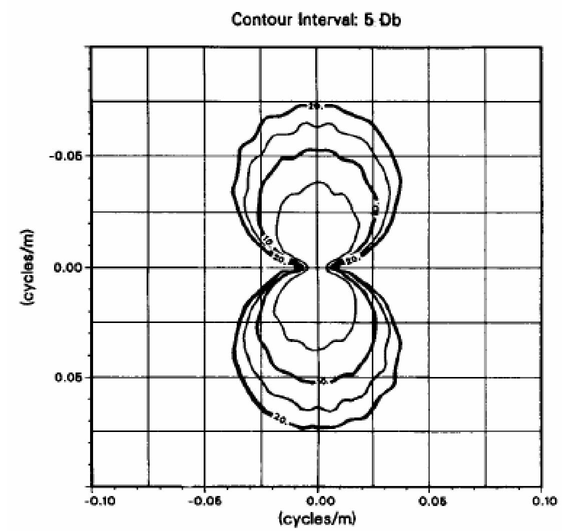




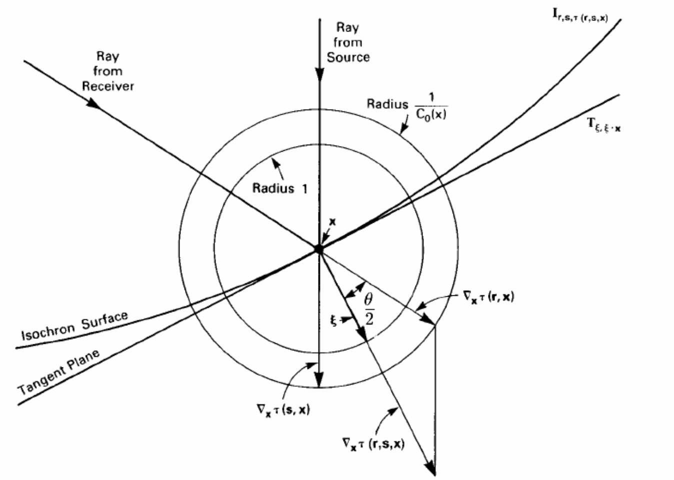
Data Space



Model Space



Acoustic GRT Inversion (Multiparameter Case)
(Beylkin & Burridge)



Now assume a 2-d acoustic world and a prestack surface-seismic geometry.

Basic GRT-inversion at fixed θ:

For any obliquity function $\mathcal{O}(\theta)$, define at each image point $\mathbf{x}$:

$$f_\theta^{\mathcal{O}}(x) = \int d\phi \, \frac{\mathcal{O}(\theta)}{A(s, \mathbf{x}, r)} U(s, r, t)$$

- U is the Hilbert transformed scattered field,
- s, r, t depend on $\phi, \mathbf{x}, \theta$

$$f_\theta^{\mathcal{O}}(x) = \int d\phi \, \frac{\mathcal{O}(\theta)}{A(s,\mathbf{x},r)} U(s,r,t)$$

- Take $\mathcal{O} = \mathcal{O}_0 = \cos^2(\frac{\theta}{2})$, Then(up to a global constant):

$$f_\theta^0 \approx \kappa + \sigma\cos(\theta)$$

(κ and σ are perturbations in bulk modulus and reciprocal density)

- $\mathcal{O}_1 = \cos(\theta)\cos^2(\frac{\theta}{2})$ gives

$$f_\theta^1 \approx \kappa\cos(\theta) + \sigma\cos^2(\theta)$$

- Integrate over θ to get material parameters:

$$f^0 = \kappa, \quad f^1 = 4\pi\sigma$$

for good geometry,

a 2x2 system in κ, σ for bad geometry.

Anisotropic GRT Inversion

1 MOTIVATION

1.1 Larut Case Study

Miller, Leaney, Borland 1994, JGR

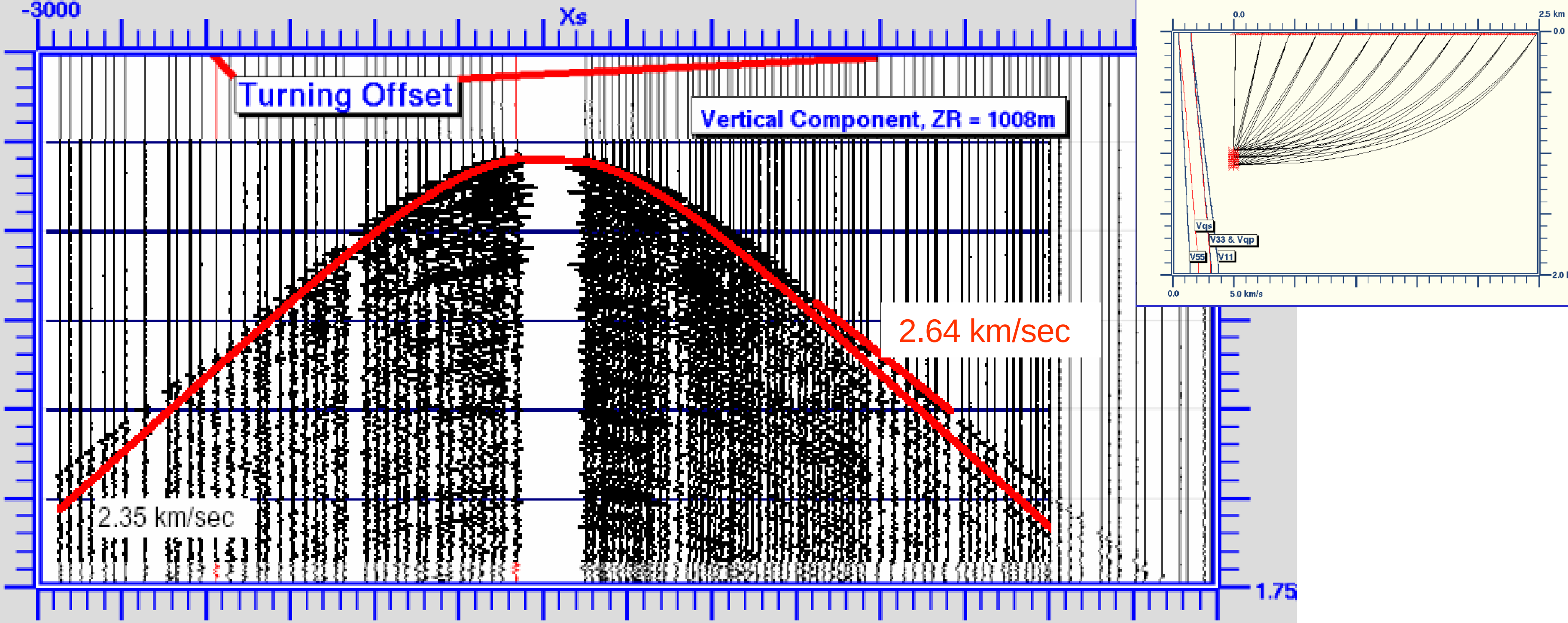
- Walkaway VSP data from a single azimuth were fit to 1% accuracy by a TIV model. Measured at z = 1 km,

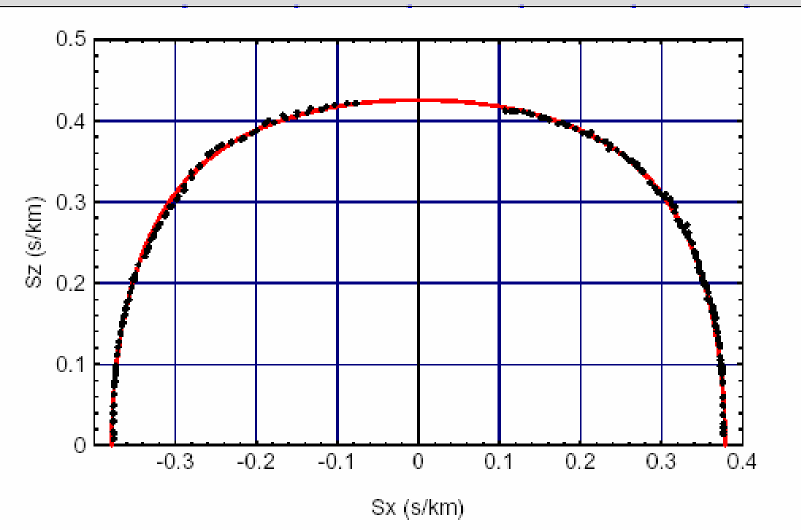
$$V_{\text{vert}} \approx V_{45^\circ} \approx .88\, V_{\text{horiz}}.$$

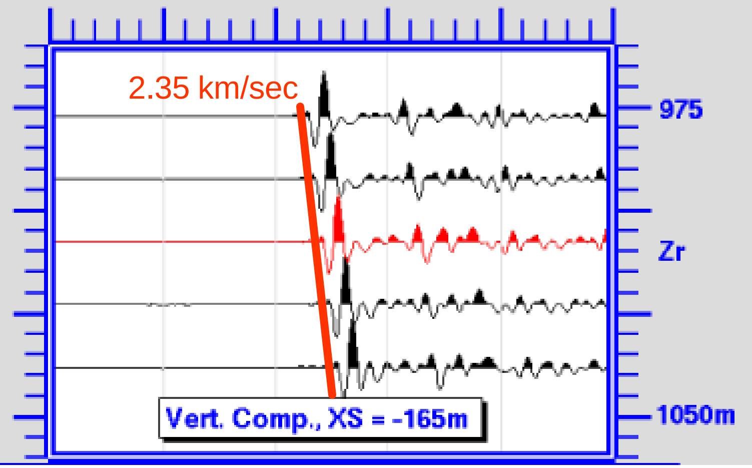
- Logs and a vertical VSP show that vertical velocities increase approximately linearly with depth from seabottom to about 3km. Similar trends for the other velocities are consistent with available data.

1.2 Computational Consideration

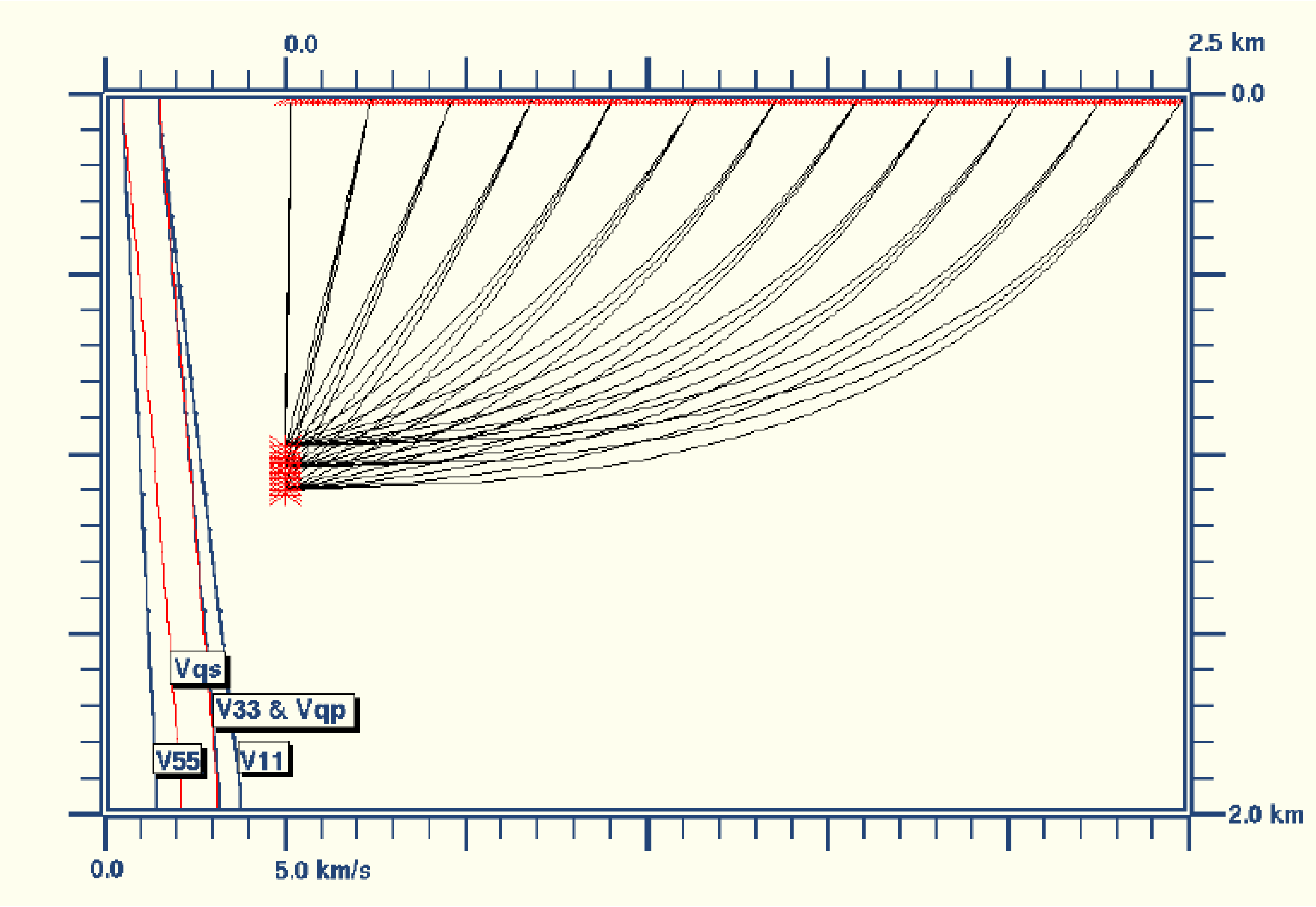
- It is practical to do 3D linearized forward modelling and inversion in a 1D background medium by precomputing all Green functions needed by diffraction integrals.







WVSP in an anisotropic medium


0.0
2.5 km
0.0
2.0 km
0.0
5.0 km/s
Vqs
V33 & Vqp
V55
V11

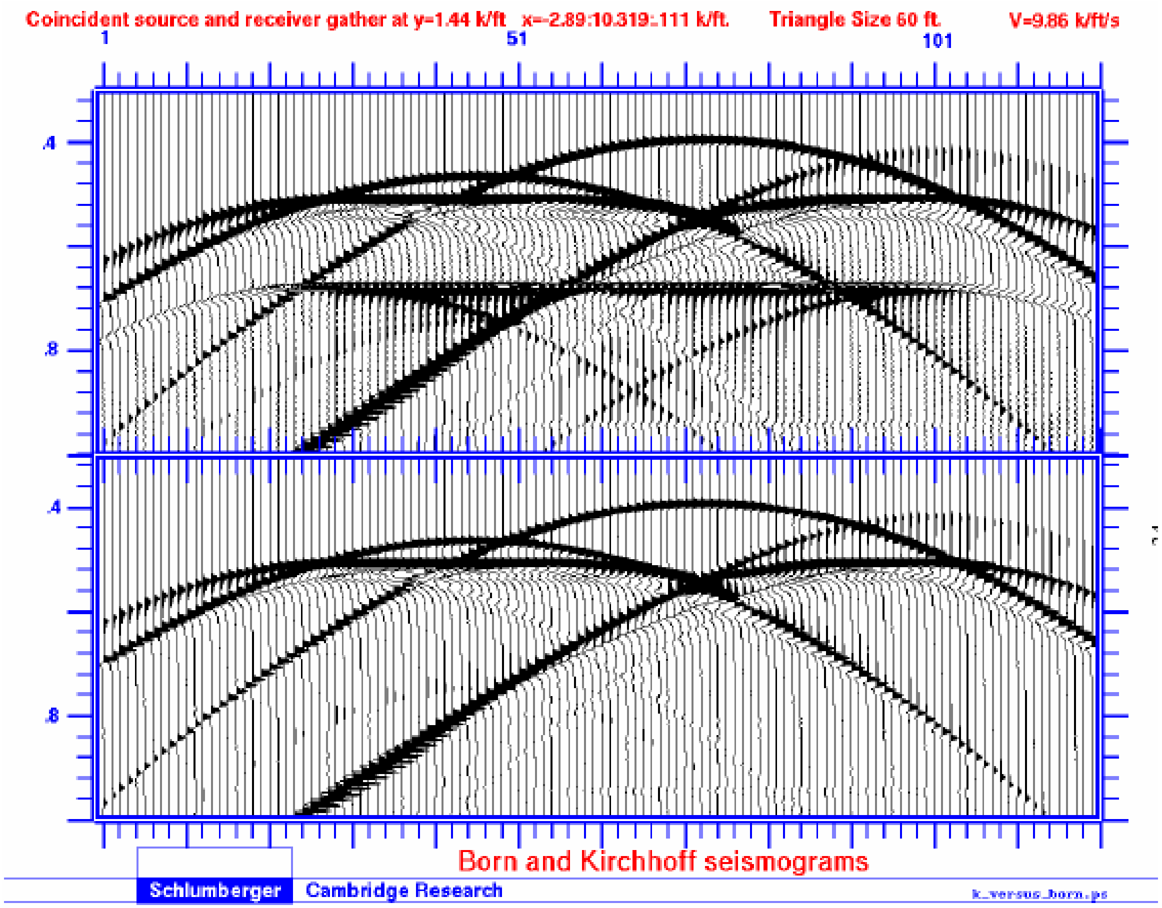


Figure 22: A coincident source and receiver profile over the French model calculated by Born volume integration (top) and Kirchhoff surface integration (bottom).

9 TI Zero-offset GRT Migration/Inversion

9.1 GRT inversion formula:

$$\langle f(\mathbf{x}_o) \rangle = \frac{1}{\pi^2} \int d^2\xi(\mathbf{s}, \mathbf{x}_o) \; \frac{|\beta(\mathbf{s}, \mathbf{x}_o)|^3}{A(\mathbf{s}, \mathbf{x}_o)^2} \; u_{sc}(\mathbf{s}, t = \tau_o).$$

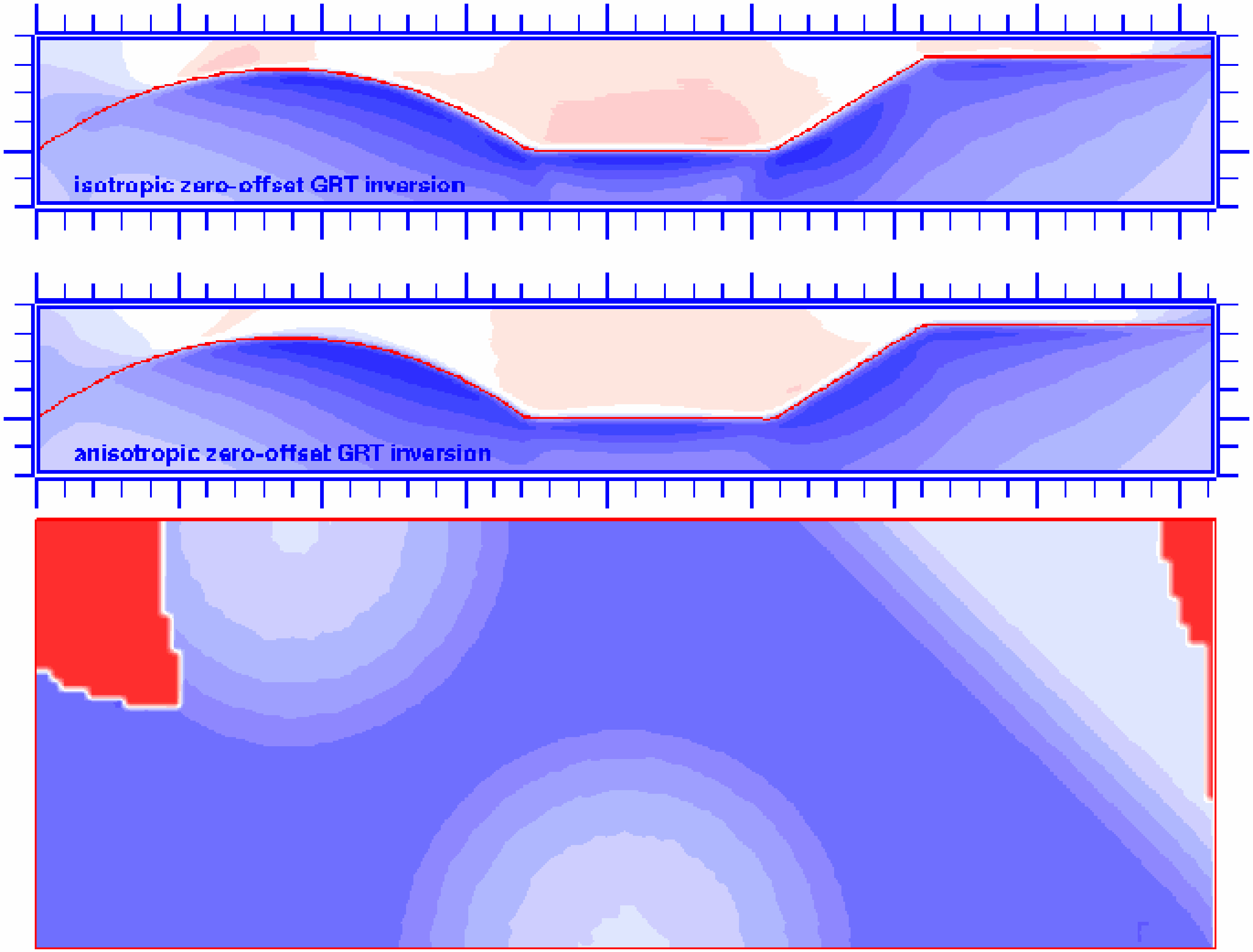
9.2 Simplification

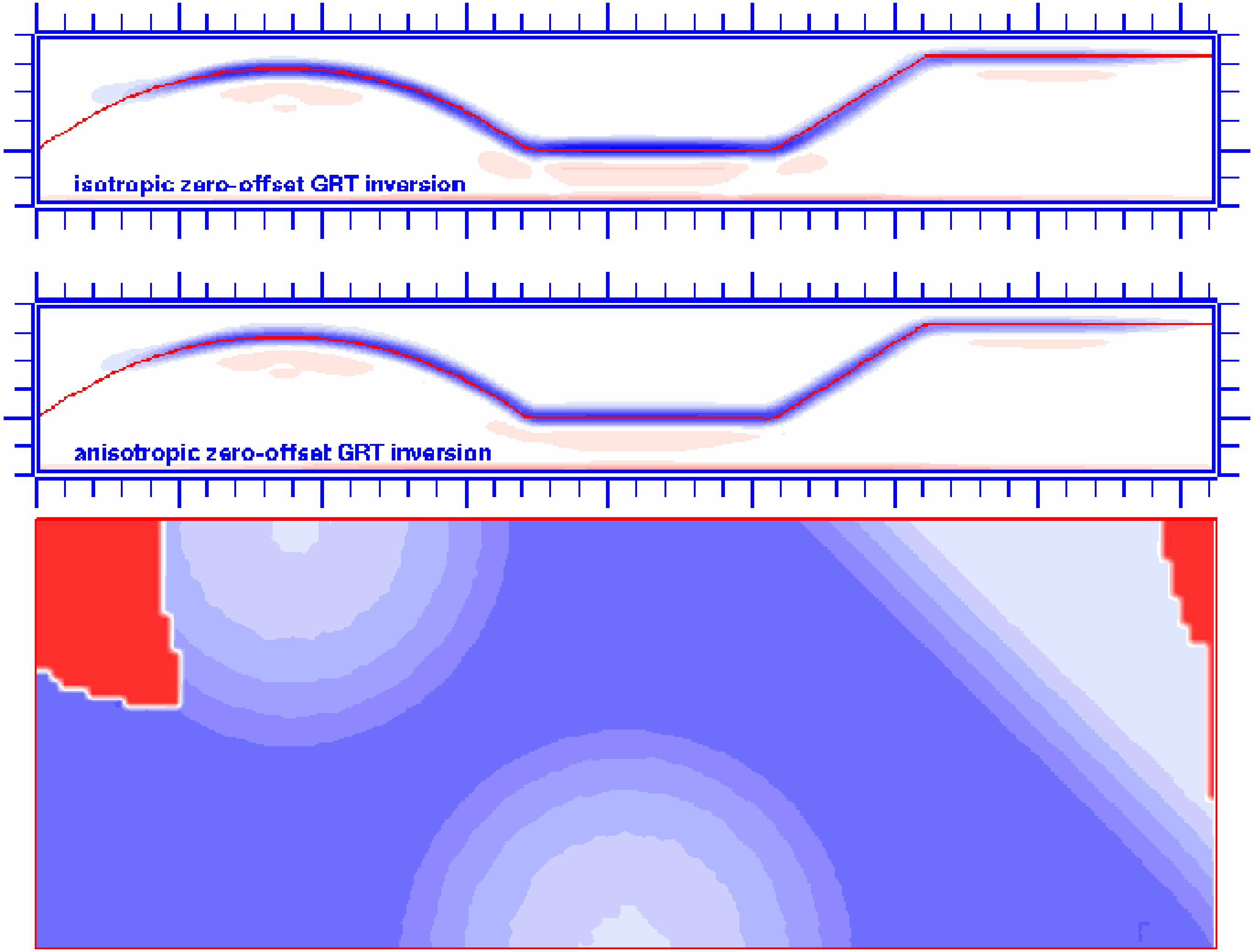
$$d^2\xi \frac{\beta^2}{A^2(\mathbf{s}, \mathbf{x}_o)} = ds_1 \, ds_2 \; \cos(\alpha)$$

where α is the vertical phase angle at the surface.

9.3 What I Calculated:

$$\int ds_1 \, ds_2 \; \cos(\alpha) \, |\beta(\mathbf{s}, \mathbf{x}_o)| \, u_{sc}(\mathbf{s}, t = \tau_o)$$


isotropic zero-offset GRT inversion
anisotropic zero-offset GRT inversion



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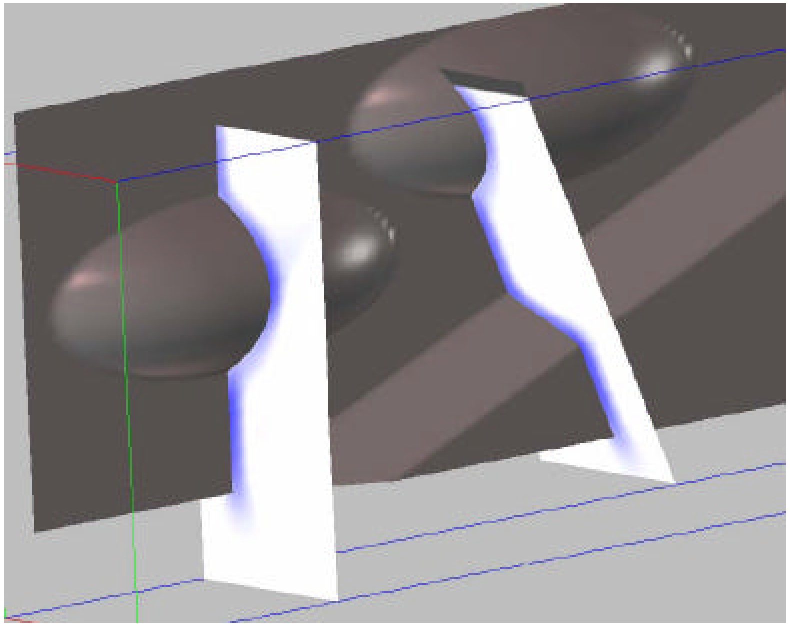
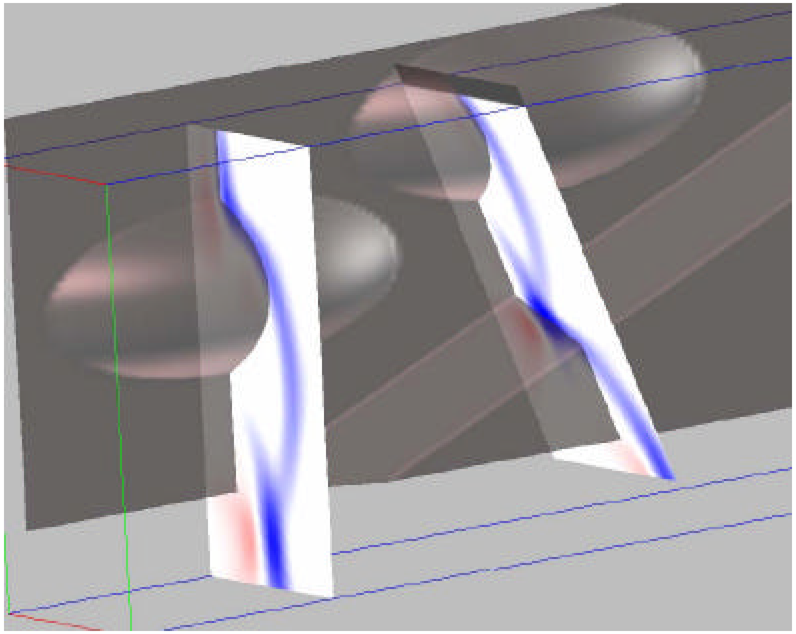
Turning-ray migration of Vertical Object



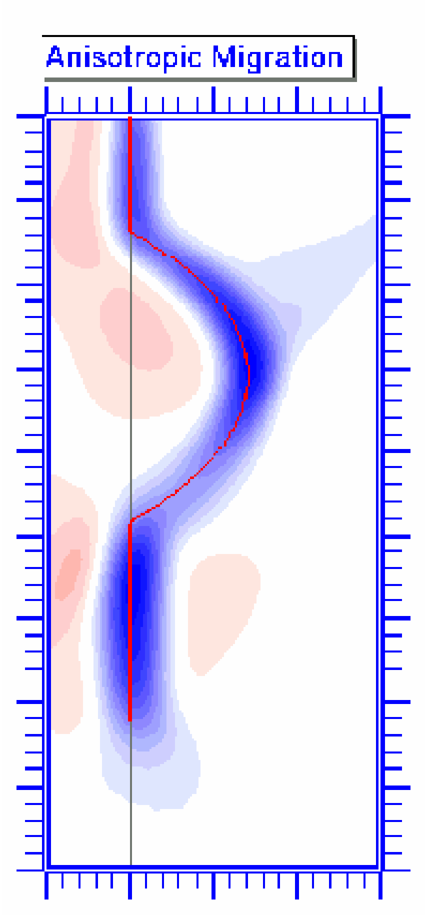
Anisotropic

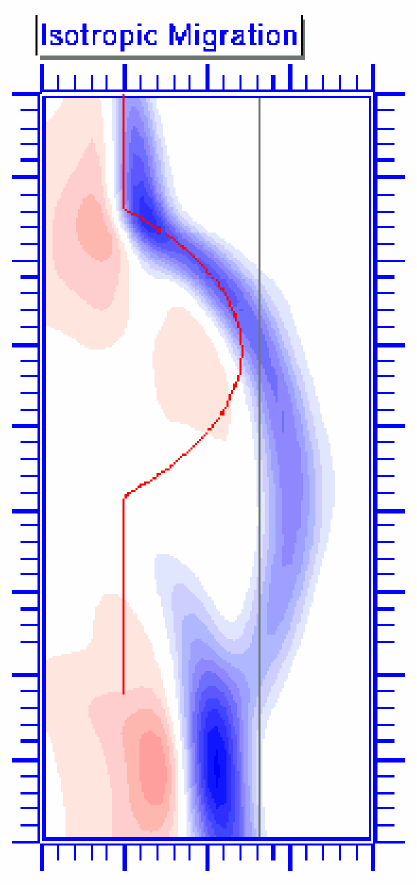


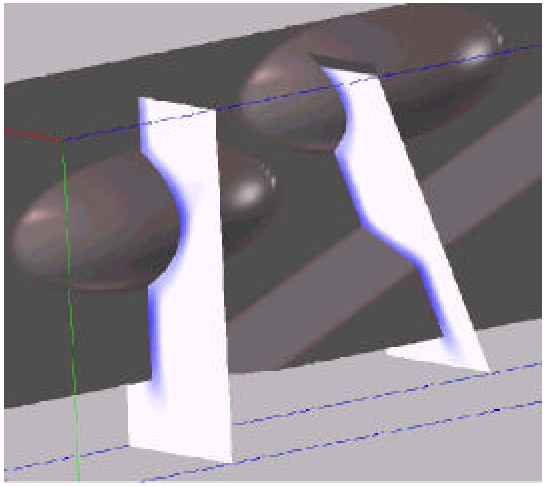
Isotropic

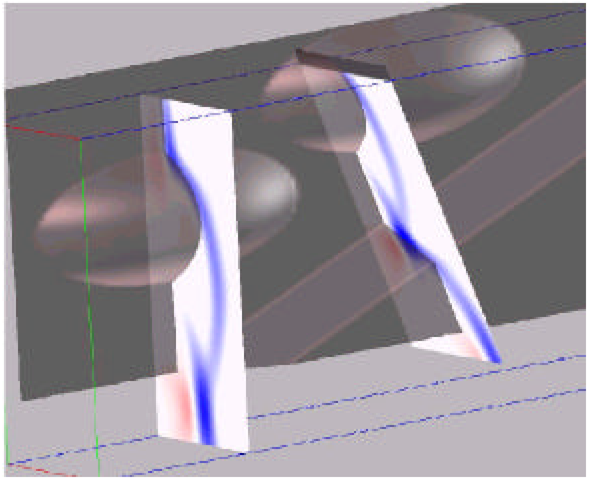
(vertical velocities)

Turning Ray Images

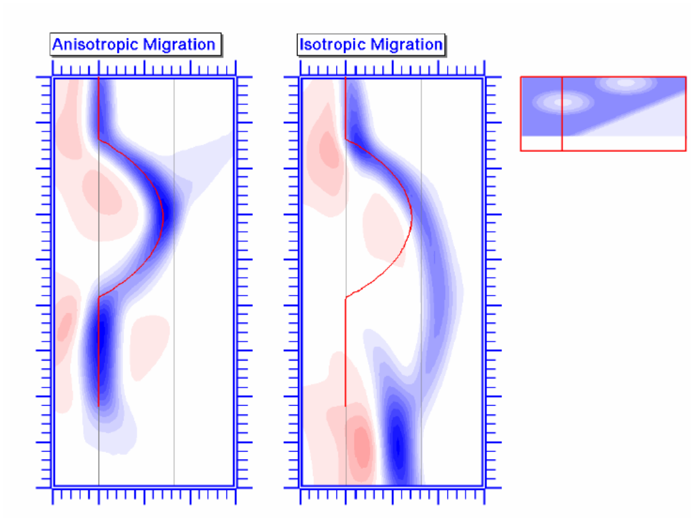


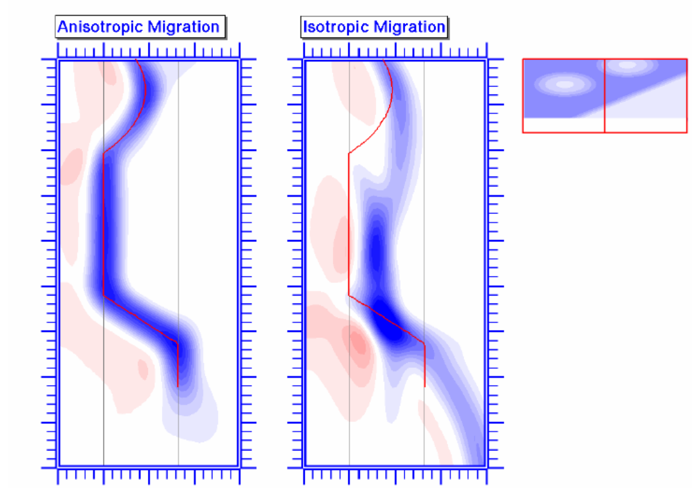




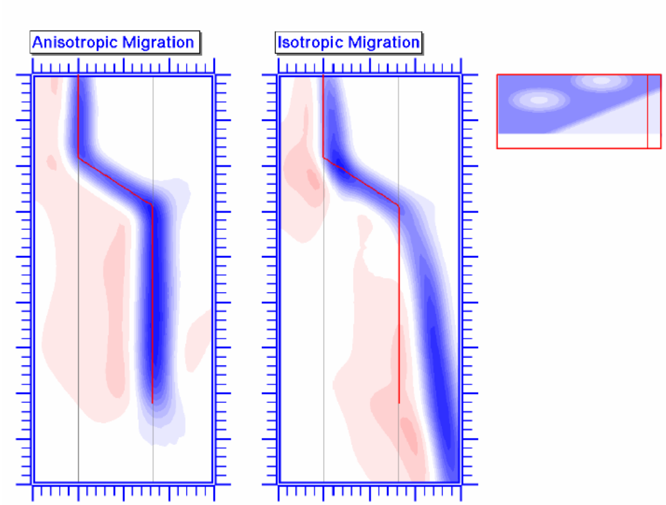


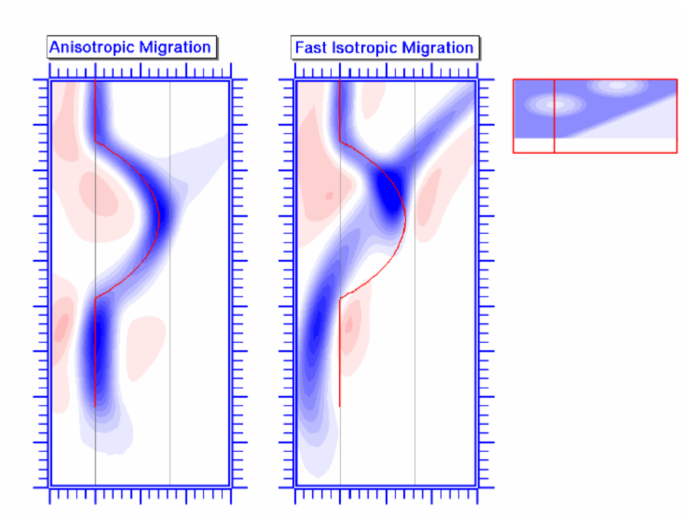
Schlumberger

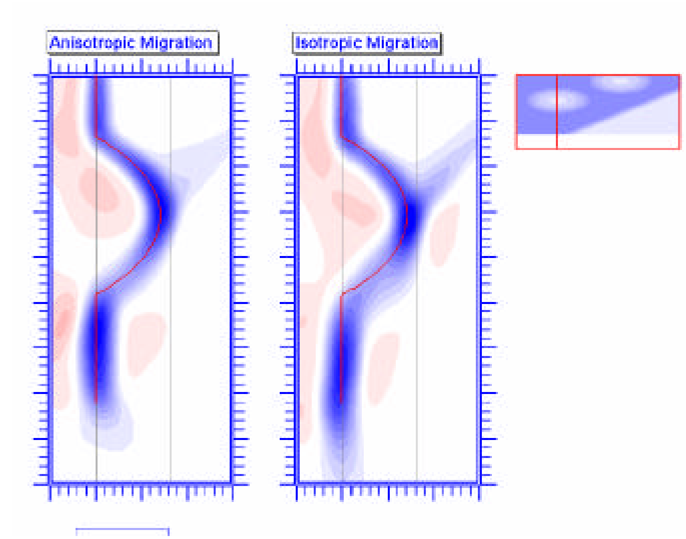




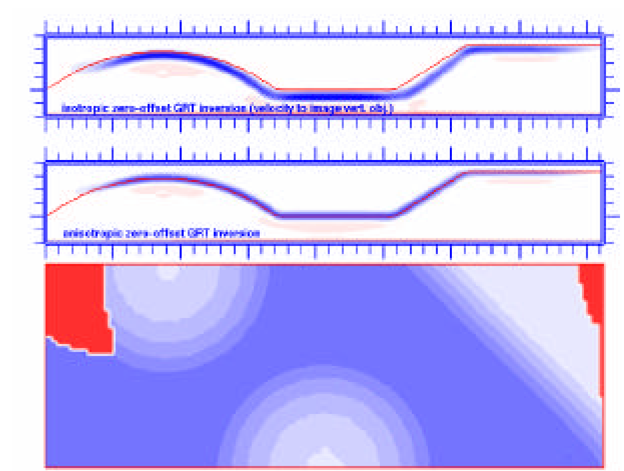
Isotropic Migration using vertical velocity profile systematically defocuses and mislocates vertical object







Isotropic Migration using a velocity profile that focuses the vertical object mislocates the horizontal object.



Multiparameter two-dimensional inversion of scattered teleseismic body waves
3. Application to the Cascadia 1993 data set

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