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Multiparameter inversion in anisotropic elastic media

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SUMMARY

In this paper, we formalize the linearized inverse scattering problem for a general anisotropic, elastic medium and describe two approaches to the construction of a stable inversion procedure. The first uses generalized Radon inversion and requires extra information limiting the independent variation of material parameters. The second uses a stationary phase approximation and requires extra information to the effect that the medium is everywhere locally stratified with known dip.

The point of common departure is the *single-scattering* or *Born approximation* to the scattered field in perfectly elastic media.

The formalism is simple to outline: the medium being modelled (or reconstructed) is thought of as a perturbation of a simpler, known, background medium. We are to find the unknown medium perturbation, given the scattered field, which is defined as the difference between the actual (total) field and the background field that would have been present if the actual medium were replaced by the background medium. If the background medium is sufficiently smooth, the background field can be well approximated within the framework of ray theory for elastic waves. If the background medium is sufficiently close to the actual medium, the scattered field can be well approximated by an integral involving the background field and linear in the medium perturbation (the Born approximation).

Within this régime we show how to find which combinations of parameters can be determined for a given arrangement of sources and receivers (acquisition geometry). Many expressions simplify when sources and receivers coincide (zero-offset), but then only one parameter may be reconstructed.

At several points we use 'generalized' linear inversion, implemented through a singular-value decomposition, which enables us to find and rank the best-determined linear combinations of the unknown parameters.

Our first method of inversion depends upon the inversion formula for the generalized Radon transform (GRT) to leading order asymptotically for high spatial frequencies. Our second method benefits from the use of further information to the effect that the medium has a locally stratified microstructure within which the medium properties vary rapidly in the direction normal to the local layering, and, assuming that the orientation of the layering can be estimated separately, the scattering problem becomes locally 1-D and the inversion procedure reduces to an amplitude versus (scattering) angle (AVA) analysis.

Key words: amplitude-versus-angle analysis, anisotropy, elasticity, generalized Radon transform, inversion, migration.

1 INTRODUCTION

General background

In this paper, we formalize the linearized inverse scattering problem for a general anisotropic, elastic medium and describe two approaches to the construction of a stable inversion procedure. The first uses generalized Radon inversion and requires extra information limiting the independent variation of material parameters. The second uses a stationary phase approximation and requires extra information to the effect that the medium is everywhere locally stratified with known dip.

The common point of departure is the *single-scattering* or *Born approximation* to the scattered field in perfectly elastic media, as introduced by Cohen & Bleistein (1977) and Bleistein & Cohen (1979).

The method is simple: the medium being modelled (or reconstructed) is thought of as a perturbation of a simpler, known, background medium. The scattered wavefield is the difference between the actual (total) field and the background field that would have been present if the actual medium were replaced by the background medium. If the background medium is sufficiently smooth, the background field can be well approximated using ray theory for elastic waves. If the background medium is sufficiently close to the actual medium, the scattered field can be well approximated by an integral involving the background field and linear in the medium perturbation (the Born approximation).

In the space–time domain, the integral operator that describes the forward scattering may be put into the form of a generalized Radon transform (GRT): the scattered displacement field at receiver r and time t due to source s is represented as an integral over an isochron surface defined as the set of points x satisfying

$$t=\tau(s,x)+\tau(x,r), \tag{1.1}$$

where $\tau(x,y)=\tau(y,x)$ is the ray-theoretical traveltime from x to y calculated in the background medium. The word 'generalized' in GRT refers to the fact that the integrals are over curved surfaces rather than planes. In the general case, the integrand is a linear combination of multiple parameters, the perturbations in the material parameters at the scattering point, with coefficients that depend on the geometry of the rays. Given a sufficient supply of seismic data and sufficient knowledge of the nature of the material perturbations (described below), the scattering medium can be approximately reconstructed by using a linear combination of generalized weighted backprojections, that is, integrals of the data over diffraction surfaces defined by the same traveltime equation (1.1), now taken as defining a relationship among t, s and r, with x fixed.

The two requirements (smooth background and small perturbation) define the limits of the method. Not all media can be decomposed in a way that satisfies both. By extending the formalism of Born forward modelling and inversion to the most general elastic media, we hope to increase the applicability of the method by widening the class of media available as background media. In particular, some media actually found in sedimentary basins (see e.g. Miller, Leaney & Borland 1994) may become tractable by these methods, once the background medium is allowed to be anisotropic.

Such techniques have been studied extensively for isotropic media. The observation that the forward linearized acoustic wave equation could be cast as a forward GRT and approximately inverted by a weighted backprojection operator was made by Norton & Linzer (1981) for coincident source/receiver geometry and by Miller, Oristaglio & Beylkin (1984) and Beylkin (1984) for other geometries. These papers tied the analysis to early work on diffraction-stack migration and to the theory of generalized Radon inversion that had been developed previously by Beylkin (1984, 1985). Beylkin & Burridge (1990) extended the method to elastic isotropic media and showed how to exploit the angle dependence of the scattering to solve a multiparameter inversion problem. Relations between the multiparameter problem and various 'scalarizations', as well as a GRT formalism for multiparameter inversion combined with dip-moveout (DMO) preprocessing, were described in Miller & Burridge (1992). The multiparameter (tensorial) inversion described in the present paper generalizes Beylkin & Burridge (1990) to anisotropic media and makes additional use of amplitudes as functions of scattering angle and azimuth to reconstruct as many parameter combinations describing the medium perturbation as the data admit.

The present work is a revised version of a 1994 Schlumberger Confidential Report (Burridge, de Hoop & Miller 1994), which was graciously released by Schlumberger and submitted for publication in 1995. It has subsequently been modified considerably in the light of a referee's helpful suggestions. The original Schlumberger Report has been partially reported, referred to, quoted, and extended extensively in subsequent publications. For instance, De Hoop *et al.* (1994) uses the techniques of this paper to study which combinations of parameters are best resolved by a particular acquisition geometry. De Hoop, Spencer & Burridge (1996) contains an extensive historical review and synthetic tests of the method. Most recently the work forms the basis of De Hoop & Bleistein (1997), in which the authors proceed to invert for reflection coefficients, which, within the Kirchhoff approximation, appear linearly in the scattering formulae even when material contrasts across interfaces are not small, a useful step towards a fully non-linear inversion.

The Born approximation for the scattering equation in anisotropic media was discussed by Gibson & Ben-Menahem (1991). Different migration techniques have been generalized to transversely isotropic media, and have been applied to real data (see Gonzalez, Lynn & Robinson 1991; Uren, Gardner & McDonald 1990). Larner & Cohen (1993) have discussed the mispositioning of reflectors when isotropic operators are used to migrate data acquired from anisotropic media.

Our inversion method is a synthesis of GRT migration and amplitude-versus-angle (AVA) analysis, and it follows an inversion formalism first introduced by Cohen & Bleistein (1977), Bleistein & Cohen (1979) and Bleistein (1987) by which boundary data may be inverted to find perturbations in material parameters as functions of position. Methods of migration go back to Hagedoorn (1954) and beyond (see Miller *et al.* 1984; Gardner 1985; the recent review by Stolt & Weglein 1985).

The multi-parameter aspects of the problem are treated using generalized linear inversion, which has been extensively studied in isotropic amplitude-versus-offset (AVO) analysis (see e.g. Van Rijssen & Herman 1991). This becomes apparent if the local dip of the discontinuity is reconstructed prior to the inversion in the framework of a stationary phase analysis, our second method. Then we can cast our formalism in a form that resembles AVA inversion. The separate problem of estimating the dip using pre-stack depth migration has been addressed by Lumley (1993).

Summary of the paper

In Section 2 we introduce some basic mathematical notation, state the anisotropic elastodynamic equation, define the Green's tensor, and introduce the perturbation scheme in which the medium perturbations act as scatterers. In Section 3 we introduce the traveltime function and the slowness, we approximate the Green's function by ray theory, and we non-dimensionalize the unknown quantities. The main result of Section 3 is the expression (3.32) for the first-order scattered field written as an integral, over an isochrone surface, of a linear combination of the non-dimensionalized unknown perturbations, the coefficients being functions of the ray theoretic quantities defined earlier. The section ends with some remarks on reducing the number of unknown parameters. In Section 4 we concentrate on a particular image point $\boldsymbol{y}$, and in preparation for use of the GRT inversion formula we set time t equal to the two-way traveltime (source s to y to receiver r) in (3.32). This causes the argument of a certain δ-function, which is a function of position, to vanish at y. We then scale this argument so that it has a unit gradient at y to obtain the integral representation for the scattered field in a new form (4.7) closely related to Beylkin's form of the Radon inversion formula. We first specialize to coincident source and receiver and find that we can then invert for only one combination of parameters for each type of scattering used (quasi-P to quasi-P, or quasi-S to quasi-S, say). We next allow source and receiver to vary independently. This enables us to use generalized linear inversion techniques, to solve for the unknown quantities listed as the vector $\mathbf{c}^{(1)}$. The results are estimates (4.18) for a single unknown parameter and (4.38) for several unknown parameters. In Section 5 we make use of extra geological information to the effect that the medium is stratified, so that locally the material parameters are rapidly varying functions of the curvilinear coordinate normal to the local layering. This extra information enables us to avoid using the GRT and leads to an inversion procedure less sensitive to deficiencies in angular coverage. Here, by means of a further asymptotic approximation to the scattered field, each data point is directly interpreted as revealing information about the neighbourhood of a particular specular reflection point, as in an AVA analysis. In Sections 4 and 5 the use of generalized inversion involving a singular value decomposition enables us to rank linear combinations of the unknown parameters in order of our ability to resolve them. The body of the paper ends with Section 6, which contains some concluding remarks. A glossary of symbols, the list of references, and several appendices follow, which deal with anisotropic ray theory, generalized inversion, and the form of certain Jacobians used to express integrals over ray directions at the image point as integrals over source and receiver positions.

2 THE BASIC EQUATIONS

In this section we will discuss the *single-scattering* or *Born approximation* to the scattered field in perfectly elastic anisotropic media.

As indicated above, the formalism is simple to outline: the medium being modelled (or reconstructed) is thought of as a perturbation of a simpler background medium. The scattered wavefield is the difference between the actual (total) field and the background field that would have been present if the actual medium were replaced by the background medium. If the background medium is sufficiently smooth, the background field can be well approximated within the framework of geometrical elasticity (ray theory). If the background medium is sufficiently close to the actual medium, the scattered field can be well approximated by an integral, the Born scattering formula, of terms involving the background field and the medium perturbations (see formula 2.21). The two requirements (smooth background, small perturbation) are sufficient for the validity of the method.

The formalism that captures this discussion in the case of a general anisotropic medium will be developed in this section. Equation (3.32), repeated below, is the key equation to be derived and explained in this and the next section:

$$u_{pq}^{(1)(NM)}(\boldsymbol{r}, \boldsymbol{s}, t) = -\int_{\mathscr{S}} \xi_p^{\sim}(\boldsymbol{r})\xi_q^{\sim}(\boldsymbol{s})\mathcal{A}^{\sim}(\boldsymbol{x}) \times \delta''[t - T^{\sim}(\boldsymbol{x})]\mathbf{w}^{\sim T}(\boldsymbol{x})\mathbf{c}^{(1)}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,. \tag{3.32}$$

Here $\mathbf{c}^{(1)}(\boldsymbol{x})$ is the vector of unknown material parameter perturbations and ξ, A, T, w depend upon $\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{x}$ and the background medium. The detailed meaning of the various symbols will become apparent as we proceed.

Notation

First we introduce some basic notation. Let

$$\boldsymbol{x} = (x_1, x_2, x_3) \text{ be a Cartesian position vector}, \tag{2.1}$$

$$\boldsymbol{s} = (s_1, s_2, s_3) \text{ be the source point}, \tag{2.2}$$

$$\boldsymbol{r} = (r_1, r_2, r_3) \text{ be the receiver point}, \tag{2.3}$$

$$t \text{ be the time}. \tag{2.4}$$

Let $_{,k}$ stand for ∂_{x_k}, let the overdot $\dot{}$ or $_{,t}$ stand for ∂_t, and $*_t$ stand for convolution in t.

Let

$$\rho(\boldsymbol{x}) \text{ be the density of the medium}, \tag{2.5}$$

$$c_{ijk\ell}(\boldsymbol{x}) \text{ be its elastic stiffness tensor}. \tag{2.6}$$

The wavefield is described by the displacement vector

$$\boldsymbol{u}(\boldsymbol{x}, t) = (u_1(\boldsymbol{x}, t), u_2(\boldsymbol{x}, t), u_3(\boldsymbol{x}, t)). \tag{2.7}$$

It is generated by a source distribution of body force with density

$$\boldsymbol{f}(\boldsymbol{x}, t) = (f_1(\boldsymbol{x}, t), f_2(\boldsymbol{x}, t), f_3(\boldsymbol{x}, t)). \tag{2.8}$$

We will employ the Einstein summation convention with regard to repeated subscripts in Cartesian tensors; see e.g. Jeffreys (1963), p. 3.

The single-scattering equation

As outlined above, we consider the scattering medium to be a perturbation of a smoother (possibly anisotropic) background medium by setting

$$\rho = \rho^{(0)} + \rho^{(1)}, \tag{2.9}$$

$$c_{ijk\ell} = c_{ijk\ell}^{(0)} + c_{ijk\ell}^{(1)}, \tag{2.10}$$

$$f_k = f_k^{(0)}. \tag{2.11}$$

$$\boldsymbol{u} = \boldsymbol{u}^{(0)} + \boldsymbol{u}^{(1)}, \tag{2.12}$$

where $\boldsymbol{u}$ and $\boldsymbol{u}^{(0)}$ satisfy respectively the full elastodynamic wave equation,

$$\rho\ddot{u}_i - (c_{ijk\ell}u_{k,\ell})_{,j} = f_i, \tag{2.13}$$

and the background elastodynamic wave equation,

$$\rho^{(0)}\ddot{u}_i^{(0)} - (c_{ijk\ell}^{(0)}u_{k,\ell}^{(0)})_{,j} = f_i^{(0)} . \tag{2.14}$$

Here, and in what follows, the double dots indicate a second partial derivative taken with respect to time. We shall assume that the material perturbations $\rho^{(1)}$, $c_{ijk\ell}^{(1)}$, and consequently the scattered wavefield $\boldsymbol{u}^{(1)}$, are small 'first-order' quantities. Then, within the single-scattering (Born) approximation, the scattered wavefield $\boldsymbol{u}^{(1)}$ satisfies

$$\rho^{(0)}\ddot{u}_i^{(1)} - (c_{ijk\ell}^{(0)}u_{k,\ell}^{(1)})_{,j} = -\rho^{(1)}\ddot{u}_i^{(0)} + (c_{ijk\ell}^{(1)}u_{k,\ell}^{(0)})_{,j} . \tag{2.15}$$

Thus $\boldsymbol{u}^{(1)}$ satisfies the background equation with forcing terms which depend upon the zero-order field and the medium perturbation. The two forcing terms may be identified as

$$\begin{aligned} g_i^{(1)} &= -\rho^{(1)}\ddot{u}_i^{(0)}, \text{ the scattering body force} \\ m_{ij}^{(1)} &= c_{ijk\ell}^{(1)}u_{k,\ell}^{(0)}, \text{ the scattering moment density}. \end{aligned} \tag{2.16}$$

The tensor $m_{ij}^{(1)}$ inherits symmetry from the $c_{ijk\ell}^{(1)}$ and so has six independent components, each representing a double force system (vector dipoles on the diagonal and double couples off the diagonal); see Gibson & Ben Menahem (1991) for a detailed description.

We consider three independent body-force point sources by taking in turn

$$f_i^{(0)} = \delta_{iq}\delta(x-s)\delta(t) \tag{2.17}$$

for $q=1, 2, 3$, which labels the three canonical force directions. Here, δ_{iq} is the Kronecker delta, while $\delta(t)$ is the Dirac distribution. Because of eqs (2.19), (2.14) and (2.17), for given s and q we may identify the displacement vector $\boldsymbol{u}^{(0)}(x, t)$ with the Green's tensor for the background equation (2.14):

$$u_i^{(0)}(x, t) = G_{iq}(x, s, t) , \tag{2.18}$$

where the dependence upon q is now explicit, and satisfies (*cf.* 2.14)

$$\rho^{(0)}\ddot{G}_{ip} - (c_{ijk\ell}^{(0)}G_{kp,\ell})_{,j} = \delta_{ip}\delta(x-s)\delta(t) . \quad G_{ip}=0 \quad \text{for} \quad t<0 . \tag{2.19}$$

Sustituting (2.18) in the right member of (2.15) and then using the Green's function again to solve (2.15), we find that

$$\begin{aligned} u_{pq}^{(1)}(r, s, t) = \int_{\mathscr{D}} [&-\rho^{(1)}\ddot{G}_{iq}(x, s, t) + (c_{ijk\ell}^{(1)}G_{kq,\ell})_{,j}(x, s, t)] \\ &*_t G_{pi}(r, x, t)\,\mathrm{d}x . \end{aligned} \tag{2.20}$$

This is the p-component of the scattered displacement field at receiver r and time t due to a concentrated instantaneous point body-force in the q direction at source point s and time 0. On taking into account (2.18), we see that the factor in brackets is the i component of the body force of (2.15), which includes the propagation from the body force in the direction q at the source s to component i at the integration point x. The other factor represents propagation from component i at the integration point x to component p at the receiver r. The contributions from the various i and x are summed and integrated. All subscripts, except p and q, are repeated and therefore summed from 1 to 3.

Integration by parts to transfer one time derivative to the other factor in the first term and the j-derivative to the other factor in the second will put (2.20) into a more symmetrical form, i.e.

$$\begin{aligned} u_{pq}^{(1)}(r, s, t) = \int_{\mathscr{D}} [&\rho^{(1)}G_{iq,t}(x, s, t) *_t G_{pi,t}(r, x, t) \\ &- c_{ijk\ell}^{(1)}G_{kq,\ell}(x, s, t) *_t G_{pi,j}(r, x, t)]\,\mathrm{d}x , \end{aligned} \tag{2.21}$$

which exhibits reciprocity between r and s when we recall that $G_{pi}(r, x, t) = G_{ip}(x, r, t)$. We shall refer to (2.21) as the *single-scattering equation*.

3 ASYMPTOTIC ANALYSIS

Asymptotic ray theory

In this section we make a ray-theoretical approximation to $\boldsymbol{G}$ in order to obtain a more usable and geometrically interpretable form of eq. (2.21), namely (3.32), or the slightly modified form (3.35).

We begin by replacing $\boldsymbol{G}$ by the leading term in its high-frequency, ray-theoretical approximation. Thus we shall write

$$G_{ip}(x, s, t) = \sum_N A^{(N)}(x, s)\xi_i^{(N)}(x)\xi_p^{(N)}(s)\delta[t-\tau^{(N)}(x, s)] . \tag{3.1}$$

Here N indexes the sheets of the slowness surface (see Appendices A and B).

In (3.1) the arrival time $\tau^{(N)}$ and the associated normalized polarization vector $\xi^{(N)}$ satisfy (*cf.* A8 in Appendix A)

$$[\rho^{(0)}\delta_{ik} - c_{ijk\ell}^{(0)}\tau_{,\ell}^{(N)}\tau_{,j}^{(N)}]\xi_k^{(N)} = 0 \quad (\text{at all } x) , \tag{3.2}$$

which implies the eikonal equation

$$\det[\rho^{(0)}\delta_{ik} - c_{ijk\ell}^{(0)}\tau_{,\ell}\tau_{,j}] = 0 \quad (\text{at all } x) . \tag{3.3}$$

As a first-order partial differential equation, (3.3) may be solved by the method of characteristics (see e.g. Garabedian 1964). For this purpose we define the slowness vector $\gamma^{(N)}$ by

$$\gamma^{(N)}(x) = \nabla_x \tau^{(N)}(x, s) . \tag{3.4}$$

Given initial values x, γ and A, the ray and transport equations provide a system of ordinary differential equations which can be solved to give $x(\tau^{(N)})$, $\gamma(\tau^{(N)})$, and $A(\tau^{(N)})$ as continuous functions of traveltime $\tau^{(N)}$. The spatial components $x(\tau^{(N)})$ of the characteristic curves $(x(\tau^{(N)}), \gamma^{(N)}(\tau^{(N)}), \tau^{(N)})$ are the rays.

The *wave front* for wave-type N passing through x due to a source at s is denoted by $\Sigma^{(N)}(x, s)$. This is the set of points x' such that $\tau^{(N)}(x', s) = \tau^{(N)}(x, s)$. Eq. (3.4) shows that, for fixed s, $\gamma^{(N)}(x)$ is normal to the wave front $\Sigma^{(N)}(x, s)$ at x. We shall assume that the region of interest in x is covered simply by the family of rays. Then $\tau^{(N)}$, $\gamma^{(N)}$ and $A^{(N)}$ may be obtained as functions of x. The simple covering by characteristics excludes caustics from consideration here, but methods exist for dealing with them if necessary.

Eq. (3.3) constrains γ to lie on the sextic surface $\mathscr{S}(x)$, called the *slowness surface*, given by

$$\det[\rho^{(0)}\delta_{ik} - c_{ijk\ell}^{(0)}\gamma_\ell\gamma_j] = 0 . \tag{3.5}$$

$\mathscr{S}(x)$ consists of three sheets $\mathscr{S}^{(N)}(x)$, $N=1, 2, 3$, each of which is a closed surface surrounding the origin. The amplitudes $A^{(N)}$

satisfy the transport equation

$$[c^{(0)}_{ijk\ell}\xi^{(N)}_i\xi^{(N)}_k(A^{(N)})^2\gamma^{(N)}_\ell]_{,j}=0\,, \tag{3.6}$$

where N again indicates the mode of propagation associated with the sheet of the slowness surface on which the corresponding slowness vector lies (see Appendix A, in particular A29). For instance, if the medium is transversely isotropic, $N=1$ might refer to quasi-P waves, $N=2$ to quasi-SV and $N=3$ to SH.

It follows from the ray equations (A22) that the characteristic, or group, velocities $\boldsymbol{v}^{(N)}=\mathrm{d}\boldsymbol{x}/\mathrm{d}\tau^{(N)}$ are normal to $\mathscr{A}^{(N)}(\boldsymbol{x})$ at $\gamma^{(N)}$ and satisfy

$$\boldsymbol{v}^{(N)}\cdot\gamma^{(N)}=1\,. \tag{3.7}$$

The *normal* or *phase* velocity $V^{(N)}$ is the component of $\boldsymbol{v}^{(N)}$ in the direction of $\gamma^{(N)}$. It is given by

$$V^{(N)}=\boldsymbol{v}^{(N)}\cdot\hat{\gamma}^{(N)}=\frac{\boldsymbol{v}^{(N)}\cdot\gamma^{(N)}}{|\gamma^{(N)}|}=\frac{1}{|\gamma^{(N)}|}\,. \tag{3.8}$$

From Eq. (3.7) it follows that

$$V^{(N)}=|\boldsymbol{v}^{(N)}|\cos\chi\,, \tag{3.9}$$

where χ is the angle between $\boldsymbol{v}^{(N)}$ and $\gamma^{(N)}$; see Fig. 1. In (3.7), (3.8) and in what follows, a dot between two vectors signifies the vector dot product.

It is shown in (A31) that $c^{(0)}_{ijk\ell}\xi^{(N)}_i\xi^{(N)}_k\gamma^{(N)}_\ell=\rho^{(0)}v^{(N)}_j$. Thus (3.6) may be rewritten as

$$\nabla\cdot[\rho^{(0)}(A^{(N)})^2\boldsymbol{v}^{(N)}]=0\,, \tag{3.10}$$

leading to

$$\frac{1}{A^{(N)}}\frac{\mathrm{d}A^{(N)}}{\mathrm{d}\tau^{(N)}}=-\frac{1}{2}\nabla\cdot(\rho^{(0)}\boldsymbol{v}^{(N)})\,, \tag{3.11}$$

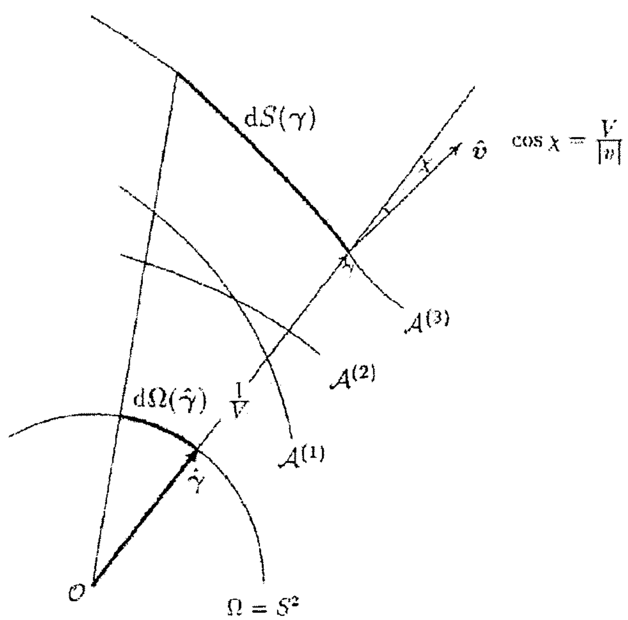

Figure 1. The central projection of the slowness direction $\hat{\gamma}$, which lies on the unit sphere Ω, into the slowness γ lying on branch $\mathscr{A}^{(3)}$ of the slowness surface $\mathscr{A}$. (We have taken N or $M=3$ merely for illustration.) The surface element $\mathrm{d}\Omega(\hat{\gamma})$ of Ω is projected onto the surface element $\mathrm{d}S(\gamma)$ of $\mathscr{A}^{(3)}$. The normal $\hat{\boldsymbol{v}}$ to $\mathscr{A}^{(3)}$, which is also the group-velocity direction, makes an angle ψ with slowness direction $\hat{\gamma}$.

the differentiation being along the ray. The amplitudes can be expressed in terms of the Jacobians (see Appendix B)

$$A^{(N)}=\frac{1}{4\pi[\rho^{(0)}(x)\rho^{(0)}(s)\mathscr{M}]^{1/2}}\,,$$

with

$$\mathscr{M}=|\boldsymbol{v}(s)|\,V(x)\,\frac{\mathrm{d}\Sigma^{(N)}(x,s)|_x}{\mathrm{d}\mathscr{A}^{(N)}|_s}\,. \tag{3.12}$$

Here $\mathrm{d}\mathscr{A}^{(N)}(s)$ and $\mathrm{d}\Sigma^{(N)}(x,s)$ are surface area elements of the surfaces $\mathscr{A}^{(N)}$ at s and $\Sigma^{(N)}(x,s)$ at x. The mapping from $\mathscr{A}^{(N)}(s)$ to $\Sigma^{(N)}(x,s)$ is defined by integrating the ray equations (A18) from $(\gamma(0),s)$ at $\tau^{(N)}=0$ to $(\gamma(\tau^{(N)}),x')$ at $\tau^{(N)}=\tau^{(N)}(x,s)$. In this process s is held fixed so that, as $\gamma(0)$ varies on $\mathscr{A}^{(N)}(s)$, x' varies on $\Sigma^{(N)}(x,s)$, and for a certain starting value γ_0 of $\gamma(0)$ at $\tau^{(N)}=0$, $x'(\tau^{(N)}(x,s))=x$. Thus in general a neighbourhood of γ_0 on $\mathscr{A}^{(N)}(s)$ is mapped onto a neighbourhood of x on $\Sigma^{(N)}(x,s)$.

We easily verify that the physical dimension of $A^{(N)}$ [to be distinguished from $\mathscr{A}^{(N)}(s)$] is $[\mathrm{time}]^2\times[\mathrm{mass}]^{-1}$, which upon multiplication by force, with dimensions $[\mathrm{mass}]\times[\mathrm{length}]\times[\mathrm{time}]^{-2}$, gives the dimension of displacement, i.e. [length]. It may be shown that $\mathscr{M}$ is symmetric in (x,s), so the expression (3.1) for A satisfies reciprocity as it should.

It is convenient here to introduce the slowness direction $\hat{\gamma}^{(N)}=\gamma^{(N)}/|\gamma^{(N)}|$. It lies on the unit sphere S^2. If we project S^2 radially onto sheet $\mathscr{A}^{(N)}(s)$ of the real slowness surface $\mathscr{A}(s)$, then from (3.8) and (3.9) we have

$$\mathrm{d}\gamma^{(N)}=\frac{|\gamma^{(N)}|^2_s}{\cos\chi|_s}\,\mathrm{d}\hat{\gamma}^{(N)}=\frac{|\boldsymbol{v}(s)|}{V^3(s)}\,\mathrm{d}\hat{\gamma}^{(N)} \tag{3.13}$$

(see Fig. 1). In (3.13) we have used the notation $\mathrm{d}\gamma^{(N)}$ and $\mathrm{d}\hat{\gamma}^{(N)}$ to denote the differential area elements on $\mathscr{A}^{(N)}$ parametrized by $\gamma^{(N)}$, and on S^2 parametrized by $\hat{\gamma}^{(N)}$. We shall use similar notation in similar situations where a vector variable of integration is constrained to lie on a surface.

The single-scattering equation

Following the notation of Aki & Richards (1980), in which $\grave{}$ denotes a downgoing wave and $\acute{}$ denotes an upgoing wave, we write

$$\grave{A}(x)=A^{(N)}(x,s)\,,\quad \acute{A}(x)=A^{(M)}(r,x)\,, \tag{3.14}$$

and then substitute eq. (3.1) into eq. (2.21) to get

$$u^{(1)}_{pq}(r,s,t)=\sum_N\sum_M u^{(1)(NM)}_{pq}(r,s,t)\,, \tag{3.15}$$

where

$$\begin{aligned}u^{(1)(NM)}_{pq}(r,s,t)=-\int_V&\grave{A}(x)\acute{A}(x)\grave{\xi}_k(x)\grave{\xi}_q(s)\acute{\xi}_p(r)\acute{\xi}_i(x)\\&\times\delta''(t-\grave{\tau}(x)-\acute{\tau}(x))\\&\times[\rho^{(1)}(x)\delta_{ki}+c^{(1)}_{ijk\ell}(x)\grave{\gamma}_\ell(x)\acute{\gamma}_j(x)]\,\mathrm{d}x\,.\end{aligned} \tag{3.16}$$

Here, $\grave{\xi}_q(s)$ is ξ_q for the incident ray in mode N from s to x and evaluated at s, and $\acute{\xi}_p(r)$ is ξ_p for the (reversed) scattered ray in mode M from r to x and evaluated at r. The quantity $u^{(1)(NM)}_{pq}(r,s,t)$ represents scattering from incident mode N to outgoing mode M.

The integrand in eq. (3.16) represents a polarization component $\xi_q^{\grave{}}(s)$ at the source, propagated to components $\xi_k^{\grave{}}(x)$ at x, scattered by the bracketed tensor to components $\xi_i^{\acute{}}(x)$, and then propagated to component $\xi_p^{\acute{}}(r)$ at the receiver. In special cases some simplifications are possible. If the background is isotropic with P velocity v and both incident and scattered modes are longitudinal, then $\gamma_i^{\acute{}}(x)=\xi_i^{\acute{}}(x)/v$, etc.

For a given source–receiver pair (s, r) and time t the integral over x in (3.16) is restricted by the support of the δ'' to the isochrone surface

$$\tau^{(N)}(x, s)+\tau^{(M)}(r, x)=t\,.$$

In (3.16) we introduced the slowness vectors at x (*cf.* 3.4):

$$\gamma^{\grave{}}(x)=\nabla_x\tau^{(N)}(x, s)\,,\quad \gamma^{\acute{}}(x)=\nabla_x\tau^{(M)}(r, x)\,, \tag{3.17}$$

the associated directions, i.e. the unit vectors:

$$\hat{\gamma}^{\grave{}}=\frac{\gamma^{\grave{}}}{|\gamma^{\grave{}}|}\,,\quad \hat{\gamma}^{\acute{}}=\frac{\gamma^{\acute{}}}{|\gamma^{\acute{}}|}\,, \tag{3.18}$$

and the normal velocities (*cf.* eq. 3.8):

$$V^{\grave{}}=\frac{1}{|\gamma^{\grave{}}|}\,,\quad V^{\acute{}}=\frac{1}{|\gamma^{\acute{}}|}\,. \tag{3.19}$$

We may now define

$$\gamma^{\grave{}\acute{}}(x)=\gamma^{\grave{}}(x)+\gamma^{\acute{}}(x)\,. \tag{3.20}$$

For quantities like this, which depend on the two rays, one from s to x in mode N ($\grave{}$), and the other from r to x in mode M ($\acute{}$), this dependence will be indicated by a $\grave{}\acute{}$, which combines both symbols as a mnemonic device. Thus we have written $\gamma^{\grave{}\acute{}}(x)$ as shorthand for $\gamma^{(NM)}(r, x, s)$. Let ν be the direction of $\gamma^{\grave{}\acute{}}(x)$, i.e.

$$\nu(x)=\hat{\gamma}^{\grave{}\acute{}}(x)=\frac{\gamma^{\grave{}\acute{}}(x)}{|\gamma^{\grave{}\acute{}}(x)|}\,. \tag{3.21}$$

The vector $\nu(x)$ is a function of s, r and x. We shall refer to it as the *isochrone normal* (see Fig. 2). The term $\delta''(t-\tau^{\grave{}}(x)-\tau^{\acute{}}(x))$ in the integrand guarantees that large contributions to the integral in eq. (3.16) arise when ν is normal to the local layering at x, i.e. when the isochrone surface is tangent to the local layering (see Section 5). ν will play a central role in considerations concerning the inversion of the generalized Radon transform in the next section.

For later use, we introduce the local ray directions (*cf.* eq. 3.7)

$$\hat{v}^{\grave{}}=\frac{v^{\grave{}}}{|v^{\grave{}}|}\,,\quad \hat{v}^{\acute{}}=\frac{v^{\acute{}}}{|v^{\acute{}}|}\,. \tag{3.22}$$

We also define the two-way traveltime $T^{\grave{}\acute{}}$:

$$T^{\grave{}\acute{}}(x)\equiv T^{(NM)}(r, x, s)\equiv\tau^{(N)}(x, s)+\tau^{(M)}(r, x)\,. \tag{3.23}$$

Then, from eqs (3.17) and (3.20) we see that

$$\nabla T^{\grave{}\acute{}}(x)=\gamma^{\grave{}\acute{}}(x)\,. \tag{3.24}$$

For fixed s, r and t, the surfaces $T^{(NM)}(r, x, s)=t$ in the space of x are the isochrone surfaces. In practice, computation of $T^{(NM)}$ will use tabulated values of $\tau^{(L)}(x, s)=\tau^{(L)}(s, x)$.

With a view to non-dimensionalizing the unknown quantities in (3.29) and (3.32) we shall need a quantity with the dimensions of velocity and independent of mode number and ray direction. One such quantity is $V_{\circ}$, the (local) normal velocity in the background medium, for γ in the 3-direction. The notation $_{\circ}$ is meant to emphasize that the quantity does not depend upon the rays or mode types. We could instead have used the normal velocity averaged over all phase angles; however, the present choice fits in naturally with borehole measurements which might be integrated into our inversion procedure. To simplify the notation, we introduce the dyadic products

$$a_{k\ell}^{\grave{}}(x)=\frac{1}{2}V_{\circ}[\xi_k^{\grave{}}(x)\gamma_\ell^{\grave{}}(x)+\xi_\ell^{\grave{}}(x)\gamma_k^{\grave{}}(x)]\,, \tag{3.25}$$

$$a_{ij}^{\acute{}}(x)=\frac{1}{2}V_{\circ}[\xi_i^{\acute{}}(x)\gamma_j^{\acute{}}(x)+\xi_j^{\acute{}}(x)\gamma_i^{\acute{}}(x)]\,, \tag{3.26}$$

which occur because of the symmetry of the $c_{ijk\ell}$, and the product of scalar amplitudes,

$$A^{\grave{}\acute{}}(x)=\rho^{(0)}(x)A^{\grave{}}(x)A^{\acute{}}(x)\,. \tag{3.27}$$

Eq. (3.16) then reduces to

$$u_{pq}^{(1)(NM)}(r, s, t)$$

$$=-\int_{\mathcal{S}}\xi_p^{\acute{}}(r)\xi_q^{\grave{}}(s)A^{\grave{}\acute{}}(x)\delta''[t-T^{\grave{}\acute{}}(x)](\rho^{(0)}(x))^{-1}$$

$$\times[\rho^{(1)}(x)\xi_i^{\grave{}}(x)\xi_i^{\acute{}}(x)+c_{ijk\ell}^{(1)}(x)V_{\circ}^{-2}(x)a_{ij}^{\acute{}}(x)a_{k\ell}^{\grave{}}(x)]\,\mathrm{d}x\,. \tag{3.28}$$

The $a_{ij}^{\acute{}}$ represent the radiation pattern of the scattering moment source density introduced in the previous section. To simplify the notation further, we group together the 22

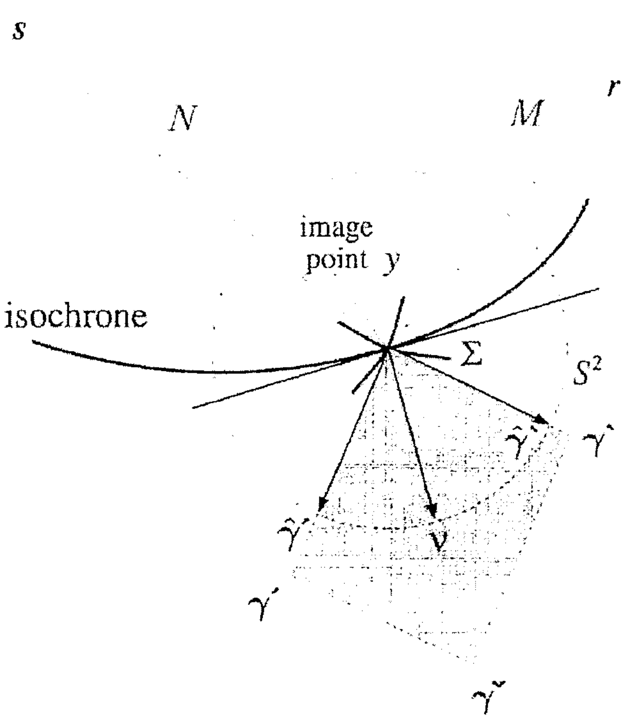

Figure 2. A 2-D representation of the image point y and the isochrone passing through it. Shown dashed are the ray in mode N from s to y and the (reversed) ray in mode M from r to y. The (reversed) wave front, the surface of equal traveltime from r, passing through y, is marked Σ. The wave front from s through y is also shown. The dashed circle represents the unit sphere S^2 of slowness directions. The vector $\gamma^{\grave{}\acute{}}$, the sum of the slownesses $\gamma^{\grave{}}$ and $\gamma^{\acute{}}$, and its direction ν are also indicated.

independent perturbations in one vector. Thus we set

$$\mathbf{c}^{(1)}(x)=\left(\frac{\rho^{(1)}(x)}{\rho^{(0)}(x)},\frac{c^{(1)}_{ijk\ell}(x)}{\rho^{(0)}(x)V^2_{\gamma}(x)}\right) \tag{3.29}$$

and define, using the symmetries of $c^{(1)}_{ijk\ell}$,

$$\mathbf{w}^{\vee}(x)=\left(\grave{\xi}_i(x)\acute{\xi}_i(x),\ \frac{1}{2}[\acute{a}_{ij}(x)\grave{a}_{k\ell}(x)+\acute{a}_{k\ell}(x)\grave{a}_{ij}(x)]\right). \tag{3.30}$$

In this 22-component notation, the independent components of $c^{(1)}_{ijk\ell}$ are naturally weighted in accordance with the rank-4 tensor contractions. Because of eqs (3.25), (3.26), (3.30) and the symmetries of $c^{(1)}_{ijk\ell}$, the tensor forming the second component of $\mathbf{w}^{\vee}$ in (3.30), which we write as

$$a^{\vee}_{ijk\ell}(x)\equiv\frac{1}{2}[\acute{a}_{ij}(x)\grave{a}_{k\ell}(x)+\acute{a}_{k\ell}(x)\grave{a}_{ij}(x)], \tag{3.31}$$

is 1/8 of the sum of $\acute{\xi}\,\acute{\gamma}\,\grave{\xi}\,\grave{\gamma}$ over the permutations $ijk\ell$, $jik\ell$, $k\ell ij$, $k\ell ji$; $ij\ell k$, $ji\ell k$, ℓkij, ℓkji of its subscripts.

Note that $\mathbf{w}^{\vee}$ is a function of x as well as of $\hat{\gamma}^{`}(x)$ and $\hat{\gamma}'(x)$ because once $x$, $N$ and $\hat{\gamma}^{`}$ are chosen, $\grave{\gamma}$, $\grave{\xi}$ are determined; therefore, by eq. (3.25), so is $\grave{a}_{k\ell}$. Similarly, M, $\hat{\gamma}'$ determine $\acute{a}_{ij}$. Thus, when x and $N^{`}$, $M'$, $\hat{\gamma}^{`}$, $\hat{\gamma}'$ are given, $a^{\vee}_{ijk\ell}$ is determined by eq. (3.30). The function $\mathbf{w}$ has been studied by Ben-Menahem & Gibson (1990) and Gibson & Ben-Menahem (1991) in the case of an isotropic background medium and is a generalization of the work of Tarantola (1986) and Wu & Aki (1985).

We now arrive, by substitution into (3.27), at the approximate expression for the NM-scattered displacement field, i.e. the scattered field in the Mth mode due to an incident field in the Nth mode:

$$u^{(1)(NM)}_{pq}(r,s,t)=-\int_{\mathscr{S}}\acute{\xi}_p(r)\grave{\xi}_q(s)A^{\vee}(x)\times\delta''[t-T^{\vee}(x)]\mathbf{w}^{\vee T}(x)\mathbf{c}^{(1)}(x)\,\mathrm{d}x\,. \tag{3.32}$$

This important equation forms the basis for the inversion schemes presented later in this paper. All quantities appearing in it, except $\mathbf{c}^{(1)}(x)$, are associated with the background medium and are regarded as known. The $u^{(1)(NM)}_{pq}(r,s,t)$ are known from the seismic data and the quantities on the right of (3.32), which depend upon the background medium which is assumed known, are obtained from a preliminary analysis. Thus the equation is regarded as an integral equation for $\mathbf{c}^{(1)}(x)$. However, $\mathbf{c}^{(1)}(x)$ is a vector with 22 unknown components. Typically the data are not sufficient for the determination of more than two, or possibly three, independent parameters. One way to address this deficiency is to make use of further information which might reduce the number of independent unknowns and improve the condition number of the problem. Before ending this section we shall show how this may be done.

Parametrization

To link the single-scattering theory with rock physics, it is advantageous to allow for reparametrizations of the medium perturbation. So far, we have based our analysis on a perturbation of the full stiffness tensor. If the stiffness tensor perturbation can be parametrized by fewer than 22 independent parameters, $d^{(1)}=(d^{(1)}_1,\ldots,d^{(1)}_P)$, say, with $P<22$, then

$$\mathbf{c}^{(1)}(x)=\left[\frac{\partial(\mathbf{c}^{(1)})}{\partial(d^{(1)})}\right]^T\Bigg|_x d^{(1)} \tag{3.33}$$

yields the perturbation of the medium. Then we may define

$$(\tilde{\mathbf{w}}^{\vee})^T=\left[(\mathbf{w}^{\vee})^T\frac{\partial(\mathbf{c}^{(1)})}{\partial(d^{(1)})}\right]^T \tag{3.34}$$

and rewrite (3.32) as

$$u^{(1)(NM)}_{pq}(r,s,t)=-\int_{\mathscr{S}}\acute{\xi}_p(r)\grave{\xi}_q(s)A^{\vee}(x)\times\delta''[t-T^{\vee}(x)]\tilde{\mathbf{w}}^{\vee T}(x)d^{(1)}(x)\,\mathrm{d}x\,. \tag{3.35}$$

The net effect is that $(\mathbf{w}^{\vee})^T$ has been replaced by $(\tilde{\mathbf{w}}^{\vee})^T$ and $\mathbf{c}^{(1)}(x)$ by $d^{(1)}(x)$.

This substitution will be implicitly assumed, if appropriate, in the remainder of this paper. It will stabilize the inverse problem introduced in the next section. Effective media may be used to reduce the number of parameters if the perturbations vary on a scale much smaller than a wavelength; the new parameters might then be, for instance, the volume fractions of sand and shale in a finely layered sand–shale sequence.

4 ANISOTROPIC INVERSION VIA AN INVERSE GENERALIZED RADON TRANSFORM

The assumption that the background medium varies only on a scale much larger than a wavelength, while the medium perturbations $\rho^{(1)}$ and $c^{(1)}_{ijk\ell}$ vary on a scale comparable with a wavelength, implies that $u^{(0)}$ is negligible compared with $u^{(1)}$ in the measurements, except for t near any $\tau^{(L)}(r,s)$, the times of arrival of the *direct* body waves from s to r. Hence for practical purposes we may treat $u^{(1)}$ as if it were in fact the deconvolved seismic data. As in the acoustic and isotropic elastic cases (see Miller *et al.* 1984; Beylkin & Burridge 1990), eq. (3.32) recasts the scattering integral as a form of generalized Radon transform, which suggests an inversion scheme based on the inversion formula for the generalized Radon transform. The Radon transform is 'generalized' in the sense that the surfaces of integration are curved, rather than plane as for the Radon transform.

Under special (possibly impractical) circumstances, the medium variation can be parametrized in terms of a scalar function of location. For example, the replacement of a small volume of liquid with gas in an otherwise known medium may be parametrized by the volume fraction replaced through an effective medium theory connecting this scalar parameter with the components of $\mathbf{c}^{(1)}(x)$. The inversion problem can then be reduced to a scalar GRT inversion, as shown below in Subsection 4.1.

Preconditioning the integral

In preparation for applying the GRT inversion formula we shall consider the family of isochrones passing through a fixed image point y. For fixed r, s and y, these are surfaces in x-space given by $T^{(NM)}(r, x, s) = T^{(NM)}(r, y, s)$. As r and s vary one obtains different members of the family. Thus in eq. (3.32) we shall set $t = T^{(NM)}(r, y, s)$. We shall also scale the argument of δ'' so that its gradient at y is a unit vector. This leads to

$$u_{pq}^{(1)(NM)}(r, s, T^{(NM)}(r, y, s))$$

$$= -\int_{\mathscr{S}} \xi_p^{\smallsmile}(r)\xi_q^{\smallsmile}(s)A^{\smallsmile}(x)\frac{1}{|\gamma^{\smallsmile}(y)|^3} \times \mathbf{w}^{\smallsmile T}(x)\mathbf{c}^{(1)}(x)\delta''\left(\frac{T^{\smallsmile}(x)-T^{\smallsmile}(y)}{|\gamma^{\smallsmile}(y)|}\right) \mathrm{d}x, \tag{4.1}$$

which is the scattered field at the arrival time associated with the point y. In Fig. 2 the relevant quantities are shown. The *obliquity factor* $1/|\gamma^{\smallsmile}|^3$ in eq. (4.1) arises because δ'' is homogeneous of degree -3.

We shall now contract with $\xi_p^{(M)}(r; y, r)\xi_q^{(N)}(s; y, s)$ and multiply both sides by $-|\gamma^{\smallsmile}(y)|^3/A^{\smallsmile}(y)$ so that the factor before $\mathbf{w}^{\smallsmile T}(x)\mathbf{c}^{(1)}(x)$ in the integrand reduces to unity when $x = y$. This may be regarded as projecting onto the unit polarization vectors associated with the given modes at source and receiver, and undoing the amplitude propagation loss and the obliquity effect.

Write

$$U^{(1)(NM)}(r, y, s)$$

$$\equiv -\frac{|\gamma^{\smallsmile}(y)|^3}{A^{\smallsmile}(y)}\xi_p^{(M)}(r; y, r)\xi_q^{(N)}(s; y, s)u_{pq}^{(1)(NM)}(r, s, T^{\smallsmile}(y))$$

$$\simeq \int_{\mathscr{S}} \frac{A^{\smallsmile}(x)}{A^{\smallsmile}(y)}\xi_p^{(M)}(r; y, r)\xi_p^{(M)}(r; x, r)\xi_q^{(N)}(s; y, s)\xi_q^{(N)}(s; x, s) \times \mathbf{w}^{(NM)^T}(r, y, s)\mathbf{c}^{(1)}(x) \times \delta''\left(\frac{T^{\smallsmile}(x)-T^{\smallsmile}(y)}{|\gamma^{\smallsmile}(y)|}\right)\mathrm{d}x. \tag{4.2}$$

As before, we have assumed that $\gamma^{\smallsmile}(y) \neq \mathbf{0}$. Otherwise, further analysis is required. From the first line of (4.2) we see that $U^{(1)(NM)}(r, y, s)$ may be calculated from the seismic data and information obtainable from knowledge of the background medium. The unknowns $\mathbf{c}^{(1)}(x)$ appear in the integral, which is closely related to a generalized Radon transform.

To emphasize the form of (4.2), write

$$F(r, x, y, s) = \frac{A^{\smallsmile}(x)}{A^{\smallsmile}(y)}\xi^{(M)}(r; y, r)\cdot\xi^{(M)}(r; x, r) \times \xi^{(N)}(s; y, s)\cdot\xi^{(N)}(s; x, s) \tag{4.3}$$

and

$$\phi(r, x, y, s) = \frac{T^{\smallsmile}(x)-T^{\smallsmile}(y)}{|\gamma^{\smallsmile}(y)|}. \tag{4.4}$$

Note that

$$F(r, x, y, s) = 1 + O(|x-y|) \tag{4.5}$$

and

$$\phi(r, x, y, s)|_{x=y} = \nu\cdot(x-y) + O(|x-y|^2), \tag{4.6}$$

with ν a unit vector. Then eq. (4.2) may be rewritten

$$U(r, y, s) \equiv -\int_{\mathscr{S}} F(r, x, y, s)\mathbf{w}^T(r, x, s)\mathbf{c}^{(1)}(x) \times \delta''[\phi(r, x, y, s)]\,\mathrm{d}x,$$

$$= \int_{\mathscr{S}} [1 + O(|x-y|)]\mathbf{w}^T(r, x, s)\mathbf{c}^{(1)}(x) \times \delta''[\nu\cdot(x-y) + O(|x-y|^2)]\,\mathrm{d}x. \tag{4.7}$$

4.1 Zero-offset scalar inversion

For clarity we have suppressed the dependence upon N and M on both sides of (4.7), but this equation forms a system of integral equations for $\mathbf{c}^{(1)}(x)$, each equation corresponding to one of the nine values of the pair N, M.

To clarify the relationship between the multiparameter inversion problem and the generalized Radon transform, it will be helpful first to consider a simpler scalar anisotropic inversion problem, *viz.* the problem of recovering a single parameter using zero-offset data.

For zero-offset data in the single mode N, we have the following simplifications:

(1) $r = s$;
(2) $N = M$;
(3) $\gamma^{\smallsmile} = \gamma' \equiv \gamma$; $\hat{\gamma}^{\smallsmile} = \hat{\gamma}' \equiv \hat{\gamma}^{\smallsmile\prime}$; $\xi^{\smallsmile} = \xi' \equiv \xi$;
(4) $\gamma^{(NN)}(s, y, s) \equiv 2\gamma^{(N)}(s, y)$, whereas $\nu = \hat{\gamma}$;
(5) $a^{\smallsmile} = a' \equiv a$; $\mathbf{w}^{(NN)}[y, \hat{\gamma}(y), \hat{\gamma}(y)] \equiv \mathbf{w}^{(N)}[y, \hat{\gamma}(y)]$, which now exhibits additional symmetries.

The single-scattering equation

Set the two-way traveltime

$$T^{(N)}(y, s) = T^{(NN)}(s, y, s). \tag{4.8}$$

Using eq. (4.2), we arrive at the approximate expression for the scattered displacement field:

$$u_{pq}^{(1)(NN)}[s, s, T^{(N)}(y, s)]$$

$$\simeq -\int_{\mathscr{S}} \xi_p^{(N)}(s)\xi_q^{(N)}(s)\rho^{(0)}(y)[A^{(N)}(y)]^2 \times \frac{1}{8|\gamma^{(N)}(s, y)|^3}\delta''[\nu(s, y)\cdot(y-x) + O(|x-y|^2)] \times \delta''[\nu(s, y)\cdot(y-x) + O(|x-y|^2)]\mathbf{w}^{(N)T}[y, \nu(s, y)]\mathbf{c}^{(1)}(x)\,\mathrm{d}x. \tag{4.9}$$

Thus (*cf*. eq. 4.7),

$$U^{(NN)}(s, y, s) = -\int_{\mathscr{S}} \mathbf{w}^{(N)T}[y, \nu(s, y)]\mathbf{c}^{(1)}(x) \times \delta''(\nu(s, y)\cdot(x-y) + O(|x-y|^2))\,\mathrm{d}x. \tag{4.10}$$

We shall assume that $\mathbf{c}^{(1)}(x)$ depends upon one scalar parameter $d^{(1)}(x)$ so that we have a special case of eq. (3.34) with $\mathbf{c}^{(1)}(x)$ depending upon $d^{(1)}(x)$ according to (3.32):

$$\mathbf{c}^{(1)}(x) = \left.\frac{\partial(\mathbf{c}^{(1)})}{\partial(d^{(1)})}\right|_x d^{(1)}(x). \tag{4.11}$$

Then (4.10) becomes

$$U^{(NN)}(s, y, s) = -\int_{\mathscr{D}} \tilde{\mathbf{w}}^{(N)T}[y, \nu(s, y)] d^{(1)}(x) \times \delta''(\nu(s, y)\cdot(x-y) + O(|x-y|^2))\,\mathrm{d}x\,, \tag{4.12}$$

where

$$\tilde{\mathbf{w}}^{(N)T}[y, \nu(s, y)] = \mathbf{w}^{(N)T}[y, \nu(s, y)] \left.\frac{\partial(\mathbf{c}^{(1)})}{\partial(d^{(1)})}\right|_x. \tag{4.13}$$

We call this the *scalarization* of the perturbation.

Inversion

For $y \in \mathscr{D}$, we may use

$$\int_{S^2} [1 + O(|x-y|)]\delta''[\nu\cdot(y-x)) + O(|x-y|^2)]\,\mathrm{d}\nu$$

$$= -8\pi^2\delta(x-y) + \text{smoother terms} \tag{4.14}$$

in a neighbourhood of y in which the isochrones through y do not form envelopes. This result may be obtained by applying the Laplacian to the easily proved formula

$$\int_{S^2} [1 + O(|x-y|)]\delta[\nu\cdot(y-x)) + O(|x-y|^2)]\,\mathrm{d}\nu$$

$$= \frac{4\pi}{|x-y|} + \text{smoother terms}\,. \tag{4.15}$$

[See Beylkin (1982), (1984) for a more rigorous approach. The suggested argument was motivated by Chapter 1 of John (1955).] In (4.14) and (4.15) $O(|x-y|)$ and $O(|x-y|^2)$ may depend upon ν and are smooth away from the diagonal $x=y$.

To invert (4.10) using (4.14) we simply take advantage of the correspondence between source position s and ray direction at the image point $\nu(s, y)$, introducing a Jacobian term and integrating with respect to ν over the unit sphere, *viz.*

$$8\pi^2\tilde{d}^{(1)}(y) = \int_{\partial S} \frac{1}{\tilde{w}^{(N)}[y, \nu(s, y)]} U^{(1)(NN)}(s, y, s) \left.\frac{\partial(\nu)}{\partial(s)}\right|_y \mathrm{d}s\,, \tag{4.16}$$

where $\tilde{w}^{(N)}$ is defined in (4.13); see also (3.34).

Since

$$\left.\frac{\partial(\nu)}{\partial(s)}\right|_y = 16\pi^2\rho^{(0)}(s)\rho^{(0)}(y)V^{(N)}(s)[V^{(N)}(y)]^3[A^{(N)}(y)]^2\nu\cdot\hat{v}_s\,, \tag{4.17}$$

the amplitude contributions to the Jacobian cancel a similar amplitude factor in $U^{(1)(NN)}$ and we have

$$\begin{aligned}\tilde{d}^{(1)}(y) &= 2\int_{\partial S} \rho^{(0)}(s)V^{(N)}(s)[V^{(N)}(y)]^3 \frac{|\gamma^{(N)}(s, y)|^3}{\tilde{w}^{(N)}[y, \nu(y)]} \\ &\quad\times \xi_p^{(N)}(s)u_{pq}^{(1)(N)}[s, s, T^{(N)}(y, s)]\xi_q^{(N)}(s)\nu\cdot\hat{v}|_s\,\mathrm{d}s \\ &= 16\int_{\partial S} \frac{\rho^{(0)}(s)V^{(N)}(s)}{\tilde{w}^{(N)T}[y, \nu(s, y)]} \\ &\quad\times \xi_p^{(N)}(s)u_{pq}^{(1)(N)}[s, s, T^{(N)}(y, s)]\xi_q^{(N)}(s)\hat{\gamma}\cdot\hat{v}|_s\,\mathrm{d}s\,.\end{aligned} \tag{4.18}$$

Eq. (4.18) is the anisotropic extension of the zero-offset acoustic GRT inversion formula derived by Miller, Oristaglio & Beylkin (1987), eq. (29).

4.2 Multi-offset multiparameter inversion

We return to the general case. The basic plan is to take advantage of the multivariable correspondence between source and receiver positions, the isochrone normal, and the scattering angles at the image point. The analysis is similar to the multiparameter isotropic elastic case treated by Beylkin & Burridge (1990), but is less well conditioned because there are more parameters to unravel and because the dependence on scattering angle of the scattering tensor $\mathbf{w}$ is not independent of ν.

Since we shall eventually employ an inversion formula (4.14), which involves an integration in ν over the unit sphere S^2, we shall parametrize r and s by ν and two further parameters: θ, the angle between $\hat{\gamma}'$ and $\hat{\gamma}`$, and $\psi$, the azimuth of the plane containing $\hat{\gamma}'$ and $\hat{\gamma}`$ about ν (like the third Euler angle). Thus

$$\cos\theta = \hat{\gamma}`\cdot\hat{\gamma}'\,. \tag{4.19}$$

Then, for fixed y, N and M, the parameters ν, θ and ψ are functions of $\hat{\gamma}'$ and $\hat{\gamma}`$ and hence of r and s. We shall assume that these functional dependences are invertible, so that

$$r = r^{(M)}(y, \hat{\gamma}`) = r^{(NM)}(y, \nu, \theta, \psi)\,, \tag{4.20}$$

$$s = s^{(N)}(y, \hat{\gamma}`) = r^{(NM)}(y, \nu, \theta, \psi)\,. \tag{4.21}$$

The corresponding differential relationships may be written

$$\mathrm{d}\hat{\gamma}`\,\mathrm{d}\hat{\gamma}' = \frac{\partial(\hat{\gamma}`, \hat{\gamma}')}{\partial(s, r)}\mathrm{d}r\,\mathrm{d}s = \frac{\partial(\hat{\gamma}`, \hat{\gamma}')}{\partial(\nu, \theta, \psi)}\,\mathrm{d}\theta\,\mathrm{d}\psi\,\mathrm{d}\nu\,. \tag{4.22}$$

We shall eventually integrate first over θ and ψ on the right side of (4.7) and then over ν. However, because the data appearing on the left side are naturally parametrized by r and s, we shall there express the integral in terms of these variables (*cf.* Beylkin & Burridge 1990).

The angles on the right-hand side potentially vary through the ranges $\theta \in [0, \pi]$, $\nu \in S^2$ and $\psi \in [0, 2\pi)$. In practice, their ranges will be restricted by the acquisition geometry. The Jacobians are taken at the image point y, which is fixed for the remainder of this subsection.

To carry out the multiparameter inversion, we introduce the covariance function $\sigma_U^2[\hat{\gamma}(y), \hat{\gamma}'(y)]$ associated with an *a posteriori* noise probability distribution from the measurements (see Appendix C) and the covariance matrix σ_c^2 associated with the *a priori* probability distribution of the material parameters. In the inversion process the normal matrix is given by

$$\lambda^{(NM)}(s, y, r) = \mathbf{w}(s, y, r)\sigma_U^{-2}\mathbf{w}^T(s, y, r) \tag{4.23}$$

(see Appendix C), where the covariance σ_U^2 may be a function of y, θ, ψ and ν. If σ_U^2 is a multiple of the identity then (4.22) may be written

$$\lambda^{(NM)}(s, y, r) = \sigma_U^{-2}\mathbf{w}(s, y, r)\mathbf{w}^T(s, y, r)\,, \tag{4.24}$$

where now σ_U^2 is interpreted as a scalar quantity.

Figs 3, 4 and 5 show the values of some of the qP–qP components of $\mathbf{w}$ as functions of scattering angle θ and azimuth

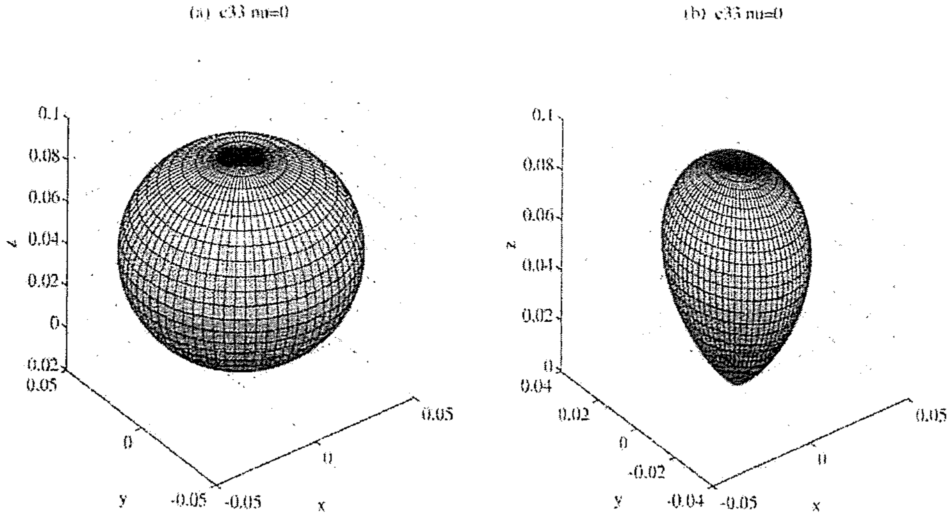

Figure 3. Total radiation pattern for a c_{3333} perturbation as a function of scattering (a) or incidence (b) angle and azimuth for a qP–qP conversion in a TI background medium (symmetry axis $\parallel$ ν).

ψ for a TI background medium when the dip ν is parallel to the symmetry axis. We integrate the normal matrix over the scattering and azimuthal angles θ and ψ at y holding ν constant (see eqs 4.22 and 4.23) and set

$$\Lambda^{(NM)}(\nu, y) = \int \lambda^{(NM)}(s, y, r)\,\frac{\partial(\hat{p}^{*}, \hat{p}')}{\partial(\nu, \theta, \psi)}\,d\theta\, d\psi + \sigma_{\mathbf{c}}^{-2}. \qquad (4.25)$$

$\Lambda^{(NM)}$ is square and is dependent only on the background medium and on the source–receiver geometry through ν, hence its values can be tabulated like the traveltimes. We shall assume here that this matrix may be inverted; see Appendix C for the maximum-likelihood interpretation. When this is not the case for certain directions ν we seek an appropriate reduction in the number of parameters as in the previous section, or we may use further information about the local layering, for instance (see Section 5). Note that for zero-offset inversion $\mathbf{w}^{(N)}$ depends on ν and y only. Hence, the matrix $\lambda^{(N)} \equiv \mathbf{w}^{(N)}(\mathbf{w}^{(N)})^T$ is of rank 1, and has only one non-vanishing eigenvalue. If the associated eigenvector is substituted for $\mathbf{c}_0^{(1)}$, we would obtain only the linear combination

$$Z(x) = (\mathbf{c}_0^{(1)})^T \mathbf{c}^{(1)}(x) \qquad (4.26)$$

of the unknown components of $\mathbf{c}^{(1)}$.

Returning to the general case, we shall now operate upon (4.7) so that (4.14) may be applied to it, at least approximately. We shall take σ_U^{-2} to be a scalar as in (4.23). First we multiply (4.7) by $d\hat{p}^{*}\,d\hat{p}'\sigma_U^{-2}\mathbf{w}(r, y, s)$ and integrate with respect to θ and

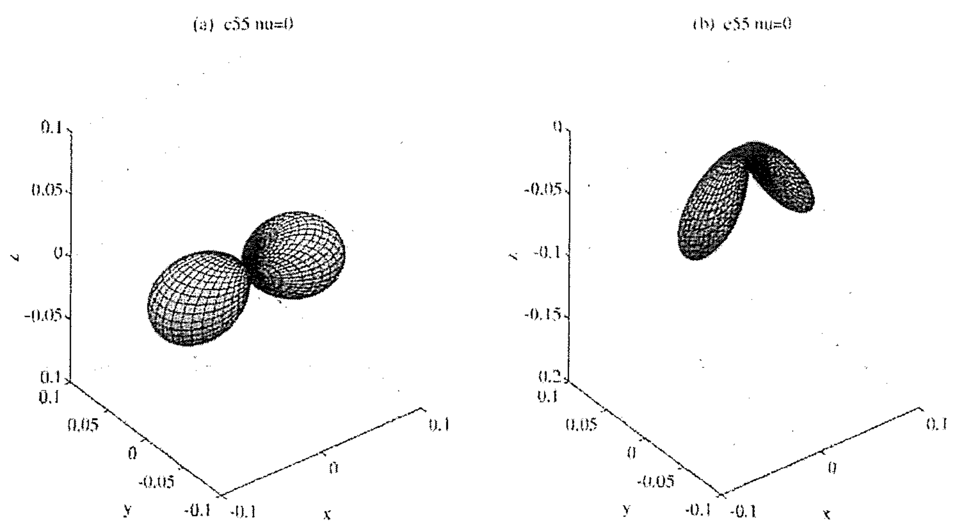

Figure 4. Total radiation pattern for a c_{1313} perturbation as a function of scattering (a) or incidence (b) angle and azimuth for a qP–qP conversion in a TI background medium (symmetry axis $\parallel$ ν).

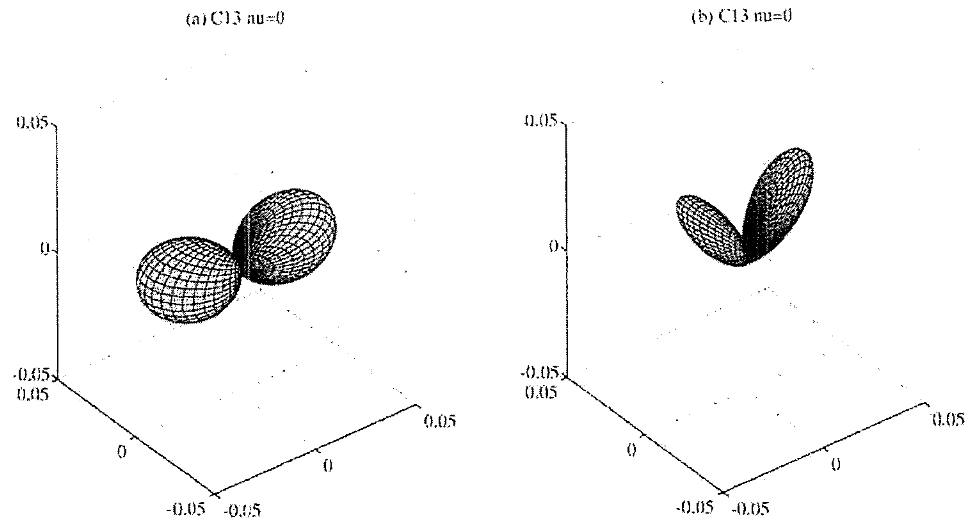

Figure 5. Total radiation pattern for a c_{1133} perturbation as a function of scattering (a) or incidence (b) angle and azimuth for a qP–qP conversion in a TI background medium (symmetry axis $\parallel \nu$).

ψ to get

$$\mathrm{d}\nu \int \mathrm{d}\theta\,\mathrm{d}\psi \frac{\partial(\hat{\gamma}^{\backprime},\hat{\gamma}')}{\partial(\nu,\theta,\psi)}\sigma_U^{-2}\mathbf{w}(r,y,s)U(r,y,s)$$

$$= \mathrm{d}\nu \int \mathrm{d}\theta\,\mathrm{d}\psi \int_{\mathscr{Y}} \mathrm{d}x\, F(r,x,y,s)\frac{\partial(\hat{\gamma}^{\backprime},\hat{\gamma}')}{\partial(\nu,\theta,\psi)}$$

$$\times \sigma_U^{-2}\mathbf{w}(r,y,s)[\mathbf{w}(r,x,s)]^T\mathbf{c}^{(1)}(x)\delta''(\Phi(r,x,y,s))$$

$$= \mathrm{d}\nu \int \mathrm{d}\theta\,\mathrm{d}\psi \int_{\mathscr{Y}} \mathrm{d}x\, F(r,x,y,s)\frac{\partial(\hat{\gamma}^{\backprime},\hat{\gamma}')}{\partial(\nu,\theta,\psi)}$$

$$\times \sigma_U^{-2}\mathbf{w}(r,y,s)[\mathbf{w}(r,y,s)^T + O(|x-y|)]\mathbf{c}^{(1)}(x)\delta''(\Phi(r,x,y,s))$$

$$= \mathrm{d}\nu \int_{\mathscr{Y}} \mathrm{d}x \int \mathrm{d}\theta\,\mathrm{d}\psi[\lambda^{(NM)}(r,y,s) + O(|x-y|)]$$

$$\times \mathbf{c}^{(1)}(x)\delta''[\nu\cdot(y-x)) + O(|x-y|^2)]$$

$$\simeq \mathrm{d}\nu \int_{\mathscr{Y}} \mathrm{d}x[\Lambda^{(NM)}(y,\nu) + O(|x-y|)]\mathbf{c}^{(1)}(x)$$

$$\times \delta''[\nu\cdot(y-x)) + O(|x-y|^2)]\,. \tag{4.27}$$

Next we multiply by the inverse $[\Lambda^{(NM)}(y,\nu)]^{-1}$ to get

$$\mathrm{d}\nu \int \frac{\partial(\hat{\gamma}^{\backprime},\hat{\gamma}')}{\partial(\nu,\theta,\psi)}\mathrm{d}\theta\,\mathrm{d}\psi[\Lambda(y,\nu)]^{-1}\sigma_U^{-2}\mathbf{w}(r,y,s)U(r,y,s)$$

$$\simeq \mathrm{d}\nu \int_{\mathscr{Y}} \mathrm{d}x[I + O(|x-y|)]\mathbf{c}^{(1)}(x)\delta''[\nu\cdot(y-x)) + O(|x-y|^2)]\,, \tag{4.28}$$

and integrate with respect to ν over S^2, using (4.14), to get

$$\iint [\Lambda^{(NM)}(y,\nu)]^{-1}\sigma_U^{-2}\mathbf{w}(r,y,s)U(r,y,s)\,\mathrm{d}\hat{\gamma}^{\backprime}\,\mathrm{d}\hat{\gamma}'$$

$$= -8\pi^2\langle \mathbf{c}^{(1)}(y)\rangle + \text{smoother terms}\,. \tag{4.29}$$

Thus

$$\langle \mathbf{c}^{(1)}(y)\rangle \simeq -\frac{1}{8\pi^2}\int_{\partial S\times\partial R} \Lambda^{(NM)^{-1}}[y,\nu(r,y,s)]\sigma_U^{-2}\mathbf{w}^{(NM)}(r,y,s)$$

$$\times U^{(NM)}(r,y,s)\left.\frac{\partial(\hat{\gamma}^{\backprime},\hat{\gamma}')}{\partial(s,r)}\right|_y \mathrm{d}r\,\mathrm{d}s\,. \tag{4.30}$$

Here, because we have neglected smoother terms in the generalized Radon transform, $\langle \mathbf{c}^{(1)}(y)\rangle$ is not precisely equal to $\mathbf{c}^{(1)}(y)$, but for the moment $\langle \mathbf{c}^{(1)}(y)\rangle$ will be taken as our best estimate for inversion in the absence of further information. We analyse this expression in detail in Appendix C, where it is shown to be the maximum-likelihood solution. For an important special case of using further information, see the next section, where we take into account the fact that the formation is layered by introducing a new curvilinear coordinate normal to the layering along which the medium varies rapidly.

To carry out the inversion we still have to evaluate the relevant Jacobians introduced in eq. (4.22). The Jacobian in the right member of eq. (4.22) occurs in the integrated normal matrix and is derived in Appendix D as (D30). Thus

$$\frac{\partial(\hat{\gamma}^{\backprime},\hat{\gamma}')}{\partial(\nu,\theta,\psi)} = \frac{\sin\theta}{1 + (|\gamma^{\backprime}||\gamma'|/|\gamma^{\backprime}|^2)(\tan\chi' - \tan\chi^{\backprime})\sin\theta}\,, \tag{4.31}$$

where (see Fig. 1)

$$\cos\chi^{\backprime} = \hat{v}^{\backprime}\cdot\hat{\gamma}^{\backprime} = \frac{V^{\backprime}}{|\gamma^{\backprime}|} \quad\text{and}\quad \cos\chi' = \hat{v}'\cdot\hat{\gamma}' = \frac{V'}{|\gamma'|}\,. \tag{4.32}$$

The Jacobian in the middle member of eq. (4.22) is directly related to quantities calculated during dynamic ray tracing. It factorizes as

$$\frac{\partial(\hat{\gamma}^{\backprime},\hat{\gamma}')}{\partial(s,r)} = \frac{\partial(\hat{\gamma}^{\backprime})}{\partial(s)}\frac{\partial(\hat{\gamma}')}{\partial(r)}\,. \tag{4.33}$$

Each factor is essentially the reciprocal of the cross-sectional area of a narrow tube of rays emanating from x. The factors can be expressed in terms of the ray amplitudes, which are inversely proportional to the square root of the cross-sectional area of the same narrow tube of rays. For a discussion of the anisotropic ray-tracing equations and the transport equations governing the variation of ray amplitudes, see Appendices A and B. The first factor on the right is

$$\frac{\partial(s)}{\partial(\hat{\gamma}^{\backprime})}=\frac{1}{16\pi^2\rho^{(0)}(s)\rho^{(0)}(y)V^{\backprime}(s)V^{\backprime 3}(y)A^{\backprime 2}(y)}\frac{\hat{\gamma}^{\backprime}(s)\cdot\hat{v}^{\backprime}(s)}{n_S(s)\cdot\hat{v}^{\backprime}(s)}, \quad (4.34)$$

where $n_S(s)$ is the unit normal to ∂S at the source s. Note that $\hat{\gamma}^{\backprime}(s)$ is the normal to the wave front at s. Similar expressions hold for $\partial(r)/\partial(\hat{\gamma}^{\backprime})$ with obvious changes.

In the case of a homogeneous background medium (in which the rays are straight), we have the following simplifications:

$$\frac{\partial(\hat{\gamma}^{\backprime})}{\partial(\hat{v}^{\backprime})}=\frac{(\hat{v}^{\backprime}\cdot\hat{\gamma}^{\backprime})V^{\backprime 2}}{\kappa^{\backprime}}, \quad \left.\frac{\partial(\hat{v}^{\backprime})}{\partial(s)}\right|_y=\frac{\hat{v}^{\backprime}\cdot n_S}{|s-y|^2}, \quad (4.35)$$

where

$\kappa^{\backprime}$ = Gaussian curvature of the slowness surface at $\hat{\gamma}^{\backprime}$.

Thus,

$$\left.\frac{\partial(s)}{\partial(\hat{\gamma}^{\backprime})}\right|_y=\frac{\kappa^{\backprime}|s-y|^2|v^{\backprime}|}{V^{\backprime 3}n_S\cdot\gamma^{\backprime}}. \quad (4.36)$$

It is now clear that the relevant Jacobians combine with the amplitudes hidden in $U^{(NM)}$ (*cf.* Miller *et al.* 1984).

Combined mode conversions

Rather than using the coordinates $(\hat{\gamma}^{\backprime}, \hat{\gamma}^{\prime})$ at y we consider the angles (ν, θ, ψ) as preferred coordinates. Then, for each N, M,

$$s=s^{(NM)}(\nu, \theta, \psi), \quad r=r^{(NM)}(\nu, \theta, \psi).$$

By ordering the integrations over (s, r) as

$$\int_{S^2}\mathrm{d}\nu\int \mathrm{d}\theta\,\mathrm{d}\phi,$$

we identify the inner integral (for fixed y) as a 3-D AVA inversion and the outer integral over ν as a migration. This particular ordering of integrals is used to derive the inversion formula (4.38) below.

To combine different modes in the inversion, instead of using (4.25) we define

$$\Lambda(\nu)=\sum_{N,M}\int \lambda^{(NM)}\frac{\partial(\gamma^{\backprime}, \gamma^{\prime})}{\partial(\nu, \theta, \psi)}\mathrm{d}\theta\,\mathrm{d}\psi+\sigma_c^{-1}. \quad (4.37)$$

The more terms included in the summation over N, M the greater the resolving power of $\Lambda(\nu)$. Thus, summing $U^{(NM)}$ over the modes and integrating the result with respect to s and r, we obtain

$$\langle c^{(1)}\rangle(y)=\frac{1}{8\pi^2}\int_{\partial R\times\partial S}\Lambda^{-1}[y, \nu(r, y, s)]\times\left[\sum_{N,M}\mathrm{w}^{(NM)}\sigma_U^{-2}U^{(NM)}(r, y, s)\left.\frac{\partial(\gamma^{\backprime}, \gamma^{\prime})}{\partial(s, r)}\right|_y\right], \mathrm{d}r\,\mathrm{d}s, \quad (4.38)$$

replacing eq. (4.30). This requires that we set $t=T^{(NM)}$ for each (NM) scattering process incorporated into the sum.

5 MULTIPARAMETER INVERSION: STRATIFIED PERTURBATION

In this section we use further information concerning the unknown medium, to the effect that it is locally stratified in the sense that $\mathbf{c}^{(1)}(x)$ may be represented in the form

$$\mathbf{c}^{(1)}(x)\to\mathbf{c}^{(1)}[\epsilon x, \phi(x)]. \quad (5.1)$$

Here ϕ is a smooth function which may be interpreted as a curvilinear coordinate whose gradient is in the direction in which $\mathbf{c}^{(1)}(x)$ varies most rapidly, and whose level surfaces are possible interfaces in the medium. ϵ is another small parameter and the first argument of $\mathbf{c}^{(1)}$ on the right represents a possible slow variation on which the medium perturbation varies laterally. Thus $\mathbf{c}^{(1)}$ varies mainly in the direction of $\nabla\phi$, but may vary slowly in other directions. By the chain rule we see that to leading order in small ϵ

$$\nabla\mathbf{c}^{(1)}[\epsilon x, \phi(x)]=\mathbf{c}^{(1)\prime}\nabla\phi, \quad \mathbf{c}^{(1)\prime}=\partial_\phi\mathbf{c}^{(1)}. \quad (5.2)$$

We will omit the first argument of $\mathbf{c}^{(1)}$ in the remainder of this section, and take into account only the leading term in the gradient of the perturbation.

The single-scattering equation

In this section we shall consider N and M to be fixed and employ the shorthand notation

$$T(x)=T^{(NM)}(r, x, s), \quad \gamma^{\backprime\prime}=\nabla_x T, \quad A_{pq}(x)=A^{\backprime\prime}(x)\xi_p^{\prime}(r)\xi_q^{\backprime}(s). \quad (5.3)$$

Then eq. (3.32) may be rewritten

$$u_{pq}^{(1)(NM)}(r, s, t)=-\int_{\mathscr{D}}A_{pq}(x)\mathbf{w}^{\backprime\prime T}(x)\mathbf{c}^{(1)}[\phi(x)]\delta''[t-T(x)]\,\mathrm{d}x. \quad (5.4)$$

To make use of (5.2) we will integrate expression (5.4) by parts using the obvious identity

$$\bar{\nu}\cdot\nabla T(x)\delta''[t-T(x)]=-\bar{\nu}\cdot\nabla\delta'[t-T(x)], \quad (5.5)$$

which holds for arbitrary $\bar{\nu}\in S^2$. Then

$$u_{pq}^{(1)(NM)}(r, s, t)=-\int_{\mathscr{D}}A_{pq}(x)[\mathbf{w}^{\backprime\prime T}\mathbf{c}^{(1)\prime}]\left.\frac{\bar{\nu}\cdot\nabla\phi}{\bar{\nu}\cdot\nabla T}\right|_x\delta'[t-T(x)]\,\mathrm{d}x. \quad (5.6)$$

Here it is assumed that the unit vector $\bar{\nu}=\bar{\nu}(x)$ is slowly varying in space, and may be chosen normal to the local *geological layering*:

$$\nu_\phi\equiv\frac{\nabla\phi}{|\nabla\phi|}. \quad (5.7)$$

Now write

$$\mathbf{c}^{(1)\prime}[\phi(x)]=\int_{\mathbb{R}}\mathbf{c}^{(1)\prime}(L)\delta[\phi(x)-L]\,\mathrm{d}L. \quad (5.8)$$

The level surfaces of ϕ, i.e. the geological interfaces, and the isochrone surfaces are illustrated in Fig. 6. Substituting eq. (5.8) into eq. (5.6) and interchanging the order of

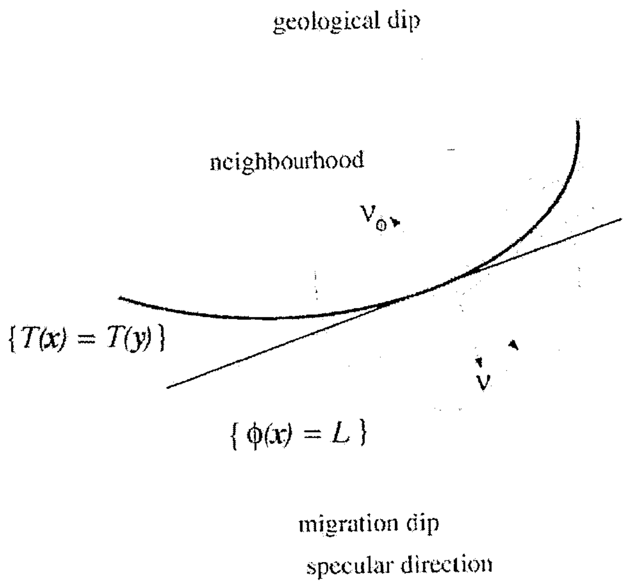

Figure 6. Locally stratified medium perturbation.

integration yields

$$
\begin{aligned}
u^{(1)(NM)}_{pq}(r,s,t) &= -\int_{\mathbb{R}} \mathrm{d}L\,\partial_t^2 \int_{\mathscr{D}} A_{pq}(x)\mathbf{w}^{\vee T}(x)\mathbf{c}^{(1)\prime}(L) \\
&\quad \times \frac{\bar{\nu}\cdot\nabla\phi}{\bar{\nu}\cdot\gamma^{\vee}}\bigg|_x \delta[\phi(x)-L]H[t-T(x)]\,\mathrm{d}x \\
&= -\int_{\mathbb{R}} \mathrm{d}L\partial_t^2 \int_{\phi=L} \mathrm{d}x\,A_{pq}(x)\mathbf{w}^{\vee T}(x)\mathbf{c}^{(1)\prime}(L) \\
&\quad \times \frac{\bar{\nu}\cdot\nabla\phi}{\bar{\nu}\cdot\gamma^{\vee}|\nabla\phi|}\bigg|_x H[t-T(x)] \\
&= -\int_{\mathbb{R}} \mathrm{d}L\,\partial_t^2 \int_{A_L} \mathrm{d}x\,A_{pq}(x)\mathbf{w}^{\vee T}(x)\mathbf{c}^{(1)\prime}(L) \\
&\quad \times \frac{\bar{\nu}\cdot\nu_\phi}{\bar{\nu}\cdot\gamma^{\vee}}\bigg|_x H[t-T(x)]\,.
\end{aligned}
\tag{5.9}
$$

In the last member of (5.9) $\int_{A_L}\mathrm{d}x$ is the surface integral with respect to x over the surface A_L given by $\phi(x)=L$. As a consequence we also have

$$
\begin{aligned}
\partial_t u^{(1)(NM)}(r,s,t) &= \xi_p'(r)\xi_q'(s)\partial_t u^{(1)(NM)}_{pq}(r,s,t) \\
&= -\int_{\mathbb{R}} \mathrm{d}L \int_{\phi=L} \mathrm{d}x\,A^{\vee}(x)\mathbf{w}^{\vee T}(x)\mathbf{c}^{(1)\prime}(L) \\
&\quad \times \frac{\bar{\nu}\cdot\nu_\phi}{\bar{\nu}\cdot\gamma^{\vee}}\bigg|_x \delta''[t-T(x)]\,.
\end{aligned}
\tag{5.10}
$$

We may identify $\mathbf{w}^{\vee T}\mathbf{c}^{(1)\prime}$ with the small contrast approximation to the reflection coefficient from incident mode N to reflected mode M across an interface locally coinciding with $\phi=L$.

We next evaluate the integral in (5.9) by the method of stationary phase. For variations of x lying on A_L the phase $T(x)$ is stationary at points $y(L)$ where ∇T is normal to A_L, i.e. where

$$\nu \equiv \frac{\nabla T}{|\nabla T|} = \nu_\phi\,. \tag{5.11}$$

To make the calculation explicit we choose local Cartesian coordinates (x_μ, z), $\mu=1,2$, in the neighbourhood of each such *specular point* $y(L)\in A_L$, so that

$$z \| \nu_\phi\,, \quad \{x_\mu\}\perp\nu_\phi\,; \tag{5.12}$$

see Fig. 6. The function $T[y(L)]$ has a derivative given by

$$\frac{\mathrm{d}T}{\mathrm{d}L} = \frac{|\nabla T|}{|\nabla\phi|}\bigg|_{y(L)}, \tag{5.13}$$

while, by the implicit function theorem, the function $L_y(t)$ satisfying

$$T(y[L_y(t)]) = t$$

exists. Indeed, several may exist if the scattered field develops caustics. Taylor expansions of the level and isochrone surfaces up to the second-order curvature terms in x_ν yield

$$
\begin{aligned}
0 &= \phi(x)-\phi(y) = |\nabla\phi(y)|z + \frac{1}{2}x_\mu\phi_{\mu\nu}(y)x_\nu\,, \\
T(x) &= T(y) + |\nabla T(y)|z + \frac{1}{2}x_\mu T_{\mu\nu}(y)x_\nu\,.
\end{aligned}
\tag{5.14}
$$

The first equality in (5.14) amounts to the equation of the level surface $\phi=L$,

$$z = -\frac{x_\mu\phi_{\mu\nu}x_\nu}{2|\nabla\phi|}. \tag{5.15}$$

Substituting this into the second equality gives

$$T(x) = T(y) + \frac{1}{2}|\nabla T(y)|x_\mu\Upsilon_{\mu\nu}(y)x_\nu\,, \tag{5.16}$$

where

$$\Upsilon_{\mu\nu} \equiv \frac{T_{\mu\nu}}{\nabla T|} - \frac{\phi_{\mu\nu}}{|\nabla\phi|}$$

for x constrained to lie on the level surface A_L. We shall assume that the matrix Υ, which may vary with the level L, is non-singular, but may be negative definite, indefinite or positive definite. The case of singular Υ leads to analysis akin to that required for caustics (coalescent points of stationary phase) and will not be treated here. However, for non-singular Υ, when t is near $T(y)$, we have the intermediate result

$$
\begin{aligned}
&\partial_t^2 \int_{\phi=L} H[t-T(x)]\,\mathrm{d}A(x) \\
&= \partial_t^2 \int_{\mathbb{R}^2} H[t-T(y) - \frac{1}{2}|\nabla T(y)|x_\mu\Upsilon_{\mu\nu}(y)x_\nu]\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= \frac{2\pi\,\delta^*[t-T(y)]}{|\gamma^{\vee}(y)|\sqrt{|\det[\Upsilon(y)]|}},
\end{aligned}
\tag{5.17}
$$

where

$$
\delta^* = \begin{cases} \delta & \text{if } \Upsilon \text{ is positive definite}\,, \\ \mathscr{H}\delta & \text{if } \Upsilon \text{ is indefinite}\,, \\ -\delta & \text{if } \Upsilon \text{ is negative definite}\,, \end{cases}
\tag{5.18}
$$

and $\mathscr{H}$ denotes the Hilbert transform. These results appear as eqs (6.2), (6.4) and (6.3) in Burridge (1963), where a time-dependent derivation is given.

On using (5.17) to evaluate the integral in eq. (5.9) we obtain

$$\int_{\mathbb{R}} \mathrm{d}L\, \partial_t^2 \int_{\phi=L} \mathrm{d}x\, \check{\mathbf{w}}^T(x)\mathbf{c}^{(1)\prime}(L) \left.\frac{\bar{\mathbf{v}}\cdot\mathbf{v}_\phi}{\bar{\mathbf{v}}\cdot\check{\gamma}}\right|_x H[t-T(x)]\,.$$

$$=2\pi \int_{\mathbb{R}} \mathrm{d}L \frac{\check{\mathbf{w}}^T[y(L)]\mathbf{c}^{(1)\prime}(L)}{|\check{\gamma}|\sqrt{|\det(\Upsilon)|}} \left.\frac{\bar{\mathbf{v}}\cdot\mathbf{v}_\phi}{\bar{\mathbf{v}}\cdot\check{\gamma}}\right|_{y(L)} \delta^*[t-T(y(L))]$$

$$=2\pi \frac{\check{\mathbf{w}}^T[y(L)]\mathbf{c}^{(1)\prime *}[L_y(t)]}{|\check{\gamma}|\sqrt{|\det(\Upsilon)|}(\mathrm{d}T/\mathrm{d}L)} \left.\frac{\bar{\mathbf{v}}\cdot\mathbf{v}_\phi}{\bar{\mathbf{v}}\cdot\check{\gamma}}\right|_{y(L_y(t))}. \tag{5.19}$$

Then, on substituting eq. (5.13) into eq. (5.19) and using the result in eq. (5.9), we see that

$$u_{pq}^{(1)(NM)}(r,s,t) = -2\pi \frac{|\nabla\phi|\acute{A}\grave{A}\check{\mathbf{w}}^T(y[L_y(t)],\grave{\gamma},\acute{\gamma})\mathbf{c}^{(1)\prime *}[L_y(t)]}{\sqrt{|\det(\Upsilon)|}\,|\check{\gamma}|^3}\,, \tag{5.20}$$

since at the specular point we have $\mathbf{v}=\mathbf{v}_\phi$. This formula is an extension to three dimensions of the 1-D *convolutional model.* In (5.19) and (5.20),

$$\mathbf{c}^{(1)\prime *}[L_y(t)] = \begin{cases} \mathbf{c}^{(1)\prime}[L_y(t)], & \text{for } \Upsilon \text{ positive definite}, \\ \mathscr{H}_t\mathbf{c}^{(1)\prime}[L_y(t)], & \text{for } \Upsilon \text{ indefinite}, \\ -\mathbf{c}^{(1)\prime}[L_y(t)], & \text{for } \Upsilon \text{ negative definite}. \end{cases} \tag{5.21}$$

If $\mathbf{c}^{(1)}$ contains a step function, the numerator represents the angle-dependent reflection coefficient in the small-contrast approximation: multiply eq. (5.20) by the amplitudes (eqs 3.12 and 3.27) according to eq. (4.2) to obtain the displacement, and substitute eq. (3.13); extracting the phase velocities at the image point from the amplitudes, we obtain the following expression for the *linearized reflection/transmission coefficients* at $y(L)$:

$$R_L^{(NM)}(.,\grave{\gamma},\acute{\gamma}) = \frac{|\nabla\phi|\check{\mathbf{w}}^T\mathbf{c}^{(1)\prime}}{[\grave{V}\acute{V}^3]^{1/2}|\check{\gamma}|^2(\bar{\mathbf{v}}\cdot\mathbf{v})^2}\,, \tag{5.22}$$

assuming that $\grave{\gamma}$ and $\acute{\gamma}$ for a given $\mathbf{v}_\phi$ satisfy Snell's law, i.e. $\grave{\gamma}+\acute{\gamma}$ is parallel to $\mathbf{v}_\phi$, [then (θ, ψ) are the only degrees of freedom in $\check{\mathbf{w}}$]. Backsubstituting (5.22) into (5.9) illustrates this:

$$\hat{\partial}_t u^{(1)(NM)}(r,s,t) = -\int_{\mathbb{R}} \mathrm{d}L \int_{\phi=L} \mathrm{d}x A'(x) R_L^{(NM)}[x,\grave{\gamma}(x),\acute{\gamma}(x)]$$

$$\times(\bar{\mathbf{v}}\cdot\mathbf{v}_\phi)(\bar{\mathbf{v}}\cdot\check{\gamma})\big|_x\,\delta''[t-T(x)]\,, \tag{5.23}$$

with

$$A'(x) = \check{A}(x)[\grave{V}(x)\acute{V}(x)^3]^{1/2}\,. \tag{5.24}$$

Eq. (5.22) provides a tool to quantify *a posteriori* the accuracy of the single-scattering or Born approximation by comparison with the solution of the full Zoeppritz equations at the specular direction. Such a comparison for a qP–qP reflection in a transversely isotropic medium (a shale overlying a gas-bearing sandstone) is shown in Fig. 7; note that the linearization breaks down for wide angles. Eq. (5.22) relates the linearized reflection/transmission coefficients to the kernels of the volume scattering formulation (eq. 4.1). The radiation patterns associated with those kernels, and oriented with respect to the geological layering, predict which medium perturbations will be mainly responsible for causing the linearization to fail; these are the ones that imply radiation patterns containing lobes parallel to the geological layering. For our example (Fig. 7), this is illustrated in Fig. 8, from which it becomes clear that in this case the perturbations in $c_{1111}=c_{11}$ and $c_{1313}=c_{55}$ are hard to determine in the linear regime.

Inversion

We can follow two approaches to carry out the linearized inversion. Either we can base the procedure on the Kirchhoff–Born-type representation eq. (5.10) or we can base it on the further asymptotically reduced representation eq. (5.20).

In the inversion of eq. (5.20), where the variations in $\grave{\gamma}$ and $\acute{\gamma}$ are constrained by the condition that the isochrone is tangent to the layering, $\mathbf{v}=\mathbf{v}_\phi$, leaving (θ, ψ) as the only degrees of freedom, the Radon transformation plays no role. The inversion formula reduces to a 3-D AVA analysis and

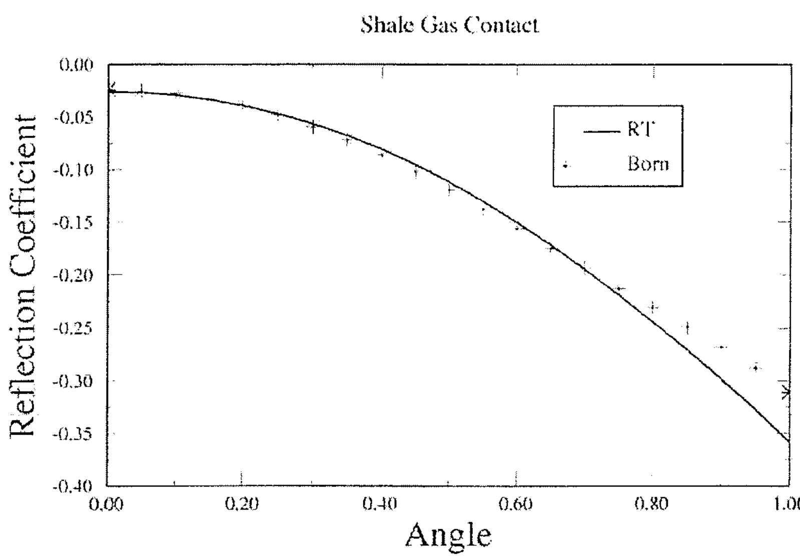

Figure 7. Exact (solid line)and linearized qP–qP reflection coefficients: an example. (The average medium properties across the interface are taken as the background.)

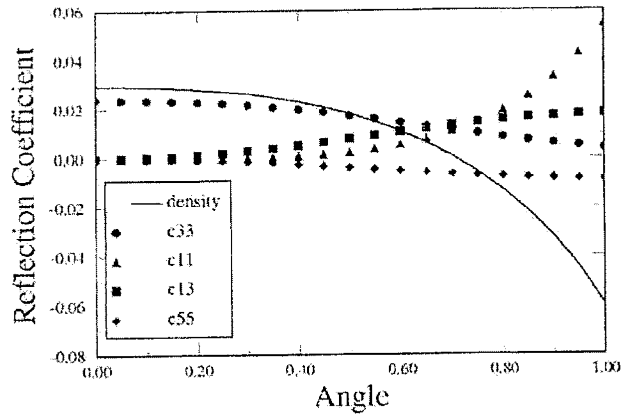

Figure 8. Contributions to the linearized qP–qP reflection coefficients; perturbations are 10 per cent (background as in Fig. 7).

corresponding techniques can be applied to invert for the medium derivatives. In accordance with our procedure, we multiply eq.(5.20) on the left by

$$-\frac{\sqrt{|\det(\Upsilon)|}\,|\gamma^{\smile}|^3}{2\pi|\nabla\phi|}\,\sigma_U^{-1}\mathbf{w}\,,$$

integrate over (θ, ψ), and invert the matrix Λ for $\nu=\nu_\phi$ as before. Thus we define

$$\Lambda(\nu)=\sum_{N,M}\int \lambda^{(NM)}\frac{\partial(\gamma^{\smile},\gamma')}{\partial(\nu,\theta,\psi)}\,\mathrm{d}\theta\,\mathrm{d}\psi+\sigma_c^{-1}\,, \tag{5.25}$$

as in (4.37), to give

$$\langle \mathbf{c}^{(1)\prime *}[L_y(t)]\rangle=-\frac{1}{2\pi}\Lambda^{-1}(\nu)\int \mathrm{d}\theta\,\mathrm{d}\psi\,|\det(\Upsilon)|^{1/2}|\gamma^{\smile}|^3$$
$$\times\,\sigma_U^{-2}\mathbf{w}^{\smile}u^{(1)(NM)}(r,s,t)\,\frac{\partial(\gamma^{\smile},\gamma')}{\partial(\nu_\phi,\theta,\psi)}\,. \tag{5.26}$$

Note that both the orientation of the layering ν_ϕ and its curvature, contained in $|\det(\Upsilon)|$, need to be known at the specular point $y(t)$ in order to carry out the procedure.

Alternatively, even when the medium has fine structure of the form (5.1), we may use the GRT as in Section 4. However, if the angular coverage is not sufficient for a good approximation using the GRT, it may be preferable to use the method of this section.

6 DISCUSSION

We have developed a procedure for linearized, multiparameter inversion that combines GRT migration with AVA analysis and that is, in principle, valid in general anisotropic media. With general anisotropy, we have excluded the cases of truly weak anisotropy (near isotropy) and the occurrence of shear-wave polarization coupling, for example near osculating singularities of the slowness surface. Further, we have excluded the occurrence of caustics in the background medium when they imply conflicting migration dips at the image point. The polarization coupling can be treated with the generalized Born approximation (Chapman & Coates 1994). The caustic problem can be addressed with Maslov's theory, and, away from conflicting dips, leads to the appropriate phase changes (through the KMAH index) and amplitude corrections in the background Green's functions. Turning rays are naturally included this way.

The extension from isotropic to anisotropic media led to a partially weak formulation of the inverse problem in terms of maximum likelihood. The explicit freedom in parametrization of the medium perturbation allowed for a reduction in the number of free parameters, which aids in stabilizing the inverse problem.

Our theory is essentially based on Born scattering developed for body-wave reflections and transmissions and thus handles general. surface and borehole seismic-acquisition geometries. It does not, however, treat head waves correctly. Although the inversion formula remains stable for any scattering angles, accuracy is lost at large angles.

Our formulation can handle realistic, irregular acquisition geometries. In practice, due to poor sampling of the scattered field, we will be restricted to applying an incomplete inverse Radon transformation; the relevant integration over the double spheres has to be supplemented in some way to compensate for missing data. In connection with this, we carried out a careful resolution analysis (see De Hoop *et al.* 1996).

In our formulation we have identified linearized reflection and transmission coefficients whenever the medium perturbation is known to jump by a small amount. This provided a way to validate the Born approximation after the inversion has been carried out. Information about the local orientation and curvature of the geological layering will further stabilize the inversion when the acquisition geometry is insufficient to reconstruct the contrast pointwise. In this context De Hoop & Bleistein (1997) have recently suggested inverting first for the reflection coefficients themselves, which appear linearly in the scattering formulae, and subsequently performing a non-linear inversion (AVA analysis) for the material parameters.

ACKNOWLEDGMENT

The authors wish to thank the referees for their helpful suggestions, which led to a much clearer exposition.

GLOSSARY OF SYMBOLS

Roman symbols	In or near eq.	Definition
$a^{\smile}_{ij}$	(3.26)	$\frac{1}{2}V_{,}(\xi_i\eta_j+\xi_j\eta_i)$
$a^{\smile}_{kl}$	(3.25)	$\frac{1}{2}V_{,}(\xi_k\eta_l+\xi_l\eta_k)$
$a^{\smile}_{ijkl}$	(3.31)	combinations appearing in $w^{\smile}$
$A^{(N)}(x, x')$	(3.1)	mode N amplitude in ray expansion of $G(x, x', t)$
$\mathcal{A}(x)$	(3.5)	slowness surface for the medium at x
$\mathcal{A}^{(N)}(x)$	(3.6)	Nth sheet of $\mathcal{A}(x)$
$A^{\smile}(x)$	(3.27)	$\rho^{(0)}(x)A^{\smile}(x)A'(x)$
$c_{ijkl}(x)=c^{(0)}_{ijkl}(x)+c^{(1)}_{ijkl}(x)$	(2.6)	elastic stiffness tensor
$c^{(0)}_{ijkl}$, $u^{(0)}$, etc.	(2.9)–(2.12)	$^{(0)}$, background values
$c^{(1)}_{ijkl}$, $u^{(1)}$, etc.	(2.9)–(2.12)	$^{(1)}$, perturbation of quantity
$\mathbf{c}^{(1)}$	(3.29)	unknown quantities to be determined
directions	(3.13)	unit vectors
direction of $\boldsymbol{v}$	(3.6)	unit vector in the direction of the vector $\boldsymbol{v}$, $\lvert\boldsymbol{v}\rvert^{-1}\boldsymbol{v}$

Roman symbols	In or near eq.	Definition
$\mathscr{D}$	(2.15)	domain of spatially rapid variations in the material parameters
f	(2.8)	body-force density (f_1, f_2, f_3).
GRT	(4.2)	generalized Radon transform
$G(x, x', t)$	(5.8)	Green's tensor
L	(5.12)	a scalar, $\phi = L$, is a typical interface
$m_{ij}^{(1)}$	(2.16)	scattering moment density
$\mathscr{M}$	(3.12)	Jacobian $\lvert v(x')\rvert V(x) \dfrac{\partial(x)}{\partial(y)}$
$n_S(s)$	(4.34)	normal to ∂S at source s
r	(2.3)	Cartesian position vector of receiver, (r_1, r_2, r_3)
∂R	(2.1)	closed surface of receiver locations
$R_L^{(NM)}(., \grave{\gamma}, \acute{\gamma})$	(5.22)	linearized reflection and transmission coefficients
s	(2.2)	Cartesian position vector of source, (s_1, s_2, s_3)
∂S	(2.1)	closed surface of source locations
$\check{T} \equiv T^{(NM)}(r, y, s)$	(3.23)	two-way traveltime $\tau^{(N)}(y, s) + \tau^{(M)}(r, y)$
$\check{T}_{\mu\nu}$	(5.3)	second derivatives of $\check{T}$
u	(2.7)	displacement vector (u_1, u_2, u_3).
$u_{pq}^{(1)(NM)}(r, s, t)$	(2.20)	scattered displacement field at r and time t in mode M in the p-direction due to the incoming field in mode N from a source at s at time 0 in the q-direction
$U^{(NM)}(r, y, s)$	(4.2)	scaled amplitude of $u_{pq}^{(1)}$
$\grave{v}(x) = v^{(N)}(x, s)$	(3.7)	group velocity vector at x on a ray from s to x in mode N
$\acute{v}(x) = v^{(M)}(x, r)$	(3.7)	group velocity vector at x on a ray from r to x in mode M
$\hat{\grave{v}}, \hat{\acute{v}}$	(3.22)	directions of $\grave{v}, \acute{v}$
V	(3.8)	normal phase velocity, a scalar $= 1/\lvert\gamma\rvert$
$V_{\gamma}^{(L)}$	(3.25)	normal velocity for γ in the 3-direction
$\check{w}$	(3.30)	coefficients of unknown quantities
x	(2.1)	Cartesian position vector (x_1, x_2, x_3)
y	(4.1)	Cartesian position vector of image point (y_1, y_2, y_3)

Greek symbols	In or near eq.	Definition
$\gamma^{(N)}(x, s)$	(3.4)	$\nabla_x \tau^{(N)}(x, s)$ slowness vector at x for a ray of type N from s
$\hat{\grave{\gamma}}$	(3.18)	direction of $\grave{\gamma}(x)$
$\check{\gamma}(x) = \gamma^{(NM)}(r, x, s)$	(3.20)	$\grave{\gamma}(x) + \acute{\gamma}(x) = \gamma^{(N)}(s, x) + \gamma^{(M)}(r, x)$
$\delta_{pq}, \delta(t)$	(2.17)	Kronecker delta or Dirac delta
$\delta(x)$	(2.17)	$\delta(x_1)\delta(x_2)\delta(x_3)$ delta
ϵ	(5.1)	small parameter
θ	(4.19)	angle between $\grave{\gamma}$ and $\acute{\gamma}$
$\grave{\kappa}$	(4.36)	Gaussian curvature of the slowness surface at $\grave{\gamma}$
$\lambda^{(NM)}$	(4.23)	normal matrix
$\Lambda^{(NM)}$	(4.25)	integral of $\lambda^{(NM)}$
$\nu, \nu^{(NM)}$	(3.21)	$\check{\gamma}$, isochrone normal, direction of $\check{\gamma}$
$\grave{\xi}(x') = \xi^{(N)}(x'; x, s)$	(3.1)	polarization for wave type N at x' on ray from s to x
$\rho(x) = \rho^{(0)}(x) + \rho^{(1)}(x)$	(2.5)	mass density
$\sigma_U^2(\hat{\grave{\gamma}}(y), \hat{\acute{\gamma}}(y))$	(4.23)	covariance function
σ_c^2	(4.23)	covariance matrix
$\Sigma(x, s)$	(3.4)	wave front through x due to an impulsive point source at position s and time 0
$\tau^{(N)}(x, s)$	(3.1)	traveltime for ray of type N from s to x
$\Upsilon_{\mu\nu}$	(5.16)	coefficients of quadratic form $\dfrac{T_{\mu\nu}}{\lvert\nabla T\rvert} - \dfrac{\varphi_{\mu\nu}}{\lvert\nabla\phi\rvert}$
ϕ	(5.1)	smooth function whose level surfaces are interfaces
$\Phi_{\mu\nu}$	(5.14)	second derivatives of ϕ
χ	(3.9)	angle between v and $\gamma(x)$
ψ	(4.19)	'third Eulerian angle' about ν

Other symbols		
´	(3.14)	(acute) pertaining to the ray from r to x (or to y) in mode M
\`	(3.14)	(grave) pertaining to the ray from s to x (or to y) in mode N
ˇ	(3.20)	(check) pertaining to both rays, r to x in mode M and s to x in mode N
ˆ	(3.13)	(hat) unit vector, or direction, $\hat{v} = v/\lvert v\rvert$
$\wedge$	(3.12)	wedge, or cross, product
$\nabla = \nabla_x$	(3.4)	gradient with respect to position x

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APPENDIX A: ANISOTROPIC RAY THEORY AND THE TRANSPORT EQUATION

In this appendix, we shall derive the transport equation (eq. 3.6) for $A^{(N)}$ from the homogeneous form of the elastodynamic wave equation (eq. 2.8), namely

$$\rho^{(0)}\ddot{u}_i - (c^{(0)}_{ijk\ell} u_{k,\ell})_{,j} = 0\,. \tag{A1}$$

We seek $\boldsymbol{u}$ in the form given by the following ansatz:

$$\boldsymbol{u} = \sum_{n=0}^{\infty} A^{(n)}(x) f_n[t - \tau(x)]\,, \tag{A2}$$

where

$$f_n' = f_{n-1}\,. \tag{A3}$$

Thus, regarded as functions of time, each term in the series eq. (A2) is one integration step smoother than the one before. The traveltime function τ is a scalar function of $\boldsymbol{x}$ in this appendix, but elsewhere it may also depend on a source point $\boldsymbol{x}'$ and a mode of propagation N, and we then write $\tau^{(N)}(\boldsymbol{x}, \boldsymbol{x}')$. Moreover, $\rho^{(0)}$ and $c^{(0)}_{ijk\ell}$, and therefore $\boldsymbol{u}$, may depend upon the parameter ϵ. Here we shall leave only the x dependence explicit.

For ease of notation later we define

$$A^{(n)}(x) = 0 \quad \text{for} \quad n < 0\,. \tag{A4}$$

To obtain equations for τ and $A^{(n)}$ we shall need to substitute (A2) into (A1), so we calculate

$$\ddot{u} = \sum_{n=0}^{\infty} A^{(n)}(x) f_{n-2}[t - \tau(x)]\,,$$
$$u_{k,\ell} = \sum_{n=0}^{\infty} (-\tau_{,\ell} A_k^{(n)} f_{n-1} + A_{k,\ell}^{(n)} f_n) \tag{A5}$$

and

$$\begin{aligned}(c^{(0)}_{ijk\ell} u_{k,\ell})_{,j} &= \sum_{n=0}^{\infty} [c^{(0)}_{ijk\ell}\tau_{,j}\tau_{,\ell} A_k^{(n)} f_{n-2} - (c^{(0)}_{ijk\ell}\tau_{,\ell}A_k^{(n)})_{,j} f_{n-1} \\ &\quad - c^{(0)}_{ijk\ell}\tau_{,j}A^{(n)}_{k,\ell} f_{n-1} + (c^{(0)}_{ijk\ell}A^{(n)}_{k,\ell})_{,j} f_n] \\ &= \sum_{n=0}^{\infty} [c^{(0)}_{ijk\ell}\tau_{,j}\tau_{,\ell}A_k^{(n)} - (c^{(0)}_{ijk\ell}\tau_{,\ell}A_k^{(n-1)})_{,j} \\ &\quad - c^{(0)}_{ijk\ell}\tau_{,j}A^{(n-1)}_{k,\ell} + (c^{(0)}_{ijk\ell}A^{(n-2)}_{k,\ell})_{,j}] f_{n-2}\,.\end{aligned} \tag{A6}$$

Substituting eqs (A5) and (A6) into (A1) leads to

$$\sum_{n=0}^{\infty} [(\rho^{(0)}\delta_{ik} - c^{(0)}_{ijk\ell}\tau_{,j}\tau_{,\ell})A_k^{(n)} + (c^{(0)}_{ijk\ell}\tau_{,\ell}A_k^{(n-1)})_{,j} + c^{(0)}_{ijk\ell}\tau_{,j}A^{(n-1)}_{k,\ell} - (c^{(0)}_{ijk\ell}A^{(n-2)}_{k,\ell})_{,j}] f_{n-2} = 0\,. \tag{A7}$$

Let us equate individual coefficients of f_{n-2} to zero successively, starting with the most singular term $n=0$. Then, for $n=0$ we get

$$(\rho^{(0)}\delta_{ik}-c^{(0)}_{ijk\ell}\tau_{,j}\tau_{,\ell})A^{(0)}_k=0\,, \tag{A8}$$

leading to

$$\det(\rho^{(0)}\delta_{ik}-c^{(0)}_{ijk\ell}\tau_{,j}\tau_{,\ell})=0\,. \tag{A9}$$

This is a first-order partial differential equation for τ. It also states that the slowness vector,

$$\gamma=\nabla\tau\,, \tag{A10}$$

lies on a sextic surface $\mathscr{A}$, say. This surface consists of three ovoid sheets $\mathscr{A}^{(N)}$, $N=1, 2, 3$, each surrounding the origin. To see this we define

$$V=\frac{1}{|\gamma|}\,,\quad \hat{\gamma}=V\gamma\,, \tag{A11}$$

so that $\hat{\gamma}$ is the unit vector in the direction of γ. Eq. (A8) may now be written as

$$c^{(0)}_{ijk\ell}\hat{\gamma}_j\hat{\gamma}_\ell A^{(0)}_k=\rho^{(0)}V^2A^{(0)}_i\,, \tag{A12}$$

so that $A^{(0)}$ is an eigenvector, belonging to the eigenvalue $\rho^{(0)}V^2$, of the positive symmetric matrix with *ik* entry $c^{(0)}_{ijk\ell}\hat{\gamma}_j\hat{\gamma}_\ell$. Thus for each direction $\hat{\gamma}$ there are three positive, not necessarily distinct, values $V^{(N)}(x, \hat{\gamma})$, $N=1, 2, 3$, of V and three orthonormal eigenvectors $\xi^{(N)}$ belonging to them. Let us assume that $V^{(1)}\geq V^{(2)}\geq V^{(3)}$. Then, as $\hat{\gamma}$ sweeps out the unit sphere, each $\gamma=(1/V^{(N)})\hat{\gamma}$ sweeps out a closed sheet $\mathscr{A}^{(N)}$ of $\mathscr{A}$ surrounding the origin. For general γ, not necessarily on $\mathscr{A}$, let $V^{(N)}(x, \gamma)$ be defined by

$$\det(\rho^{(0)}(V^{(N)})^2\delta_{ik}-c^{(0)}_{ijk\ell}\gamma_j\gamma_\ell)=0\,. \tag{A13}$$

Then the $V^{(N)}(x, \gamma)$ are positive homogeneous of degree 1 in γ and the surface $\mathscr{A}^{(N)}$ is given by

$$V^{(N)}(x, \gamma)=1\,. \tag{A14}$$

Also $V^{(N)}(x, \hat{\gamma})$ are the $V^{(N)}$ already defined by (A12).

Now let us return to the solution of (A8) for τ. We may write it in the alternative form

$$V^{(N)}(x, \nabla\tau)=1\,. \tag{A15}$$

On solving this by the method of characteristics one obtains

$$\frac{\mathrm{d}x_i}{V^{(N)}_{,\gamma_i}}=\frac{\mathrm{d}\tau}{\gamma_i V^{(N)}_{,\gamma_i}}=-\frac{\mathrm{d}\gamma_i}{V^{(N)}_{,x_i}}\,. \tag{A16}$$

By the homogeneity of $V^{(N)}$ in γ and Euler's theorem we have

$$\gamma_i V^{(N)}_{,\gamma_i}=V^{(N)}=1\,, \tag{A17}$$

so (A16) leads to the Hamiltonian system

$$\dot{x}_i=V^{(N)}_{,\gamma_i}\,,\quad \dot{\gamma}_i=-V^{(N)}_{,x_i}\,, \tag{A18}$$

where we have written $\dot{}$ for $\mathrm{d}/\mathrm{d}\tau$. The paths $(x(\tau), \gamma(\tau))$ are the *characteristics* of (A15), and since this is the characteristic equation of the elastodynamic equation, they are sometimes referred to as the *bicharacteristics* of that equation. The traces $x(\tau)$ of these curves in ordinary space are the *rays*, and (A18) are called the *ray equations* of the system. We shall assume that the region of interest is simply covered by a (two-parameter) family of rays. Then, for fixed N, τ and γ are defined uniquely as functions of x. Let $v=\dot{x}$. Then from (A17) and (A18) we see that v is normal to the surface $\mathscr{A}^{(N)}$ and

$$\gamma\cdot v=1\,, \tag{A19}$$

leading to

$$\hat{\gamma}\cdot v=V^{(N)}\,. \tag{A20}$$

So, v is normal to $\mathscr{A}^{(N)}$ and the component of $v=\dot{x}$ along the direction $\hat{\gamma}$ is $V^{(N)}=1/|\gamma|$.

Next we equate to zero the coefficient of f_{-1} in eq. (A7), so setting $n=1$ we obtain

$$(\rho^{(0)}\delta_{ik}-c^{(0)}_{ijk\ell}\gamma_j\gamma_\ell)A^{(1)}_k+(c^{(0)}_{ijk\ell}\gamma_\ell A^{(0)}_k)_{,j}+c^{(0)}_{ijk\ell}\gamma_j A^{(0)}_{k,\ell}=0\,. \tag{A21}$$

If we contract this with $A^{(0)}_i$ the first term vanishes by (A8) and we get

$$A^{(0)}_i(c^{(0)}_{ijk\ell}\gamma_\ell A^{(0)}_k)_{,j}+c^{(0)}_{ijk\ell}\,A^{(0)}_i\gamma_j A^{(0)}_{k,\ell}=0\,. \tag{A22}$$

However, the symmetry of $c^{(0)}_{ijk\ell}$ shows that

$$c^{(0)}_{ijk\ell}\,A^{(0)}_i\gamma_j A^{(0)}_{k,\ell}=A^{(0)}_{i,j}\,c^{(0)}_{ijk\ell}\gamma_\ell A^{(0)}_k\,. \tag{A23}$$

Hence, combining this with eq. (A22), we find that

$$(c^{(0)}_{ijk\ell}\,A^{(0)}_i A^{(0)}_k\gamma_\ell)_{,j}=0\,. \tag{A24}$$

However, $A^{(0)}$ is a multiple of the normalized eigenvector ξ, so we may write $A^{(0)}=A^{(0)}\xi$ in (A24) to get

$$[c^{(0)}_{ijk\ell}\,\xi_i\xi_k\,\gamma_\ell(A^{(0)})^2]_{,j}=0\,. \tag{A25}$$

We shall now reduce this equation slightly by making use of the ray equations. First notice, from the fact that ξ is a normalized eigenvector, that

$$c^{(0)}_{ijk\ell}\gamma_j\gamma_\ell\xi_k=\rho^{(0)}\xi_i\,, \tag{A26}$$

and then

$$c^{(0)}_{ijk\ell}\gamma_j\gamma_\ell\xi_i\xi_k=\rho^{(0)}\,. \tag{A27}$$

Let us suppose that $\gamma=\gamma(s)$ on $\mathscr{A}^{(N)}$ is parametrized by s so that on differentiating with respect to s we obtain a vector γ' tangent to $\mathscr{A}^{(N)}$. Then differentiating (A27) we get

$$c^{(0)}_{ijk\ell}\gamma'_j\gamma_\ell\xi_i\xi_k+c^{(0)}_{ijk\ell}\gamma_j\gamma'_\ell\xi_i\xi_k+c^{(0)}_{ijk\ell}\gamma_j\gamma_\ell\xi'_i\xi_k+c^{(0)}_{ijk\ell}\gamma_j\gamma_\ell\xi_i\xi'_k=0\,, \tag{A28}$$

but by (A26) and the normalization $\xi_k\xi_k=1$,

$$c^{(0)}_{ijk\ell}\gamma_j\gamma_\ell\xi'_i\xi_k+c^{(0)}_{ijk\ell}\gamma_j\gamma_\ell\xi_i\xi'_k=\rho^{(0)}(\xi'_i\xi_i+\xi_k\xi'_k)=0\,. \tag{A29}$$

Hence (A28) reduces to

$$c^{(0)}_{ijk\ell}\gamma'_j\gamma_\ell\xi_i\xi_k+c^{(0)}_{ijk\ell}\gamma_j\gamma'_\ell\xi_i\xi_k=2\,c^{(0)}_{ijk\ell}\gamma'_j\gamma_\ell\xi_i\xi_k=0 \tag{A30}$$

by the symmetry of $c^{(0)}_{ijk\ell}$. However, γ' is any vector tangent to $\mathscr{A}^{(N)}$. Hence, if we define v by

$$\rho^{(0)}v_j=c^{(0)}_{ijk\ell}\gamma_\ell\xi_i\xi_k\,, \tag{A31}$$

v is a vector normal to $\mathscr{A}^{(N)}$, and by (A27) $v_j\gamma_j=1$. Thus, by comparing this v with v of (A19), we find that they are equal. So, finally, (A25) may be written as

$$\nabla\cdot[\rho^{(0)}A^{(0)^2}v]=0\,. \tag{A32}$$

On differentiating this product we get

$$v\cdot\nabla[\rho^{(0)}A^{(0)^2}]+\nabla\cdot v\,[\rho^{(0)}A^{(0)^2}]=0\,, \tag{A33}$$

which on using $v=\mathrm{d}x/\mathrm{d}\tau$ may be rewritten

$$\frac{\mathrm{d}[\rho^{(0)}A^{(0)^2}]}{\mathrm{d}\tau}+\nabla\cdot v\,[\rho^{(0)}A^{(0)^2}]=0\,, \tag{A34}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\log[\rho^{(0)}A^{(0)^2}]=-\nabla\cdot v\,. \tag{A35}$$

On integrating (A32) over a narrow tube of rays terminated by small patches σ_1 and σ_2 of wave fronts, we find that

$$\rho_1^{(0)}V_1A_1^{(0)2}\sigma_1=\rho_2^{(0)}V_2A_2^{(0)2}\sigma_2\,, \tag{A36}$$

where the subscripts $_{1,2}$ indicate evaluation on $\sigma_{1,2}$, so that

$$A^{(0)}=\frac{C}{\sqrt{\rho^{(0)}V\sigma}}\,, \tag{A37}$$

where C is a constant and σ is the area cut out by the narrow tube of rays on the wavefront at the point where $\rho^{(0)}$ and V are evaluated.

APPENDIX B: SOLUTION OF THE TRANSPORT EQUATION

In this appendix, we shall solve the transport equation (3.6) for $A^{(N)}$. We begin by integrating the left member of the equation over the interior of a narrow ray tube terminated at one end by the source, at s, and at the other by the wave front $\Sigma=\Sigma^{(N)}(x)$, through x, and parametrized by x^{Σ}. Let the rays emanating from the source be parametrized by the slowness vector $\gamma^{\mathscr{S}}=\gamma^{(N)}(s)$ in the surface $\mathscr{S}=\mathscr{S}^{(N)}(s)$. Then, by (A37), we find that

$$\rho^{(0)}V^{(N)}\frac{\partial(x^{\Sigma})}{\partial(\gamma^{\mathscr{S}})}A^{(0)2} \tag{B1}$$

is conserved along a ray. Thus

$$A^{(0)}(x,s)=\frac{B(s)}{\left[\rho^{(0)}(x)V^{(N)}(x)\,\dfrac{\partial(x^{\Sigma})}{\partial(\gamma^{\mathscr{S}})}\right]^{1/2}}\,. \tag{B2}$$

To find the constant $B(s)$ in the latter expression, we allow x to approach s and make use of

$$\lim_{t\to 0}\frac{x-s}{t}=v^{(N)}(s)\,.$$

From eq. (B2), we obtain

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t^2}\frac{1}{A^{(0)^2}}&=\frac{1}{B^2(s)}\lim_{t\to 0}\frac{1}{t^2}\rho^{(0)}(x)V^{(N)}(x)\frac{\partial(x^{\Sigma})}{\partial(\gamma^{\mathscr{S}})}\\&=\frac{\rho^{(0)}(s)V^{(N)}(s)}{B^2(s)}\frac{\partial(v^{(N)})}{\partial(\hat{v}^N)}\frac{\partial(\hat{v}^N)}{\partial(\gamma^{\mathscr{S}})}\,,\end{aligned} \tag{B3}$$

with (*cf*. eq. 4.36)

$$\frac{\partial(v^N)}{\partial(\hat{v}^N)}=\frac{|v^{(N)}|^3}{V^{(N)}} \tag{B4}$$

(*cf*. 3.13) and

$$\frac{\partial(\hat{v}^N)}{\partial(\gamma^{\mathscr{S}})}=\kappa^{\mathscr{S}}\,. \tag{B5}$$

where $\kappa^{\mathscr{S}}$ is the Gaussian curvature of the slowness surface $\mathscr{S}$ at $\gamma^{\mathscr{S}}$.

Substituting eqs (B4)–(B5) into eq. (B3) yields

$$\lim_{t\to 0}\frac{1}{t^2}\frac{1}{A^{(0)^2}}=\frac{\rho^{(0)}(s)}{B^2(s)}|v^{(N)}(x)|^3\kappa^{\mathscr{S}}\,. \tag{B6}$$

Restating in the notation of this paper eq. (6.7) of Burridge (1967) for the field due to a point source in a *uniform* medium, modified slightly by a factor of $1/\rho^{(0)}$ to allow for a different definition of the source, we have

$$G_{kl}(t,x)\sim\frac{-1}{4\pi\rho^{(0)}\kappa^{\mathscr{S}1/2}|x-s|}\frac{\xi_k^{(N)}\xi_l^{(N)}}{|v^{(N)}|}\delta[t-\gamma^{\mathscr{S}}\cdot(x-s)]\,. \tag{B7}$$

Identifying the first factor as $A^{(N)}$ we obtain

$$\lim_{t\to 0}tA^{(0)}=\frac{1}{4\pi\rho^{(0)}\kappa^{\mathscr{S}1/2}|v^{(N)}|^2}\,. \tag{B8}$$

Then, comparing this with eq. (B6) leads to

$$\frac{B(s)}{[\rho^{(0)}(s)|v^{(N)}(s)|^3\kappa^{\mathscr{S}}]^{1/2}}=\frac{1}{4\pi\rho^{(0)}(s)\kappa^{\mathscr{S}1/2}|v^{(N)}(s)|^2}\,, \tag{B9}$$

so that

$$B(s)=\frac{1}{4\pi[\rho^{(0)}(s)|v^{(N)}(s)|]^{1/2}}\,. \tag{B10}$$

Finally, substituting eq. (B10) into eq. (B2), we obtain

$$A^{(0)}(x,s)=\frac{1}{4\pi\left[\rho^{(0)}(x)\rho^{(0)}(s)|v^{(N)}(s)|V^{(N)}(x)\,\dfrac{\partial(x^{\Sigma})}{\partial(\gamma^{\mathscr{S}})}\right]^{1/2}}\,, \tag{B11}$$

in agreement with eq. (3.12).

APPENDIX C: THE GENERALIZED INVERSE—A BAYESIAN ANALYSIS

We consider eqs (4.2) prior to applying the generalized Radon inversion, with y and v fixed. This may be formalized as a generalized linear inverse problem of the form

$$U_\eta=Z_{\eta J}c_J+n_\eta \tag{C1}$$

for the medium parameter perturbations c. Here n_η is random Gaussian noise. J is a compound index which includes not only the discrete index, summed from 1 to 22, but also possibly the dependence upon x integrated over $\mathscr{D}$, and η is a multi-index including dependence upon N, M, θ, ψ. We suppress the explicit dependence upon y and v since these are fixed for the moment. c may also contain a directional derivative of $c^{(1)}$, U contains $U^{(1)}$ and factors such as the appropriate Jacobians, and the functions Z contain w^T and any further Jacobians dependent on (θ, v, ψ), J and possibly x. The parameters θ and ψ parametrize the source and receiver locations through the functions $s(y,\theta,v,\psi)$ and $r(y,\theta,v,\psi)$, which may be computed in the course of ray tracing.

We begin by rewriting (C1) in matrix notation:

$$U=Zc+n\,. \tag{C2}$$

We shall suppose that the probability density of $n = U - Zc$ is

$$P\{n\} = P\{U|c\} \frac{1}{\sqrt{(2\pi)^{d_U} \det(\sigma_U^2)}} \exp\left(-\frac{1}{2} n^T \sigma_U^{-2} n\right), \qquad \text{(C3)}$$

where σ_U^2 is the covariance matrix of (the components of) n and d_U is the dimension of U and hence also of n.

We shall assume an *a priori* probability density for c, also Gaussian:

$$P\{c\} = \frac{1}{\sqrt{(2\pi)^{d_c} \det(\sigma_c^2)}} \exp\left(-\frac{1}{2} c^T \sigma_c^{-2} c\right), \qquad \text{(C4)}$$

where σ_c^2 is the covariance matrix of (the components of) c and d_c is the dimension of c.

The conditional probability density of c given U is by Bayes's theorem

$$P\{c|U\} = \frac{P\{U|c\}P\{c\}}{P\{U\}}, \qquad \text{(C5)}$$

and the maximum-likelihood estimate of c is the value $c^{\rm ml}$ of c which maximizes $P\{U|c\}P\{c\}$, since $P\{U\}$ is independent of c. Thus

$$c^{\rm ml} = \operatorname{argmax}_c \left\{ \exp\left[-\frac{1}{2}(U^T - c^T Z^T)\sigma_U^{-2}(U - Zc)\right] \times \exp\left[-\frac{1}{2} c^T \sigma_c^{-2} c\right] \right\}, \qquad \text{(C6)}$$

i.e.

$$c^{\rm ml} = \operatorname{argmin}_c \left\{ \frac{1}{2}(U^T - c^T Z^T)\sigma_U^{-2}(U - Zc) + \frac{1}{2} c^T \sigma_c^{-2} c \right\}. \qquad \text{(C7)}$$

By setting to zero the derivative with respect to c of this quadratic form, we are led to the linear system

$$(Z^T \sigma_U^{-2} Z + \sigma_c^{-2})c = Z^T \sigma_U^{-2} U. \qquad \text{(C8)}$$

Putting in the subscripts this may be rewritten as

$$(Z_{\eta,J} \sigma_{U,\eta\eta'}^{-2} Z_{\eta'} + \sigma_{C,JJ'}^{-2})c_{J'} = Z_{\eta,J} \sigma_{U,\eta\eta'}^{-2} U_{\eta'}. \qquad \text{(C9)}$$

In (C9) repeated indices imply summation over discrete components or integration over continuum components. Thus repeated J' implies summation over the 22 subscripts of c and integration over x. The repeated η' implies summation over (selected values of) N and M and integration over the variables θ and ψ through r and s. If it is assumed that $\sigma_{U,\eta\eta'}^2$ is a multiple of the identity, then we may rewrite (C9) as

$$(\sigma_U^{-2} Z_{\eta,J} Z_{\eta,J'} + \sigma_{C,JJ'}^{-2})c_{J'} = Z_{\eta,J} \sigma_{U,\eta\eta'}^{-2} U_{\eta'}. \qquad \text{(C10)}$$

Compare this with the use of (4.23), (4.25) in (4.30) and the summations and integrations used in those equations where only one pair NM is used, and again with the use of (4.23), (4.37) in (4.38), where there is a summation over multiple values of the pair NM.

APPENDIX D: VOLUME FORMS ON SPHERES

In this appendix we will generalize the results of Burridge & Beylkin (1988) on the volume form on $S^2 \times S^2$ (*cf.* eq. 4.22). We treat the problem independently from the main text and hence introduce a separate notation. Set $\xi = \hat{\gamma}'$, $\eta = \hat{\gamma}'$; then

$$\nu = \lambda(\xi, \eta)\xi + \mu(\xi, \eta)\eta, \qquad \text{(D1)}$$

where (*cf.* eq. 3.21)

$$\lambda = \frac{|\gamma'|}{|\gamma''|}, \quad \mu = \frac{|\gamma'|}{|\gamma''|}. \qquad \text{(D2)}$$

We have

$$\xi \in S^2, \quad \eta \in S^2 \quad \text{and} \quad \nu \in S^2. \qquad \text{(D3)}$$

We introduce the angle θ between the unit slowness directions as

$$\cos\theta = \xi \cdot \eta, \quad \theta \in [0, \pi). \qquad \text{(D4)}$$

In view of eq. (D3) we have the constraint

$$\lambda^2 + \mu^2 + 2\lambda\mu\cos\theta = 1. \qquad \text{(D5)}$$

Further, we introduce the unit vector

$$\zeta = \frac{1}{\sin\theta}(\xi \wedge \eta) \wedge \nu = \frac{(\xi \cdot \nu)\eta - (\eta \cdot \nu)\xi}{\sin\theta}. \qquad \text{(D6)}$$

The vectors ξ, η, ν and ζ lie in the same plane; also $\zeta \perp \nu$. Note that for ν fixed, $\zeta \in S^1$. We shall analyse the transformation $(\xi, \eta) \to (\nu, \theta, \psi)$, where ψ denotes the angular displacement of ζ, and evaluate the associated Jacobian.

First, let ξ and η vary in their *own* plane, i.e. the plane they initially span. The associated infinitesimal angular displacements of the relevant vectors will be denoted by the superscript $^{\parallel}$. Then, in terms of angles u, v in a fixed reference frame, we write

$$\xi = \begin{pmatrix} \sin u \\ 0 \\ \cos u \end{pmatrix}, \quad \xi_{,u} = \begin{pmatrix} \cos u \\ 0 \\ -\sin u \end{pmatrix}, \quad \eta = \begin{pmatrix} \sin v \\ 0 \\ \cos v \end{pmatrix}, \quad \eta_{,v} = \begin{pmatrix} \cos v \\ 0 \\ -\sin v \end{pmatrix}, \qquad \text{(D7)}$$

while $\lambda = \lambda(u, v)$ and $\mu = \mu(u, v)$. Note that

$$v - u = \theta. \qquad \text{(D8)}$$

In general, from eqs (D1) and (D7) it follows that for in-plane variations

$$d\nu = (\lambda_{,u}du + \lambda_{,v}dv)\xi + \lambda\xi_{,u}du + (\mu_{,u}du + \mu_{,v}dv)\eta + \mu\eta_{,v}dv. \qquad \text{(D9)}$$

We introduce the unit vector (*cf.* eq. D5)

$$\nu' = \lambda(u, v)\xi_{,u} + \mu(u, v)\eta_{,v}. \qquad \text{(D10)}$$

Note that

$$\xi_{,u} \perp \xi, \quad \eta_{,v} \perp \eta, \quad \nu' \perp \nu, \qquad \text{(D11)}$$

while ξ, $\xi_{,u}$, η, $\eta_{,v}$, ν and ν' all lie in the same plane.

Since $\nu \cdot d\nu = 0$, the angular displacement $d\nu^{\parallel}$ of ν is given by

$$d\nu^{\parallel} = \nu' \cdot d\nu = [\lambda^2 + \lambda\mu_{,u}(\xi_{,u} \cdot \eta) + \mu\lambda_{,u}(\xi \cdot \eta_{,v}) + \lambda\mu(\xi_{,u} \cdot \eta_{,v})]\,du + [\lambda\mu(\xi_{,u} \cdot \eta_{,v}) + \lambda\mu_{,v}(\xi_{,u} \cdot \eta) + \mu\lambda_{,v}(\xi \cdot \eta_{,v}) + \mu^2]\,dv. \qquad \text{(D12)}$$

On the other hand, using eq. (D8),

$$d\theta = dv - du. \tag{D13}$$

In our notation $du = d\xi^{\parallel}$ and $dv = d\eta^{\parallel}$. Combining eqs (D12) and (D13) leads to the Jacobian

$$\frac{\partial(v^{\parallel}, \theta)}{\partial(u, v)} = \begin{vmatrix} \lambda^2 + (\lambda\mu_{,u} - \lambda_{,u}\mu)\sin\theta + \lambda\mu\cos\theta & \mu^2 + (\lambda\mu_{,v} - \lambda_{,v}\mu)\sin\theta + \lambda\mu\cos\theta \\ -1 & 1 \end{vmatrix}$$

$$= \lambda^2 + \mu^2 + 2\lambda\mu\cos\theta + [\lambda(\mu_{,u} + \mu_{,v}) - (\lambda_{,u} + \lambda_{,v})\mu]\sin\theta. \tag{D14}$$

Using eq. (D5), this results in

$$\frac{\partial(v^{\parallel}, \theta)}{\partial(\xi^{\parallel}, \eta^{\parallel})} = 1 + \lambda\mu\left[(\partial_u + \partial_v)\log\left(\frac{\mu}{\lambda}\right)\right]\sin\theta. \tag{D15}$$

Second, consider the case where ξ and η are varied *perpendicular* to the plane they initially span. The resulting infinitesimal changes will be denoted by the superscript $^{\perp}$. We first write the analogue of (D9):

$$d\mathbf{v} = d\lambda\,\boldsymbol{\xi} + \lambda\, d\boldsymbol{\xi}^{\perp} + d\mu\,\boldsymbol{\eta} + \mu\, d\boldsymbol{\eta}^{\perp}. \tag{D16}$$

Because the variations of $\mathbf{v}$ due to in-plane variations of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are also in the $\boldsymbol{\xi}$–$\boldsymbol{\eta}$ plane, for the purpose of calculating the Jacobians we only need the component of $d\mathbf{v}$ perpendicular to that plane. This component is

$$dv^{\perp} = \lambda(\xi, \eta)\, d\xi^{\perp} + \mu(\xi, \eta)\, d\eta^{\perp}. \tag{D17}$$

Similarly, using (D6) instead of (D1), we find that

$$d\zeta^{\perp} = \frac{\cos\alpha\, d\eta^{\perp} - \cos\beta\, d\xi^{\perp}}{\sin\theta}, \tag{D18}$$

where α and β are defined by

$$\cos\alpha = \boldsymbol{\xi}\cdot\mathbf{v}, \quad \cos\beta = \boldsymbol{\eta}\cdot\mathbf{v}. \tag{D19}$$

Note that

$$\alpha + \beta = \theta. \tag{D20}$$

The sine rule applied to the triangle with sides $\lambda\boldsymbol{\xi}$, $\mu\boldsymbol{\eta}$ and $\mathbf{v}$ gives

$$\frac{\sin\alpha}{\mu} = \frac{\sin\beta}{\lambda} = \sin\theta. \tag{D21}$$

Substituting (D21) into (D17) then yields

$$dv^{\perp} = \frac{\sin\beta\, d\xi^{\perp} + \sin\alpha\, d\eta^{\perp}}{\sin\theta}. \tag{D22}$$

Note that $d\zeta^{\perp} = d\psi$. Combining (D22) and (D18) yields

$$\frac{\partial(v^{\perp}, \zeta^{\perp})}{\partial(\xi^{\perp}, \eta^{\perp})} = \frac{1}{\sin^2\theta}\begin{vmatrix} \sin\beta & \sin\alpha \\ -\cos\beta & \cos\alpha \end{vmatrix} = \frac{1}{\sin\theta}. \tag{D23}$$

Putting (D15) and (D23) together, we get

$$\frac{\partial(\mathbf{v}, \theta, \psi)}{\partial(\boldsymbol{\xi}, \boldsymbol{\eta})} = \frac{\partial(v^{\parallel}, \theta)}{\partial(\xi^{\parallel}, \eta^{\parallel})}\frac{\partial(v^{\perp}, \zeta^{\perp})}{\partial(\xi^{\perp}, \eta^{\perp})} = \frac{1 + \lambda\mu\left[(\partial_u + \partial_v)\log\left(\frac{\mu}{\lambda}\right)\right]\sin\theta}{\sin\theta}. \tag{D24}$$

Thus

$$\frac{\partial(\boldsymbol{\xi}, \boldsymbol{\eta})}{\partial(\mathbf{v}, \theta, \psi)} = \frac{\sin\theta}{1 + \lambda\mu\left[(\partial_u + \partial_v)\log\left(\frac{\mu}{\lambda}\right)\right]\sin\theta}. \tag{D25}$$

In this final expression we can substitute

$$\frac{\mu}{\lambda} = \frac{V(\boldsymbol{\xi})}{V(\boldsymbol{\eta})} = \frac{V(\boldsymbol{\xi}(u))}{V(\boldsymbol{\eta}(v))}, \tag{D26}$$

where V denotes the phase velocity as before, so

$$\begin{aligned}(\partial_u + \partial_v)\log\left(\frac{\mu}{\lambda}\right) &= \partial_u \log V(\boldsymbol{\xi}) - \partial_v \log V(\boldsymbol{\eta}) \\ &= -\partial_u \log|\boldsymbol{\gamma}^{\backprime}| + \partial_v \log|\boldsymbol{\gamma}'|.\end{aligned} \tag{D27}$$

Returning to the notation of the main text yields

$$\partial_u \log|\boldsymbol{\gamma}^{\backprime}| = \frac{|\boldsymbol{\gamma}^{\backprime}|_{,u}}{|\boldsymbol{\gamma}^{\backprime}|} = \tan\chi^{\backprime} \quad \text{with} \quad \cos\chi^{\backprime} = \hat{\mathbf{v}}^{\backprime}\cdot\hat{\boldsymbol{\gamma}}^{\backprime}. \tag{D28}$$

$$\partial_v \log|\boldsymbol{\gamma}'| = \frac{|\boldsymbol{\gamma}'|_{,v}}{|\boldsymbol{\gamma}'|} = \tan\chi' \quad \text{with} \quad \cos\chi' = \hat{\mathbf{v}}'\cdot\hat{\boldsymbol{\gamma}}'. \tag{D29}$$

Then, on substituting (D28) and (D29) into (D27), and the result into (D25), making use of (D2), we obtain

$$\frac{\partial(\boldsymbol{\gamma}^{\backprime}, \boldsymbol{\gamma}')}{\partial(\mathbf{v}, \theta, \psi)} = \frac{\sin\theta}{1 + (|\boldsymbol{\gamma}^{\backprime}||\boldsymbol{\gamma}'|/|\boldsymbol{\gamma}^{\backprime\prime}|^2)(\tan\chi' - \tan\chi^{\backprime})\sin\theta}, \tag{D30}$$

which is (4.31).