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A SELECTOR FOR EQUIVALENCE RELATIONS WITH G_δ ORBITS

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ABSTRACT. Assume X is a Polish space and E is an open equivalence on X such that every equivalence class is a G_δ set. We show that there is a G_δ transversal for E. It follows that for any separable C^*-algebra A, there is a Borel cross-section for the canonical map $\operatorname{Irr}(A) \to \operatorname{Prim}(A)$.

Let X be a Polish space, E an open equivalence relation on X. It is known [**4**, Corollary 2] that if all equivalence classes (orbits) are closed, then there is a G_δ transversal for E. We will show that this result holds under the weaker hypothesis that each orbit is G_δ. The existence of a Borel selector in the special case where each orbit is both F_σ and G_δ was established by Kallman and Mauldin [**3**]. Their result in turn extends the selector theorem in Effros [**2**]. Both the main theorem and Corollary 2 were conjectured in [**3**].

Remark on terminology. Given a space X and equivalence E, let π: $X \to X/E$ be the canonical projection. A *cross-section* is a map s: $X/E \to X$ such that $\pi \circ s$ is the identity. A *selector* is a map f: $X \to X$ which factors as a composition $f = s \circ \pi$ with s a cross-section. A selector is continuous (respectively Borel measurable) if and only if the associated cross-section is continuous in the quotient topology (resp. Borel measurable in the quotient Borel structure). A *transversal* is a subset of X which meets each orbit in a singleton. If f is a continuous (resp. 1-Borel measurable) selector, then Image(f) is a closed (resp. G_δ) transversal. The converse does not hold in general.

Our main theorem is slightly more general than the result promised in the first paragraph.

THEOREM 1. *Let X be a Polish space, $\mathcal{K}$ a countable basis for the topology on X. Suppose E is an equivalence on X such that*

(i) *For every $O \in \mathcal{K}$, the E-saturation of O is both F_σ and G_δ,*

(ii) *Every E-orbit is G_δ.*

Then there is a selector for E which is Borel measurable at level 1.

It follows immediately that the associated transversal is G_δ and the quotient Borel structure on X/E is standard in the sense of Mackey [**6**].

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The proof proceeds in four lemmas. We first give a sufficient condition for an equivalence to have a continuous cross-section. Then we introduce a new topology X' on the underlying set of X such that X' and E satisfy that condition.

LEMMA 1. *Let R be an open equivalence on a Polish space Z. If the quotient topological space Z/R is T_1 and zero-dimensional, then there exists a continuous cross-section for R.*

PROOF. In [5] Kuratowski and Ryll-Nardzewski prove the following general selection theorem:

Let Y be an arbitrary set, $\mathfrak{L}$ a field of subsets of Y. Suppose F is a function on Y to the closed subsets of Z such that, for every open $G \subseteq Z$, $\{y\colon F(Y) \cap G \neq \varnothing\} \in \mathfrak{L}_\sigma$. Then there is a function $f\colon Y \to S$ satisfying:

(a) *$f(y) \in F(y)$ for all $y \in Y$.*

(b) *$f^{-1}(G) \in \mathfrak{L}_\sigma$ for all open sets $G \subseteq Z$.*

Our proposition follows immediately by taking $Y = Z/R$, $\mathfrak{L} = \{C \subseteq Z/R\colon C \text{ is clopen}\}$, $F =$ the identity function on Z/R. $\square$

Now fix X, E and $\mathcal{H}$ satisfying the hypothesis of the theorem. For $B \subseteq X$, B^+ denotes the E-saturation of B.

LEMMA 2. *$\{O^+\colon O \in H\}$ separates orbits.*

PROOF. Suppose x and y are inequivalent elements of X. Let $[y]$ be the E-orbit of y, $\overline{[y]}$ its closure. There are two cases to consider (compare [3, Lemma 4].)

Case 1. $x \notin \overline{[y]}$. Then for some $O \in \mathcal{H}$, $x \in O$ and $O \subseteq \sim\overline{[y]}$, so $[x] \subseteq O^+$, $[y] \cap O^+ = \varnothing$.

Case 2. $x \in \overline{[y]}$. $[y]$ is dense in $\overline{[y]}$ and by (ii) both $[y]$ and $[x]$ are G_δ. It follows from the Baire Category Theorem that $[x]$ is not dense in $\overline{[y]}$. Choose $O \in \mathcal{H}$ such that $O \cap [y] \neq \varnothing$, $O \cap [x] = \varnothing$. Then $[y] \subseteq O^+$, $[x] \cap O^+ = \varnothing$. $\square$

Let $\mathcal{S} = \{O^+\colon O \in \mathcal{H}\} \cup \{\sim O^+\colon O \in \mathcal{H}\}$. Let X' be the space with the same underlying set as X but with the topology generated by $\mathcal{H} \cup \mathcal{S}$.

LEMMA 3. *X' is Polish.*

PROOF. Enumerate $\mathcal{S}$ as $\{A_n\colon n \in \omega\}$ in such a way that for each natural number n, $A_{2n+1} = \sim A_{2n}$. Given $n \in \omega$, define $G_n \subseteq X \times 2^\omega$ by

$$G_n = \Big\{(x, \xi)\colon \big[(\xi(2n) = 1 \,\&\, x \in A_{2n}) \text{ or } (\xi(2n) = 0 \,\&\, x \in A_{2n+1})\big] \\ \&\big[\xi_{2n} = 1 \leftrightarrow \xi_{2n+1} = 0\big]\Big\}$$

By (i) each G_n is G_δ. Let $G = \bigcap_{n\in\omega} G_n$. Given $x \in X'$, define $\xi_x \in 2^\omega$ by setting $\xi_x(n) = 1 \leftrightarrow x \in A_n$. The map $x \mapsto (x, \xi_x)$ is easily seen to be a homeomorphism from X' to G. Since G is G_δ in $X \times 2^\omega$, both G and X' are Polish. $\square$

LEMMA 4. *E is an open equivalence on X', and the quotient space X'/E is T_1 and zero-dimensional.*

PROOF. Let $\mathbb{S}'$ be the closure of $\mathbb{S}$ under finite intersections. Let O be open in X'. Then $O = B \cup C$ where B is open in X and C is a union of elements of $\mathbb{S}'$. Write $B = \bigcup_{i \in \omega} O_i$ with each $O_i \in \mathcal{K}$. Then

$$O^+ = B^+ \cup C^+ = B^+ \cup C = \bigcup_{i \in \omega} O_i^+ \cup C.$$

Thus O^+ is a union of elements of $\mathbb{S}'$. It follows that E is an open equivalence on X' and that $\{C/E: C \in \mathbb{S}'\}$ is a clopen basis for X'/E. It follows from Lemma 2 and the definition of X' that orbits are closed in X', so X'/E is T_1. □

Now we can apply Lemma 1 to obtain a continuous cross-section s: $X'/C \to X'$. Let $f = s \circ \pi$ be the associated continuous selector. If O is open in X', then $f^{-1}(O)$ is open in X', hence $f^{-1}(O)$ is F_σ in X. Thus, f is Borel measurable at level 1 with respect to X. The image of f is a G_δ transversal for E and s is a Borel isomorphism from the quotient Borel space X/E to Image(f). The theorem is established.

COROLLARY 1. *Let X be Polish, Y an arbitrary T_0 space, g: $X \to Y$ continuous, open, and onto. Then the Borel space associated to Y is standard and there is a* 1-*Borel function f: $Y \to X$ such that $g \circ f$ is the identity.*

PROOF. In the proof of Theorem 1 note that when E is open, the cross-section s is 1-Borel with respect to the quotient topology on X/E. Now to obtain the corollary, let $E = \{(x, z): f(x) = f(z)\}$. Apply Theorem 1 and note that Y is homeomorphic to X/E. □

A particularly interesting case of Corollary 1 arises in the study of C^*-algebras. For details of the following definitions and remarks see e.g. Dixmier [**1**, §3]. Let A be a separable C^*-algebra. Prim(A) is the space of primitive ideals of A with the Jacobson topology; it is T_0. For $n \leqslant \aleph_0$, $\mathrm{Irr}_n(A)$ is the space of irreducible representations of A on the Hilbert space of dimension n, with the topology of simple weak convergence. Irr(A) is the disjoint union of the spaces Irr_n, $n \leqslant \aleph_0$; it is Polish. The map K: Irr(A) $\to$ Prim(A) which sends each representation to its kernel is continuous and open. $\hat{A}$ is the quotient topological space Irr(A)/U where U is the relation of unitary equivalence. The quotient topology coincides with the weakest topology making $\hat{K}$: $[x] \mapsto \mathrm{Ker}(x)$ continuous.

We let E be the equivalence relation on Irr(A) induced by K, and let $\hat{E}$ be the corresponding equivalence on $\hat{A}$. In [**3**] Kallman and Mauldin obtained in the conclusion of Corollary 3 under the additional assumption that B is Borel.

COROLLARY 2. *There is a Borel cross-section s*: Prim(A) $\to$ Irr(A) *for E.*

COROLLARY 3. *Suppose $B \subseteq \hat{A}$ is $\hat{E}$-invariant and $\hat{K}_{\restriction B}$ is* 1-1. *Then there is a Borel cross-section s*: $B \to$ Irr(A) *for U.*

Let π be the canonical mapping from $\mathrm{Irr}(A)$ to $\hat{A}$. Let $B \subseteq \hat{A}$ be Borel in the quotient Borel structure. According to Moore's definition [8], A is locally type I in B provided $K_{\restriction B}$ is 1-1 and there is a cross-section $s\colon B \to \mathrm{Irr}(A)$ which is measurable with respect to the quotient Borel structure on B. Corollary 3 shows that the second clause is redundant when $\pi^{-1}(B)$ is saturated with respect to E. Clearly it would be enough to assume that there is a Borel cross-section p for E with $\pi^{-1}(B) \subseteq \mathrm{Image}(p)$. Conversely, suppose A is locally type I in B with Borel cross-section $s\colon B \to \mathrm{Irr}(A)$. Then B is standard and $\hat{K}(B)$ is a Borel subset of $\mathrm{Prim}(A)$. If $p\colon \mathrm{Prim}(A) \to \mathrm{Irr}(A)$ is any Borel cross-section for E, we can define a new Borel cross-section p' by setting $p'(x) = s(y)$ if $x = \hat{K}(y)$ for some $y \in B$, $p'(x) = p(x)$ otherwise. Thus, we can characterize "locally type I" in terms of cross-sections for E:

COROLLARY 4. *Let $B \subseteq \hat{A}$ be Borel (in the quotient Borel structure).*
A is locally type I *in B if and only if there is a Borel cross-section p:* $\mathrm{Prim}(A) \to \mathrm{Irr}(A)$ *such that $\pi \circ p \circ \hat{K}$ is the identity map on B.*

REMARKS. I. While Lemma 1 does not appear to be corollary to any of the results proved by Kuratowski and Maitra in [4], it is essentially similar. Note, for example, that the proof of Lemma 1 is easily modified to give their Corollary 2. (Take the field generated by the invariant open sets for $\mathfrak{L}$.) Compare also Maitra and Rao [7].

II. The proof of Lemma 2 made no use of hypothesis (i). It shows: *If X is any topological space of weight κ and E is an equivalence on X such that* (a) *the saturation of every open set is Borel and* (b) *every orbit is strictly Baire (almost open in its closure) and a relative Baire space; then the Borel space X/E is κ-separated.*

III. The conjecture stated in [3] differs somewhat from our theorem. It is proposed there that a Borel measurable selector exists provided only that orbits are absolute G_δ and the saturations of open sets are Borel. We have been unable to prove or refute this version of the conjecture.[2] Note that it implies its own relativization to any Borel subspace. It would show that all reference to selectors could be omitted in the definition of "locally type I".

We do have a small piece of evidence supporting the general conjecture. Suppose s were a Borel selector. Then setting $B^* = s^{-1}(B)$ we could conclude

(1) Every Borel subset B of X has a Borel invariantization, i.e. a Borel set B^* such that $\sim(\sim B)^+ \subseteq B^* \subseteq B^+$.

In fact (1) holds under the hypothesis of the conjecture and can be established by setting

$$B^* = \{x\colon B \cap [x] \text{ is comeager in the subspace } [x]\}.$$

To prove that B^* is Borel when B is, one defines for each $U \in \mathcal{K}$,

$$B^{*U} = \{x\colon U \cap [x] \neq \varnothing \;\&\, B \cap U \cap [x] \text{ is comeager in } U \cap [x]\}$$

[2](Added in proof June 15, 1978.) This version of the conjecture has recently been established by S. M. Srivastava [10].

and then proceeds by induction on the complexity of B, imitating Vaught [**9**]. Note, however, that the results in [**9**] show that it is possible to have Borel invariantizations in cases where Borel selectors cannot exist.

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