Invariant Descriptive Set Theory and the Topological Approach to Model Theory

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Abstract

We study various types of topological spaces with equivalence relations ("topological equivalence spaces") which arise in connection with model theory and we apply topological results and methods to the study of languages and structures.

Most of our model theoretic applications derive from consideration of the natural topological space formed by the set of countable structures of any fixed countable similarity type. Given a similarity type ρ, for illustration consisting of a single binary relation, we identify the structure (ω, R) with the characteristic function of R and form the usual topological product space $X_\rho = 2^{\omega\times\omega}$.

Logic deals primarily with sets $B \subseteq X_\rho$ which are closed under isomorphism, i.e. invariant under the equivalence relation $I = \{(R,S): (\omega,R) \cong (\omega,S)\}$. While the study of the topological equivalence space (X_ρ, I) in connection with model theory dates from the thirties, the subject has received increased attention since the intensive study of the language $L_{\omega_1\omega}$ was commenced in the early sixties. One indication of the close connection between the topological equivalence space (X_ρ, I) and infinitary logic is given by Lopez-Escobar's result: $B \subseteq X_\rho$ _is an I-invariant Borel set if and only if_ $B = X_\rho \cap \mathrm{Mod}(\sigma)$ _for_ some sentence $\sigma \in L_{\omega_1\omega}(\rho)$.

Salient features of the space (X_ρ, I) include the following: (i) (X_ρ, I) is a Polish (separable, completely metrizable) topological space and I is a $\underset{\sim}{\Sigma}^1_1$ (analytic) subset of $X_\rho^2 = X_\rho \times X_\rho$; (ii) I is induced by a "Polish action", i.e., a continuous action on the Polish space X_ρ by a Polish topological group, viz. $\omega!$, the group of permutations of the natural numbers given the relative topology from the Baire space ω^ω.

In the first two chapters of this work we study in turn the spaces which satisfy each of these hypotheses. In chapter III we apply some of the material from chapter II together with some additional results proved just for the logic spaces to obtain new facts in model theory.

In chapter I we study spaces (X,E) such that X is a Polish space and the equivalence relation E is a $\underset{\sim}{\Sigma}^1_1$ (analytic) subset of X^2. We introduce an invariant version of the prewellordering property and apply it to prove that the collections of E-invariant $\underset{\sim}{\Sigma}^1_2$ (PCA) and $\underset{\sim}{\Pi}^1_1$ (CA) sets have the reduction property. Assuming projective determinacy, these results are extended to $\underset{\sim}{\Sigma}^1_{2n+2}$ and $\underset{\sim}{\Pi}^1_{2n+1}$ for all $n \in \omega$. An invariant uniformization principle is also considered and shown to follow for $\underset{\sim}{\Sigma}^1_n$, $n \geq 2$ from the axiom of constructibility. With suitable restrictions on X, effective versions of all the results are obtained. These effective results are proved in a "set theoretically primitive recursive" context which has a wider applicability than the traditional "lightface descriptive set theory."

The invariant $\underset{\sim}{\Pi}^1_1$ prewellordering and reduction theorem is due to Solovay based on a conjecture of the author. The results on reduction extend theorems of Vaught and Moschovachis, who proved invariant $\underset{\sim}{\Pi}_1^1$ reduction for the spaces (X_ρ, I), cf. Vaught [44]; Vaught, who proved invariant $\underset{\sim}{\Pi}_1^1$ and $\underset{\sim}{\Sigma}_2^1$ reduction for Polish actions, see Vaught [46] and Burgess, who proved invariant reduction for pairs of $\underset{\sim}{\Pi}_1^1$ sets under the hypotheses of Chapter I, see Burgess-Miller [11]. The result on uniformization was obtained jointly with Burgess and appeared in [11] as did much of the material of Chapter I. It extends unpublished work of Kuratowski.

Chapter II deals with Polish actions and, more generally, with spaces (X,G,J) such that G is a nonmeager topological group with a countable basis, X is a topological space, and $J\colon G \times X \to X$ is a Borel map which defines an action of G on X. The main tool and subject of interest in this chapter is the $*$-transform, $B \mapsto B^* = \{x\colon \{g\colon J(g,x) \in B\}$ is comeager$\}$, which was introduced in Vaught [45]. We contribute both to the basic theory of the transform and to the list of applications of the transform to the theory of group actions. Perhaps the most important result from this chapter is the invariant version of the well-known strong $\underset{\sim}{\Pi}_\alpha^0$ separation theorem of Hausdorff and Kuratowski (cf. Kuratowski [26] or Addison [3]). $\underset{\sim}{\Pi}_\alpha^0(X)$ is the αth multiplicative level of the Borel hierarchy on X (so $\underset{\sim}{\Pi}_2^0 = G_\delta$, $\underset{\sim}{\Pi}_3^0 = F_{\sigma\delta}$, etc.). We prove II.4.3: If J is continuous in each variable, X is Polish, E_J is the equivalence relation $\{(x,y)\colon (\exists g)(J(g,x) = y)\}$ and $1 \le \alpha < \omega$, then disjoint E_J-invariant $\underset{\sim}{\Pi}_{\alpha+1}^0$ sets can be separated by a countable alternated union of E_J-invariant $\underset{\sim}{\Pi}_\alpha^0$ sets, a fortiori by an invariant $\underset{\sim}{\Delta}_{\alpha+1}^0$ set. For $\alpha = 1$ the result is proved for the much wider class of equivalence spaces (X,E) such that E is lower semicontinuous (open).

In chapter III we apply topological results and methods to model theory, obtaining several new results and giving new proofs of several known theorems. In the latter case, the topological proofs are generally shorter than previously known arguments. Moreover, they make explicit a causal connection between classical topological theorems and their model theoretic analogues.

Topics discussed in this chapter include a $\underset{\sim}{\Pi}'^{0}_{\alpha}$ separation theorem, a recent global definability theorem of M. Makkai, a generalization of a result on the definability of invariant Borel functions due to Lopez-Escobar and a result on continuous selectors for elementary equivalence. Our treatment of the $\underset{\sim}{\Pi}'^{0}_{\alpha}$ separation theorem illustrates most of the types of results proved in the chapter. $\underset{\sim}{\Pi}'^{0}_{\alpha}$ denotes the αth level of the natural hierarchy on formulas of $L_{\omega_1\omega}$ in which, for example, $\underset{\sim}{\Pi}'^{0}_{2}$ classes have the form $\mathrm{Mod}(\bigwedge_n \forall \underset{\sim}{x} \bigvee_m \exists \underset{\sim}{y}\, \theta_{nm}(\underset{\sim}{x}\underset{\sim}{y}))$ with each θ_{nm} finitary, quantifier-free, *cf*. Vaught [46]. The basic $\underset{\sim}{\Pi}'^{0}_{\alpha}$ separation theorem is an unpublished result of Reyes. We give two new proofs of the basic theorem: *Disjoint* $\underset{\sim}{\Pi}'^{0}_{\alpha+1}$ *classes* $(1 \leq \alpha)$ *can be separated by a countable alternated union of* $\underset{\sim}{\Pi}'^{0}_{\alpha}$ *classes*. One proof is based on II.4.3; the second is a model theoretic translation of the classical topological proof. We use some ideas of D. A. Martin to obtain an admissible version of the theorem and we apply an approximation theorem of J. Keisler to obtain the well-known finitary version (Shoenfield's $\underset{\sim}{\forall}^{0}_{n}$ interpolation theorem). We also apply the result to study the complexity of $L_{\omega_1\omega}$ definitions of isomorphism types. We prove several theorems. The following is typical:

II.4.2: <u>If a complete</u> $L_{\omega\omega}$ <u>theory</u> T <u>has a countable model</u> $\mathfrak{A}$ <u>such that the isomorphism type of</u> $\mathfrak{A}$ <u>is</u> $\underset{\sim}{\Sigma}'^{0}_{2}$-over-$L_{\omega\omega}$, <u>then</u> T <u>is</u> ω-<u>categorical</u>. This extends a theorem of M. Benda. $\underset{\sim}{\Sigma}'^{0}_{2}$-over-$L_{\omega\omega}$ classes have the form $\mathrm{Mod}(\bigvee_{n} \exists \underset{\sim}{\check{x}} \bigwedge_{m} \forall \underset{\sim}{\check{y}} \theta_{nm}(\check{x},\check{y}))$, each $\theta_{nm} \in L_{\omega\omega}$.

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CHAPTER 0: INTRODUCTION

A brief summary of this work may be found in the preceding Abstract.

Descriptive set theory deals with Borel and projective sets in metrizable spaces and especially in the spaces ω^ω, 2^ω and other "Polish" (separable, completely metrizable) spaces. This "classical" work goes back to Lebesgue, Lusin, Suslin, Sierpinski and others. The best known reference is Kuratowski [26]. The first application of descriptive set theory to logic was made by Kuratowski in 1933 in [25] where he defined the infinitary language $L_{\omega_1\omega}$ and showed that the collection of well-orderings is not an $L_{\omega_1\omega}$-elementary class. Since that time and especially during the last fifteen years, the connections between classical descriptive set theory and model theory have been studied by many authors. See, for example, Addison [2-4], Scott [38], Lopez-Escobar [28], Grzegorczyk, et. al., [16], Morley [33] and Vaught[44-46].

Some authors, notably Addison and his students, have expounded the analogies which exist between the classical theory of Polish spaces and the model theory of the finitary predicate calculus, $L_{\omega\omega}$. Other authors, such as Lopez-Escobar in [28] and Vaught in [44] have found similar analogies with the model theory of $L_{\omega_1\omega}$. In fact, topological considerations were an important factor leading Scott and Ryll-Nardzewski to propose $L_{\omega_1\omega}$ as the "natural" infinitary first-order language in the early sixties (cf. Scott [38]).

Most recently, Vaught in [45] and [46] introduced a powerful method to show that many infinitary results can be derived from their classical counterparts. As we shall remark in III §2 below, the corresponding finitary theorems often follow by an approximation theorem of J. Keisler. Moreover, the theorems in logic are obtained as special cases of theorems about Polish actions or certain other kinds, of action spaces.

This is the type of result with which we will be primarily concerned. It accomplishes several things. First, it gives a "causal" explanation for the analogies found by previous authors. Second, by showing that certain results in model theory are special cases of general results on equivalence spaces or action spaces, it enables us to compare the restrictiveness of the hypotheses under which the general results are obtained. On the basis of this comparison, we can then classify some results as more "model theoretic" than others. Finally, it leads us to new theorems about actions or equivalence spaces as generalizations of results in model theory and to new theorems in model theory as invariant versions of classical theorems of descriptive set theory.

For a summary of the topics to be considered in this work we refer the reader to the abstract and to the table of contents. In the remainder of the introductory chapter we will establish some notational conventions and review some of the basic definitions which we will require.

Sets and Topological Spaces

ON is the collection of ordinals, each ordinal being the set of preceding ordinals. Cardinals are initial ordinals, ω and ω_1 are the first two infinite cardinals. α, β, γ will always be ordinals, λ will always be a limit ordinal, κ will always be a cardinal. $\overline{\overline{B}}$ is the cardinality of B. $\mathcal{P}(A) = \{B: B \subseteq A\}$ and $\mathcal{P}_{(\kappa)}(A) = \{B \in \mathcal{P}(A): \overline{\overline{B}} < \kappa\}$. $(a,b) = \{\{a\},\{a,b\}\}$ and $\langle B_i : i \in I\rangle = \{(i,B_i): i \in I\}$. ${}^{B}A$ is the set of functions on B to A.

A topological space $X = (|X|, \mathcal{T})$ is a pair such that $|X|$ is a set and $\mathcal{T} \subseteq \mathcal{P}(|X|)$ contains $\emptyset$ and $|X|$ and is closed under finite intersections and arbitrary unions. $\mathcal{T}$ is the collection of open subsets of X. $\mathcal{B}(X)$, the collection of Borel subsets of X is the smallest collection which includes $\mathcal{T}$ and is closed under complementation and countable unions. The Borel hierarchy on X is defined recursively by the conditions

$$\begin{aligned}
\Sigma^0_1(X) &= \mathcal{T}\\
\Pi^0_\alpha(X) &= \{\sim A: A \in \Sigma^0_\alpha(X)\}\\
\Pi^0_{(\alpha)}(X) &= \bigcup \{\Pi^0_\beta(X): \beta < \alpha\}\\
\Sigma^0_\alpha(X) &= \{\bigcup \Phi : \Phi \in \mathcal{P}_{(\omega_1)}(\Pi^0_{(\alpha)}(X))\}
\end{aligned}$$

Π^0_2, Π^0_3, etc. were classically known as G_δ, $F_{\sigma\delta}$, etc. $\Delta^0_\alpha(X) = \Pi^0_\alpha(X) \cap \Sigma^0_\alpha(X)$.

A function f on X to a topological space Y is Borel measurable, (respectively α-Borel), if $f^{-1}(A) \in \mathcal{B}(X)$, ($\Sigma^0_\alpha(X)$), whenever $A \in \Sigma^0_1(Y)$. f is a Borel isomorphism, ((α,β)-generalized homeomorphism), if f is 1-1, onto, and both f and f^{-1} are Borel, (f is α-Borel, f^{-1} is β-Borel).

Note. Our $1+\alpha$-Borel maps are "measurable at level α" in the terminology of Kuratowski. That is because Σ^0_1 is the 0th additive level of his hierarchy.

$^{\omega}\omega$ and 2^{ω} are the topological spaces formed on the sets $^{\omega}\omega$, $^{\omega}2$ by taking the product topology over the discrete spaces ω and 2.

If I is any index set and G is a function on $^{I}P(X)$ to $P(X)$, we say G is an I-Boolean operation provided $s^{-1}(G(\langle A_i : i \in I\rangle) = G(\langle s^{-1}(A_i) : i \in I\rangle)$ whenever s is a function on X to X. Given a Boolean operation G and $\Gamma \subseteq P(X)$ we define $G(\Gamma) = \{G(A) : A \in {}^{I}\Gamma\}$ Clearly, each class $G(\underset{\sim}{\Sigma}^0_1(X))$ is closed under inverse continuous images. Working with W. Wadge's theory of reducibility by continuous functions, R. Steele and R. Van Wesep have recently showed that in a certain natural sense, for "almost all" $\Gamma \subseteq B(2^{\omega})$ such that Γ closed under inverse continuous images, Γ has the form $\Gamma = G(\underset{\sim}{\Sigma}^0_1(X))$ where G is a ω-Boolean operation.

$A \subseteq X$ is nowhere dense if the closure of A includes no non-empty open subset. A is meager (of first category) if A is a countable union of nowhere dense sets. A is almost open (has the Baire property) if there exists an open set O such that the symmetric difference $O \Delta A$ is meager. X is a Baire space if no non-empty open subset of X is meager. X is separable if X includes a countable dense subset.

X is a Polish space if X is separable and admits a complete metrization. X is a Suslin (analytic) space if X is a metrizable continuous image of some Polish space. X is a Lusin (absolutely Borel) space if X is a metrizable, continuous one-one image of some Polish space.

Given a product space $X \times Y$ let $\pi_1 : X \times Y \to X$ be the projection mapping $(x,y) \mapsto x$. For any X, the projective hierarchy on X is defined by setting

$$\mathbf{\Sigma}^1_1(X) = \{\pi_1(A) : A \in \mathcal{B}(X \times Y) \text{ for some Polish } Y\}$$
$$\mathbf{\Pi}^1_n(X) = \{\sim A : A \in \mathbf{\Sigma}^1_n(X)\}$$
$$\mathbf{\Sigma}^1_{n+1}(X) = \{\pi_1(A) : A \in \mathbf{\Pi}^1_n(X \times Y) \text{ for some Polish } Y\}$$
$$\mathbf{\Delta}^1_n(X) = \mathbf{\Sigma}^1_n(X) \cap \mathbf{\Pi}^1_n(X).$$

$\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$, and $\mathbf{\Sigma}^1_2$ subsets of Polish spaces were classically known as analytic, coanalytic, and PCA sets respectively.

There are many useful normal form results for Polish, Lusin and Suslin spaces. The following is a partial list (cf. Kuratowski [26]): Every Lusin space is a continuous one-one image of a closed subset of ${}^\omega\omega$ and of a $\mathbf{\Pi}^0_2$ subset of ${}^\omega 2$. Every uncountable zero-dimensional Polish space is a union of a countable set and a set homeomorphic to ${}^\omega\omega$. All uncountable Lusin spaces are Borel isomorphic and every metrizable space which is Borel isomorphic to a Lusin space is Lusin. Every $\mathbf{\Pi}^0_2$ subspace of a Polish space is Polish. If a subspace of a Polish space is Polish then it is $\mathbf{\Pi}^0_2$. Every Borel subspace of a Lusin space is Lusin. If a subspace of a Lusin space is Lusin, then it is Borel. Every $\mathbf{\Sigma}^1_1$ subset of a Suslin space is Suslin. If a subspace of a Suslin space is Suslin, it is $\mathbf{\Sigma}^1_1$. A subset A of a Polish space X is Suslin if and only if it can be obtained by the operation (A) applied to Borel sets; that is, $A = \bigcup_{\xi \in {}^\omega\omega} \bigcap_n B_{\xi \restriction n}$ for some collection $\{B_s : s \in S_q\} \subseteq \mathcal{B}(X)$. $S_q = \bigcup\{{}^n\omega : n \in \omega\}$ is the set of finite sequences of natural numbers. One important effect of these normal forms is to reduce questions about Polish, Lusin, and Suslin spaces to questions about Borel and projective subsets of ${}^\omega\omega$ or ${}^\omega 2$.

Actions and Equivalence Relations

A <u>topological group</u> $G = (|G|, \mathcal{T}, \circ)$ is a triple such that

$(|G|,\mathcal{T})$ is a topological space, $(|G|,\circ)$ is a group, and the function $(g,h) \mapsto g \circ h^{-1}$ is a continuous map on the product space $G \times G$ to G. G is a Polish group, Lusin group, etc. if the space $(|G|,\mathcal{T})$ is Polish, Lusin, etc.

Given $J: G \times X \to X'$, we define $J^g: X \to X'$ for $g \in G$ and $J^x: G \to X'$ for $x \in X$ by the condition $J^g(x) = J^x(g) = J(g,x)$. If G is a group and the map $g \mapsto J^g$ is a homomorphism on G to the group of permutations of X, then $\mathcal{J} = (X,G,J)$ is an action. If X is a Polish space, G is a Polish group, and J is continuous, then $\mathcal{J}$ is a Polish action. When no confusion will arise, we write gh for $g \circ h$ and gx for $J(g,x)$. Given an action of G on X, we obtain an action of G on $\mathcal{P}(X)$ by setting $gA = \{gx: x \in A\}$.

Every action $\mathcal{J} = (X,G,J)$ induces an equivalence relation $E_{\mathcal{J}} = \{(x,y): (\exists g \in G)(gx = y)\}$. If X and G are Suslin and J is Borel (a fortiori, if $\mathcal{J}$ is a Polish action), then $E_{\mathcal{J}}$ is easily seen to be a $\underset{\sim}{\Sigma}{}^1_1$ subset of $X \times X$.

Given an equivalence relation E on a set X, we say that $A \subseteq X$ is E-invariant if $x \in A$ and yEx implies $y \in A$. For arbitrary $A \subseteq X$, $A^{+E} = \{y: (\exists x \in A)(yEx)\}$ and $A^{-E} = \{y: (\forall x)(xEy \Rightarrow x \in A\}$. When no confusion will result we write A^+ and A^- for A^{+E} and A^{-E}. A^+ and A^- are respectively the smallest invariant set including A and the largest invariant set included in A. For $x \in X$, $[x]_E = \{x\}^+$ is the E-equivalence class or orbit of x. If $B^\#$ is an invariant set such that $B^- \subseteq B^\# \subseteq B^+$ we say $B^\#$ is an E-invariantization of B.

X/E is the set of E-equivalence classes and π_E is the projection map $x \mapsto [x]_E$. When X is a topological space, X/E is topologized by giving it the strongest topology such that π_E is continuous. E is lower semicontinuous (respectively upper semicontinuous) if A^+ is open (closed) whenever A is open (closed). An equivalent condition is that π_E be an open (closed) mapping. Bourbaki [9] refers to lower (upper) semicontinuous equivalences as open (closed) equivalences. We have chosen our terminology, which agrees with that in Kuratowski [26] when equivalence classes are closed, to avoid confusion with equivalences whose graphs are open or closed.

When X is Suslin, we say E is a $\underset{\sim}{\Sigma}_n^1$ equivalence on X provided E is an equivalence on X and E is a $\underset{\sim}{\Sigma}_n^1$ subset of $X \times X$.

If E_1 and E_2 are equivalences on X and Y respectively, then $E_1 \times E_2 = \{((x_1,y_1),(x_2,y_2)) \colon x_1 E_1 x_2 \;\&\; y_1 E_2 y_2\}$ is the product equivalence on $X \times Y$. $\underset{\sim}{1}$ will always be the identity equivalence. If $\mathcal{J}_1$ and $\mathcal{J}_2$ are actions of G on X and Y respectively, then the product action $\mathcal{J}_1 \times \mathcal{J}_2$ of G on $X \times Y$ is defined by setting $g(x,y) = (gx,gy)$. Note that $E_{\mathcal{J}_1 \times \mathcal{J}_2}$ is not generally the same as $E_{\mathcal{J}_1} \times E_{\mathcal{J}_2}$.

Logic

If $n \in \omega$ and R is any set then $\underline{R} = (1,(R,n))$ is an n-ary relation symbol and $n = n(\underline{R})$ is the arity of $\underline{R}$. For any set c, $\underline{c} = (1,(c,0))$ is a constant symbol.

A similarity type is a set of relation symbols and constant symbols. If ρ is a similarity type, then R_ρ is the set of relation symbols of ρ and C_ρ is the set of constant symbols of ρ.

For any set $A \neq \emptyset$, $|X_{\rho,A}|$ is the set

$$\prod_{\underline{R} \in R_\rho} A^{n(\underline{R})}2 \quad \times \quad C_{\rho_A}$$

and $X_{\rho,A}$ is the corresponding topological product space formed over the discrete topologies on 2 and A. If $S \in X_{\rho,A}$ then (A,S) is a ρ-<u>structure</u> <u>with</u> <u>universe</u> A. V_ρ is the class of all ρ-structures and X_ρ, the <u>canonical</u> <u>logic</u> <u>space</u> <u>of</u> <u>type</u> ρ, is $X_{\rho,\omega}$.

Since we will be interested in questions of effectiveness for $L_{\omega_1\omega}$, we adopt a standard set theoretic "arithmetization" of the language as follows:

The set of symbols $L_{\omega_1\omega}(\rho)$ contains each symbol in ρ, plus variables $\underline{v}_n = (3,(n,0))$ for each $n \in \omega$, and logical connectives $\underset{\sim}{\equiv} = (0,(\emptyset,2))$, $\neg = 4$, $\underset{\sim}{\bigvee} = 5$, $\underset{\sim}{\exists} = 6$, $\underset{\sim}{\bigwedge} = 7$, $\underset{\sim}{\forall} = 8$.

Atomic formulas are $\tau_1 \underset{\sim}{\equiv} \tau_2 = (9,(\equiv,(\tau_1,\tau_2)))$ and $\underline{R}(\tau_1,\ldots,\tau_{n(\underline{R})}) = (9,(\underline{R},(\tau_1,\ldots,\tau_{n(\underline{R})}))$ where each τ_i is a variable or constant symbol and $\underline{R}$ is a relation symbol. $\neg\phi$ is $(4,\phi)$, $\underset{\sim}{\bigvee}\Theta$ is $(5,\Theta)$, $\underset{\sim}{\bigwedge}\Theta$ is $(7,\Theta)$, $\underset{\sim}{\exists}v\phi$ is $(6,(v,\phi))$, $\underset{\sim}{\forall}v\phi$ is $(8,(v,\phi))$.

If ϕ is an atomic formula, then ϕ, $\neg\phi$ are <u>subbasic</u>' formulas. A <u>basic</u> ' formula is a conjunction $\underset{\sim}{\bigwedge}\Theta$ where Θ is a finite set of subbasic' formulas. An <u>open</u>' $(\underset{\sim}{\Sigma}'^0_1)$ formula is an arbitrary (possibly uncountable) disjunction $\underset{\sim}{\bigvee}\Theta$ where each $\theta \in \Theta$ is of the form $\underset{\sim}{\exists}v_1,\ldots,\underset{\sim}{\exists}v_n M$ where $v_1,\ldots,v_n$ are variables and M is basic'.

Here and below, disjunctions and conjunctions can only be formed which have finitely many free variables. The set $L_{\omega_1\omega}(\rho)$ of infinitary (first order) formulas of type ρ is the smallest set which includes the set of open' formulas and contains $\neg\phi$, $\underset{\sim}{\bigvee}\Theta$, $\underset{\sim}{\bigwedge}\Theta$, $\underset{\sim}{\exists} v\phi$, $\underset{\sim}{\forall} v\phi$ when it includes $\Theta \cup \{\phi\}$, v is a variable, and Θ is countable.

These definitions correspond to those in Vaught [46]. They differ from the usual definitions of $L_{\omega_1\omega}$ in allowing some uncountable disjunctions when ρ is uncountable. They have the virtue of being more natural from the topological standpoint. We obtain a hierarchy on $L_{\omega_1\omega}(\rho)$ which is analogous to the Borel hierarchy by defining the $\underset{\sim}{\Sigma}'^{o}_{1}$ formulas as above, and then recursively defining

$$\underset{\sim}{\Pi}'^{o}_{\alpha}(\rho) = \{\neg\phi : \phi \in \underset{\sim}{\Sigma}'^{o}_{\alpha}(\rho)\}$$

$$\underset{\sim}{\Pi}'^{o}_{(\alpha)}(\rho) = \bigcup\{\underset{\sim}{\Pi}'^{o}_{\beta}(\rho) : \beta < \alpha\}$$

$$\underset{\sim}{\Sigma}'^{o}_{\alpha}(\rho) = \{\underset{\sim}{\bigvee}\Theta : \Theta \text{ is countable and each } \theta \in \Theta \text{ is of the form}$$

$\underset{\sim}{\exists} v_1 \ldots \underset{\sim}{\exists} v_k \psi$ where $k \in \omega$, each v_i is a variable, and $\psi \in \underset{\sim}{\Pi}'^{o}_{(\alpha)}(\rho)\}$

A subset L of $L_{\omega_1\omega}(\rho)$ is a <u>fragment</u> if L contains every atomic formula and L is closed under subformulas, finite conjunctions and disjunctions, and negations.

The finitary predicate calculus $L_{\omega\omega}(\rho)$ is the fragment of $L_{\omega_1\omega}(\rho)$ obtained by restricting all disjunctions to be finite. The usual prefix hierarchy on $L_{\omega\omega}$ is defined by setting

$$\underset{\sim}{\forall}^{o}_{n}(\rho) = \underset{\sim}{\Pi}'^{o}_{n}(\rho) \cap L_{\omega\omega}(\rho)$$

and

$$\underset{\sim}{\exists}^{o}_{n}(\rho) = \underset{\sim}{\Sigma}'^{o}_{n}(\rho) \cap L_{\omega\omega}(\rho).$$

An n-<u>formula</u> is a formula with free variables among $\underline{v}_o, \ldots, \underline{v}_{n-1}$. A 0-formula is a <u>sentence</u>. A <u>propositional formula</u> is one with no variables at all. $\phi\binom{a}{b}$ is the result of replacing a by b in ϕ.

We will use standard abbreviations ($\phi \wedge \psi$ for $\underset{\sim}{\bigwedge}\{\phi,\psi\}$, $\phi \to \psi$ for $\neg(\phi \wedge \neg\psi)$, etc.) to simplify formal expressions.

$$(\underset{\approx}{\exists}!\underline{v}_i)(\phi) \quad \text{abbreviates} \quad (\underset{\approx}{\exists}\underline{v}_i)(\phi \wedge (\underset{\sim}{\forall}\underline{v}_j)(\phi(\tfrac{\underline{v}_i}{\underline{v}_j}) \to \underline{v}_i \approx \underline{v}_j))$$

where j is the smallest such that $\underline{v}_j$ does not occur in ϕ.

$$(\underset{\approx}{\exists}^{\neq}v_1,\ldots,v_n)(\phi) \quad \text{abbreviates} \quad (\underset{\approx}{\exists}v_1,\ldots,v_n)(\phi \wedge \bigwedge_{i<j\le n} v_i \not\approx v_j).$$

Given similarity types ρ and ρ_1, $\rho + \rho_1$ is the result of adding the symbols of ρ_1 to ρ. It is defined by setting

$$\rho_1^{\rho} = \{(i,((\rho,a),n): \quad (i,(a,n)) \in \rho_1\}$$

and then defining $\rho+\rho_1 = \rho \cup \rho_1^{\rho}$.

A $\underset{\sim}{\Sigma}'^1_n$ (2nd order) <u>formula of type</u> ρ is an expression of the form

$$\underset{\approx}{\exists}\rho_1\underset{\sim}{\forall}\rho_2\underset{\approx}{\exists}\rho_3\cdots(\underset{\approx}{\exists} \text{ or } \underset{\sim}{\forall})\rho_n\phi$$

where $\rho_1,\ldots,\rho_n$ are countable similarity types and $\phi \in L_{\omega_1\omega}(\rho+\rho_1+\ldots+\rho_n)$.

In Vaught [44] $\underset{\sim}{\Sigma}'^1_n$ and $\underset{\sim}{\Pi}'^1_n$ were denoted as $\underset{\approx}{\exists}^1_n$ and $\underset{\sim}{\forall}^1_n$ respectively. We will use the expressions $\underset{\sim}{\forall}^1_n$, $\underset{\approx}{\exists}^1_n$ to denote the corresponding classes of <u>finitary</u> 2nd-order formulas obtained by restricting ϕ to belong to $L_{\omega\omega}(\rho+\rho_1+\ldots+\rho_n)$ in the definition of $\underset{\sim}{\Pi}'^1_n(\rho)$, $\underset{\sim}{\Sigma}'^1_n(\rho)$.

For $K \subseteq V_\rho$, $K^{(n)} \subseteq V_{\rho \cup \{\underline{0},\ldots,\underline{n-1}\}}$ is defined by the equation $K^{(n)} = \{(A,S,a_0,\ldots,a_{n-1}): a_0,\ldots,a_{n-1} \in A \ \& \ (A,S) \in K\}$. If ϕ is an n-formula, (A,S) is a structure, and $a_0,\ldots,a_{n-1}$ are elements of A which satisfy ϕ in (A,S) (in the obvious sense), we say $(A,S,a_0,\ldots,a_{n-1})$ is a model of ϕ and write $(A,S,a_0,\ldots,a_{n-1}) \in \mathrm{Mod}^{(n)}(\phi)$ or $(A,S,a_0,\ldots,a_{n-1}) \models \phi$. When $n = 0$ we drop the superscript. $\mathrm{Mod}^K(\phi) = \mathrm{Mod}(\phi) \cap K$. If Γ is a collection of ρ-formulas and $K \subseteq V_\rho$, we define $\Gamma(K^{(n)}) = \{\mathrm{Mod}^{(n)}(\phi) \cap K^{(n)}: \phi \text{ is an n-formula in } \Gamma\}$. We say K is $\underset{\sim}{\Sigma}'^0_\alpha$, $\underset{\sim}{\Sigma}'^1_n$, etc., if $K \in \underset{\sim}{\Sigma}'^0_\alpha(V_\rho)$, $\underset{\sim}{\Sigma}'^1_n(V_\rho)$, etc..

$\mathcal{J}_\rho = (X_\rho, \omega!, J_\rho)$, <u>the canonical logic action of type</u> ρ, is defined as follows:

$\omega!$ is the group of permutations of ω given the relative topology as a subspace of ω^ω.

For $g \in \omega!$, $S \in X_\rho$, $gS = J_\rho(g,S)$ is the usual isomorph of S under g. Thus, for each $\underline{R} \in \mathcal{R}$ and each $\underline{c} \in \mathcal{C}_\rho$

$$(gS)_{\underline{R}}(i_1,\ldots,i_{n(\underline{R})}) = S_{\underline{R}}(g^{-1}(i_1),\ldots,g^{-1}(i_{n(\underline{R})}))$$

$$(gS)_{\underline{c}} = g(S_{\underline{c}}).$$

It is easily seen that $\mathcal{J}_\rho$ is a Polish action whenever ρ is countable. $I_\rho = E_{\mathcal{J}_\rho}$ is the usual isomorphism relation on X_ρ.

Let $\rho_N = \{\underline{n}: n \in \omega\}$. If ρ is disjoint from ρ_N, then Borel and projective subsets of X_ρ are naturally described by propositional formulas of type $\rho \cup \rho_N$ as follows. An <u>atomic name</u> is an expression $\underline{R}(\underline{i}_1,\ldots,\underline{i}_{n(\underline{R})})$ or $\underline{c} = \underline{i}$ where $\underline{R},\underline{c} \in \rho$ and $\underline{i},\underline{i}_1,\ldots,\underline{i}_{n(\underline{R})}) \in \rho_N$. ϕ is a ρ-<u>name</u> if ϕ is propositional and every atomic subformula of ϕ is an atomic name. If $\phi \in L_{\omega_1\omega}(\rho \cup \rho_N)$, $\utilde{\Sigma}'^{0}_{\alpha}(\rho \cup \rho_N)$, $\utilde{\Sigma}'^{1}_{n}(\rho \cup \rho_N)$, etc., we say ϕ is a <u>Borel</u> ρ-<u>name</u>, $\utilde{\Sigma}^0_\alpha$-ρ-<u>name</u>, $\utilde{\Sigma}^1_n$-ρ-<u>name</u>, etc.. ϕ <u>names</u> the set $[\phi] = \{S: (\omega,S,0,1,\ldots) \models \phi\}$. Clearly, $B \subseteq X_\rho$ is Borel, $\utilde{\Sigma}^0_\alpha$, $\utilde{\Sigma}^1_n$, etc. if and only if B has a Borel-name, $\utilde{\Sigma}^0_\alpha$-name, $\utilde{\Sigma}^1_n$-name, etc.

<u>We assume throughout the dissertation that ρ_N is disjoint from all other similarity types which are mentioned</u>.

Subsets of X_ρ are also defined by arbitrary sentences of type ρ. Thus, if θ is a (first or second order) n-formula of type ρ, we identify $S \in X_\rho$ with (ω,S) and set $[\![\theta^{(n)}]\!] = \mathrm{Mod}^{(n)}(\theta) \cap X^{(n)}_\rho$. It is apparent that $[\![\theta^{(n)}]\!]$ is invariant under the canonical equivalence on $X^{(n)}_\rho = X_\rho \times \omega^n$. It is also apparent that $[\![\theta^{(n)}]\!]$ is $\utilde{\Sigma}^0_\alpha$, Borel,

$\underset{\sim}{\Sigma}^1_n$, etc. when θ is $\underset{\sim}{\Sigma}'^0_\alpha$, $L_{\omega_1\omega}$, $\underset{\sim}{\Sigma}'^1_n$, etc. (A $\underset{\sim}{\Sigma}^0_\alpha$-name etc. for $[\![\theta^{(n)}]\!]$ may be obtained by inductively replacing subformulas of the form $\underset{\sim}{\exists} v\phi$ by disjunctions $\underset{\sim}{\bigvee}\{\phi(\frac{v}{\underline{i}}): i \in \omega\})$. Thus, B is I_ρ-invariant $\underset{\sim}{\Sigma}^0_\alpha$, $\underset{\sim}{\Sigma}^1_n$, etc. whenever B is $\underset{\sim}{\Sigma}'^0_\alpha$, $\underset{\sim}{\Sigma}'^1_n$, etc.

For each of the Borel and projective classes the converse of the above holds and we have the identities "$\underset{\sim}{\Sigma}'^1_n(X_\rho)$ = invariant $\underset{\sim}{\Sigma}^1_n(X_\rho)$," "$L_{\omega_1\omega}(X_\rho)$ = invariant $\mathcal{B}(X_\rho)$," "$\underset{\sim}{\Sigma}'^0_\alpha(X_\rho)$ = invariant $\underset{\sim}{\Sigma}^0_\alpha(X_\rho)$." If θ is a $\underset{\sim}{\Sigma}^1_n$-$\rho$-name, then the equation

$$[\theta]^{+I_\rho} = [\![(\underset{\sim}{\exists}\rho_N)(\theta \wedge (\underset{\sim}{\forall}\underline{v}_o)(\underset{\sim}{\bigvee}\{\underline{v}_o = \underline{i}: \ i \in \omega\}) \wedge \underset{\sim}{\bigwedge}\{\underline{i} \neq \underline{j}: \ i < j < \omega\})]\!]$$

indicates that $[\theta]^+$ is $\underset{\sim}{\Sigma}'^1_n$. Since $B = B^+$ when B is invariant, the first identity follows. The second, for countable ρ, follows from the first and the Lopez-Escobar interpolation theorem: $\underset{\sim}{\Sigma}'^1_1(V_\rho) \cap \underset{\sim}{\Pi}'^1_1(V_\rho) = L_{\omega_1\omega}(V_\rho)$. The third identity is a recent result due to Vaught ([46]). It refines the second and extends it to arbitrary ρ. In chapter III we will add to the list of identities by proving "countable alternated union of $\underset{\sim}{\Pi}'^0_\alpha$ sets = invariant countable alternated union of $\underset{\sim}{\Pi}^0_\alpha$ sets." These sets coincide with the invariant

$\underset{\sim}{\Delta}^{0}_{\alpha+1}$ sets when ρ is countable.

Effectiveness

In chapter I and chapter III we will prove "effective" versions of topological and model theoretic results. For us it will be most convenient to formalize the concept of "effectiveness" in terms of hereditarily countable sets, admissible sets and primitive recursive set functions.

The canonical reference for primitive recursive set functions is Jenson-Karp [20]. We recall the basic definitions: A set function is primitive recursive (prim) if it can be obtained from the initial functions by substitution and recursion as follows:

Initial functions:

(i) $P_{n,i}(x_1,\ldots,x_n) = x_i$; $1 \leq n < \omega$, $1 \leq i \leq n$

(ii) $F(x,y) = x \cup \{y\}$

(iii) $C(x,y,u,v) = x$ if $u \in v$, y otherwise

Substitution:

(a) $F(\check{v},\check{w}) = G(\check{v},H(\check{v}),\check{w})$; $\check{v} = (v_1,\ldots,v_m)$, $\check{w} = (w_1,\ldots,w_n)$

(b) $F(\check{v},\check{w}) = G(H(\check{v}),\check{w})$

Recursion:

$F(w,\check{v}) = G(\bigcup\{F(u,\check{v}): u \in w\},w,\check{v})$

If $x_1,\ldots,x_k$ are sets, a function F is prim$(x_1,\ldots,x_k)$ if there exists a prim function G such that for all $v_1,\ldots,v_m$, $F(v_1,\ldots,v_m) = G(v_1,\ldots,v_m,x_1,\ldots,x_k)$. y is prim$(x_1,\ldots,x_k)$ if the constant function $F(v) = y$ is prim$(x_1,\ldots,x_k)$. It is apparent that y is prim$(x_1,\ldots,x_k)$ if and only if $y = G(x_1,\ldots,x_k)$ for some prim function G. The set of all such y is the prim-closure of $(x_1,\ldots,x_k)$.

A set A is transitive if $y \in x \in A$ implies $y \in A$. Given a set x, the transitive closure, $Tc(x)$, of x is the smallest transitive set A such that $x \in A$. x is hereditarily countable ($x \in HC$) if $\overline{\overline{Tc(x)}} \leq \omega$.

As an example, note that the definition $TC(y) = y \cup \bigcup\{TC(z): z \in y\}$ shows that the function $F(y) = TC(y)$ is primitive recursive.

If ρ is hereditarily countable then $X_\rho \subseteq HC$ and all of the first and second order formulas of type ρ are hereditarily countable. Moreover, all of the syntactical notions ("θ is a formula," "ϕ is a subformula of ψ," etc.) we will use are easily seen to be set theoretically primitive recursive (cf. Cutland [14] or Barwise [7]). In particular, the function $\theta \mapsto \theta^N$ which maps ρ-formulas to ρ-names by replacing variables by special constants is prim(ρ) for $\rho \in HC$. -- Consider, for illustration the case $\rho = \{\underline{\varepsilon}\}$ where $\underline{\varepsilon}$ is binary. Then $\theta \mapsto \theta^N$ is defined by the recursive conditions:

$$(\underline{\varepsilon}(\underline{v}_i, \underline{v}_j))^N = \underline{\varepsilon}(\underline{i}, \underline{j})$$
$$(\neg \phi)^N = \neg(\phi^N), \quad (\underset{\sim}{\bigvee}\Theta)^N = \underset{\sim}{\bigvee}\{\theta^N: \theta \in \Theta\}, \quad \text{dually for } \underset{\sim}{\bigwedge}$$
$$(\underset{\sim}{\exists}\underline{v}_i\phi)^N = \bigvee\{\phi^N(\tfrac{\underline{i}}{\underline{j}}): j \in \omega\}, \quad \text{dually for } \underset{\sim}{\forall}$$
$$(\underset{\sim}{\exists}\rho_1\phi)^N = \underset{\sim}{\exists}\rho_1\phi^N, \quad \text{dually for } \underset{\sim}{\forall}$$

The language of set theory is $L_{\omega\omega}(\{\underline{\varepsilon}\})$ where $\underline{\varepsilon}$ is binary. $\phi \in L_{\omega\omega}(\{\underline{\varepsilon}\})$ is Δ_o if every quantification in ϕ is restricted; that is, of the form $(\underset{\sim}{\forall} v)(v\underline{\varepsilon}w \rightarrow \ldots)$ or of the form $(\underset{\sim}{\exists} v)(v\underline{\varepsilon}w \wedge \ldots)$.

The set of Σ-formulas is the smallest which includes the Δ_o formulas and is closed under finite conjunctions and disjunctions, restricted universal quanification and arbitrary existential quantification.

A set A is admissible if A is transitive, prim-closed and satisfies the $\underset{\sim}{\Sigma}$-reflection principle:

If θ *is* Σ, $a_1,\ldots,a_n \in A$ *and* $(A,\varepsilon,a_1,\ldots,a_n) \models \theta$, *then for some transitive* $b \in A$, $a_1,\ldots,a_n \in b$ *and* $(b,\varepsilon,a_1,\ldots,a_n) \models \theta$.

A subset X of A is $\underset{\sim}{\Sigma}$-definable on A ($X \in \underset{\sim}{\Sigma}(A)$) if for some $n \in \omega$, some $\check{b} \in A^n$, and some $n+1$-formula $\phi \in \Sigma$, $X = \{a: (A,\varepsilon,\check{b},a) \models \phi\}$. X is $\underset{\sim}{\Delta}$-definable ($X \in \underset{\sim}{\Delta}(A)$) if both X and $A \sim X$ are $\underset{\sim}{\Sigma}$-definable.

All the facts we require about admissible sets may be found in [6] or [23].

For $\mathcal{A} \subseteq HC$, $\rho \in \mathcal{A}$, "Γ" = "$\underset{\sim}{\Sigma}^0_\alpha$", "$\underset{\sim}{\Sigma}^1_n$", etc., we define $\Gamma[\mathcal{A}](X_\rho) = \{[\phi]: \phi \in \mathcal{A}$ and ϕ is a Γ-ρ-name$\}$.

In I §2, we will require some standard results about the conventional "lightface" classes Σ^1_n, Π^1_n as found e.g. in [39]. It is known that if ρ is finite, $x \in X_\rho$, and $A = A_x$, the smallest admissible set containing x, then our classes $\underset{\sim}{\Sigma}^0_\alpha[A](X_\rho)$, $\underset{\sim}{\Sigma}^1_n[A](X_\rho)$ coincide with the lightface classes $\Sigma^0_\alpha(\text{Hyp } x)$, $\Sigma^1_n(x)$. Thus, the approach via prim-closed and admissible sets, subsumes and refines the lightface approach for our purposes.

The only results connecting "lightface" with "prim-closed" which are required for our arguments are the following obvious facts:

(i) $\Pi^1_1(2^{\omega\times\omega}) \subseteq \underset{\sim}{\Pi}^1_1[\mathcal{A}_\omega](2^{\omega\times\omega})$ where $\mathcal{A}_\omega$ is the prim-closure of $\{\omega\}$.

(ii) $\underset{\sim}{\forall}^1_1(2^{\omega\times\omega}) \subseteq \Pi^1_1(2^{\omega\times\omega})$.

Numbered Items

Certain statements in the body of the dissertation will be numbered. To assist the reader we have assigned these numbers in logical order rather than in the order of appearance in the text.

CHAPTER I: PROJECTIVE EQUIVALENCE RELATIONS AND INVARIANT PREWELLORDERINGS

Moschovakis [35] and Vaught [44] established invariant $\underset{\sim}{\Pi}^1_1$ and $\underset{\sim}{\Sigma}^1_1$ reduction theorems (and, implicitly, corresponding invariant prewellordering theorems) for canonical logic actions. In [46] Vaught did the same for arbitrary Polish actions. The main subject of this chapter (§1 and §2) is a proof that these results hold for arbitrary $\underset{\sim}{\Sigma}^1_1$ equivalence relations on arbitrary Suslin spaces. For suitable spaces X the results are obtained in a very effective "primitive recursive" version, (Theorem 2.1) -- as were the effective theorems of [44]. Assuming projective determinacy, (PD), the same arguments, (which are based on the ordinary Π^1_1 prewellordering theorem), extend without alteration to yield invariant reduction theorems for $\underset{\sim}{\Pi}^1_{2n+1}$ and $\underset{\sim}{\Sigma}^1_{2n+2}$, all $n \in \omega$.

The present chapter is the latest version of work begun jointly with John Burgess in [11]. That paper contained proofs of the invariant $\underset{\sim}{\Sigma}^1_2$ prewellordering and reduction theorems (due to the author) and of the invariant $\underset{\sim}{\Pi}^1_1$ reduction theorem for pairs (due to Burgess). At that time we conjectured the full invariant $\underset{\sim}{\Pi}^1_1$ prewellordering theorem, but could prove only a very special case of the theorem for $\underset{\sim}{\Pi}^1_{2n+1}$ ($n \geq 1$) assuming PD ([11] 4.2). The conjecture (and, hence, full invariant $\underset{\sim}{\Pi}^1_1$ reduction) was established by R. Solovay based on an idea of the author (see our remark III, p.40 for more details). This argument forms a central part of the proof of 2.1 below. Solovay's proof is sufficient to establish a corresponding "lightface" theorem -- as is a second argument (for invariant $\underset{\sim}{\Pi}^1_1$ prewellordering) due to Burgess [10]. The additional argument of §2 which establishes the stronger "primitive recursive" part of 2.1 is new. It is closely related to the methods of Vaught [44].

§ 3 contains a discussion of the invariant uniformization principle. Assuming $V = L$ we show that the principle holds for $\underset{\sim}{\Sigma}^1_n$ $(n \geq 2)$. We prove a general result on counterexamples which is related to previous work of Dale Myers.

INVARIANT DESCRIPTIVE SET THEORY AND

THE TOPOLOGICAL APPROACH TO MODEL THEORY

by

Douglas Edward Miller

Abstract

We study various types of topological spaces with equivalence relations ("topological equivalence spaces") which arise in connection with model theory and we apply topological results and methods to the study of languages and structures.

Most of our model theoretic applications derive from consideration of the natural topological space formed by the set of countable structures of any fixed countable similarity type. Given a similarity type ρ, for illustration consisting of a single binary relation, we identify the structure (ω,R) with the characteristic function of R and form the usual topological product space $X_\rho = 2^{\omega\times\omega}$.

Logic deals primarily with sets $B \subseteq X_\rho$ which are closed under isomorphism, i.e. invariant under the equivalence relation $I = \{(R,S)\colon (\omega,R) \cong (\omega,S)\}$. While the study of the topological equivalence space (X_ρ, I) in connection with model theory dates from the thirties, the subject has received increased attention since the intensive study of the language $L_{\omega_1\omega}$ was commenced in the early sixties. One indication of the close connection between the topological equivalence space (X_ρ, I) and infinitary logic is given by Lopez-Escobar's result: $B \subseteq X_\rho$ _is an I-invariant Borel set if and only if_ $B = X_\rho \cap \mathrm{Mod}(\sigma)$ _for some sentence_ $\sigma \in L_{\omega_1\omega}(\rho)$.

Salient features of the space (X_ρ, I) include the following: (i) (X_ρ, I) is a Polish (separable, completely metrizable) topological space and I is a $\undertilde{\Sigma}^1_1$ (analytic) subset of $X_\rho^2 = X_\rho \times X_\rho$; (ii) I is induced by a "Polish action", i.e., a continuous action on the Polish space X_ρ by a Polish topological group, viz. $\omega!$, the group of permutations of the natural numbers given the relative topology from the Baire space ω^ω.

In the first two chapters of this work we study in turn the spaces which satisfy each of these hypotheses. In chapter III we apply some of the material from chapter II together with some additional results proved just for the logic spaces to obtain new facts in model theory.

In chapter I we study spaces (X,E) such that X is a Polish space and the equivalence relation E is a $\undertilde{\Sigma}^1_1$ (analytic) subset of X^2. We introduce an invariant version of the prewellordering property and apply it to prove that the collections of E-invariant $\undertilde{\Sigma}^1_2$ (PCA) and $\undertilde{\Pi}^1_1$ (CA) sets have the reduction property. Assuming projective determinacy, these results are extended to $\undertilde{\Sigma}^1_{2n+2}$ and $\undertilde{\Pi}^1_{2n+1}$ for all $n \in \omega$. An invariant uniformization principle is also considered and shown to follow for $\undertilde{\Sigma}^1_n$, $n \geq 2$ from the axiom of constructibility. With suitable restrictions on X, effective versions of all the results are obtained. These effective results are proved in a "set theoretically primitive recursive" context which has a wider applicability than the traditional "lightface descriptive set theory."

The invariant $\undertilde{\Pi}^1_1$ prewellordering and reduction theorem is due to Solovay based on a conjecture of the author. The results on reduction extend theorems of Vaught and Moschovachis, who proved invariant $\underset{\sim}{\Pi}^1_1$ reduction for the spaces (X_ρ, I), cf. Vaught [44]; Vaught, who proved invariant $\underset{\sim}{\Pi}^1_1$ and $\underset{\sim}{\Sigma}^1_2$ reduction for Polish actions, see Vaught [46] and Burgess, who proved invariant reduction for pairs of $\underset{\sim}{\Pi}^1_1$ sets under the hypotheses of chapter I, see Burgess-Miller [11]. The result on uniformization was obtained jointly with Burgess and appeared in [11] as did much of the material of chapter I. It extends unpublished work of Kuratowski.

Chapter II deals with Polish actions and, more generally, with spaces (X,G,J) such that G is a nonmeager topological group with a countable basis, X is a topological space, and $J\colon G \times X \to X$ is a Borel map which defines an action of G on X. The main tool and subject of interest in this chapter is the *-transform, $B \mapsto B^* = \{x\colon \{g\colon J(g,x) \in B\}$ is comeager$\}$, which was introduced in Vaught [45]. We contribute both to the basic theory of the transform and to the list of applications of the transform to the theory of group actions. Perhaps the most important result from this chapter is the invariant version of the well-known strong $\underset{\sim}{\Pi}^0_\alpha$ separation theorem of Hausdorff and Kuratowski (cf. Kuratowski [26] or Addison [3]). $\underset{\sim}{\Pi}^0_\alpha(X)$ is the αth multiplicative level of the Borel hierarchy on X (so $\underset{\sim}{\Pi}^0_2 = G_\delta$, $\underset{\sim}{\Pi}^0_3 = F_{\sigma\delta}$, etc.). We prove II.4.3: _If_ J _is continuous in each variable,_ X _is Polish,_ E_J _is the equivalence relation_ $\{(x,y)\colon (\exists g)(J(g,x) = y)\}$ _and_ $1 \leq \alpha < \omega$, _then disjoint_ E_J_-invariant_ $\underset{\sim}{\Pi}^0_{\alpha+1}$ _sets can be separated by a countable alternated union of_ E_J_-invariant_ $\underset{\sim}{\Pi}^0_\alpha$ _sets, a fortiori by an invariant_ $\underset{\sim}{\Delta}^0_{\alpha+1}$ _set._ For $\alpha = 1$ the result is proved for the much wider class of equivalence spaces (X,E) such that E is lower semi-continuous (open).

In chapter III we apply topological results and methods to model theory, obtaining several new results and giving new proofs of several known theorems. In the latter case, the topological proofs are generally shorter than previously known arguments. Moreover, they make explicit a causal connection between classical topological theorems and their model theoretic analogues.

Topics discussed in this chapter include a $\underset{\sim}{\Pi}{}'^{0}_{\alpha}$ separation theorem, a recent global definability theorem of M. Makkai, a generalization of a result on the definability of invariant Borel functions due to Lopez-Escobar and a result on continuous selectors for elementary equivalence. Our treatment of the $\underset{\sim}{\Pi}{}'^{0}_{\alpha}$ separation theorem illustrates most of the types of results proved in the chapter. $\underset{\sim}{\Pi}{}'^{0}_{\alpha}$ denotes the αth level of the natural hierarchy on formulas of $L_{\omega_1\omega}$ in which, for example, $\underset{\sim}{\Pi}{}'^{0}_{2}$ classes have the form $\mathrm{Mod}(\bigwedge_{n} \underset{\sim}{\forall} \check{x} \bigvee_{m} \underset{\sim}{\exists} \check{y}\, \theta_{nm}(\check{x}\check{y}))$ with each θ_{nm} finitary, quantifier-free, <u>cf</u>. Vaught [46]. The basic $\underset{\sim}{\Pi}{}'^{0}_{\alpha}$ separation theorem is an unpublished result of Reyes. We give two new proofs of the basic theorem: <u>Disjoint</u> $\underset{\sim}{\Pi}{}'^{0}_{\alpha+1}$ <u>classes</u> $(1 \leq \alpha)$ <u>can be separated by a countable alternated union of</u> $\underset{\sim}{\Pi}{}'^{0}_{\alpha}$ <u>classes</u>. One proof is based on II.4.3; the second is a model theoretic translation of the classical topological proof. We use some ideas of D. A. Martin to obtain an admissible version of the theorem and we apply an approximation theorem of J. Keisler to obtain the well-known finitary version (Shoenfield's $\underset{\sim}{\forall}^{0}_{n}$ interpolation theorem). We also apply the result to study the complexity of $L_{\omega_1\omega}$ definitions of isomorphism types. We prove several theorems. The following is typical:

II.4.2: <u>If a complete</u> $L_{\omega\omega}$ <u>theory</u> T <u>has a countable model</u> $\mathfrak{A}$ <u>such that the isomorphism type of</u> $\mathfrak{A}$ <u>is</u> $\underset{\sim}{\Sigma}'^{0}_{2}$-over-$L_{\omega\omega}$, <u>then</u> T <u>is</u> ω-<u>categorical</u>. This extends a theorem of M. Benda. $\underset{\sim}{\Sigma}'^{0}_{2}$-over-$L_{\omega\omega}$ classes have the form $\mathrm{Mod}(\bigvee_{n} \underset{\sim}{\exists} \check{x} \bigwedge_{m} \underset{\sim}{\forall} \check{y} \theta_{nm}(\check{x},\check{y}))$, each $\theta_{nm} \in L_{\omega\omega}$.

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My interest in descriptive set theory developed from a course given by Ashok Maitra. I first learned that logic is a branch of general topology from Dana Scott.

Special thanks are due to Robert Solovay for permission to include a result of his in Chapter I, and to John Burgess for permission to include several results which we obtained jointly. These results also appear in Chapter I.

I owe an extraordinary debt of gratitude to my advisor, Professor Robert L. Vaught. The importance of Professor Vaught's ideas to the material of this dissertation will be obvious to any reader; the influence of Professor Vaught on my mathematical education cannot be overestimated. He has freely given encouragment, advice, and criticism throughout my years as a graduate student.

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CHAPTER 0: INTRODUCTION

A brief summary of this work may be found in the preceding Abstract.

Descriptive set theory deals with Borel and projective sets in metrizable spaces and especially in the spaces ω^ω, 2^ω and other "Polish" (separable, completely metrizable) spaces. This "classical" work goes back to Lebesgue, Lusin, Suslin, Sierpinski and others. The best known reference is Kuratowski [26]. The first application of descriptive set theory to logic was made by Kuratowski in 1933 in [25] where he defined the infinitary language $L_{\omega_1\omega}$ and showed that the collection of well-orderings is not an $L_{\omega_1\omega}$-elementary class. Since that time and especially during the last fifteen years, the connections between classical descriptive set theory and model theory have been studied by many authors. See, for example, Addison [2-4], Scott [38], Lopez-Escobar [28], Grzegorczyk, et. al., [16], Morley [33] and Vaught[44-46].

Some authors, notably Addison and his students, have expounded the analogies which exist between the classical theory of Polish spaces and the model theory of the finitary predicate calculus, $L_{\omega\omega}$. Other authors, such as Lopez-Escobar in [28] and Vaught in [44] have found similar analogies with the model theory of $L_{\omega_1\omega}$. In fact, topological considerations were an important factor leading Scott and Ryll-Nardzewski to propose $L_{\omega_1\omega}$ as the "natural" infinitary first-order language in the early sixties (cf. Scott [38]).

Most recently, Vaught in [45] and [46] introduced a powerful method to show that many infinitary results can be derived from their classical counterparts. As we shall remark in III §2 below, the corresponding finitary theorems often follow by an approximation theorem of J. Keisler. Moreover, the theorems in logic are obtained as special cases of theorems about Polish actions or certain other kinds, of action spaces.

This is the type of result with which we will be primarily concerned. It accomplishes several things. First, it gives a "causal" explanation for the analogies found by previous authors. Second, by showing that certain results in model theory are special cases of general results on equivalence spaces or action spaces, it enables us to compare the restrictiveness of the hypotheses under which the general results are obtained. On the basis of this comparison, we can then classify some results as more "model theoretic" than others. Finally, it leads us to new theorems about actions or equivalence spaces as generalizations of results in model theory and to new theorems in model theory as invariant versions of classical theorems of descriptive set theory.

For a summary of the topics to be considered in this work we refer the reader to the abstract and to the table of contents. In the remainder of the introductory chapter we will establish some notational conventions and review some of the basic definitions which we will require.

Sets and Topological Spaces

ON is the collection of ordinals, each ordinal being the set of preceding ordinals. Cardinals are initial ordinals, ω and ω_1 are the first two infinite cardinals. α, β, γ will always be ordinals, λ will always be a limit ordinal, κ will always be a cardinal. $\overline{\overline{B}}$ is the cardinality of B. $P(A) = \{B: B \subseteq A\}$ and $P_{(\kappa)}(A) = \{B \in P(A): \overline{\overline{B}} < \kappa\}$. $(a,b) = \{\{a\},\{a,b\}\}$ and $\langle B_i: i \in I\rangle = \{(i,B_i): i \in I\}$. ${}^{B}A$ is the set of functions on B to A.

A <u>topological space</u> $X = (|X|,T)$ is a pair such that $|X|$ is a set and $T \subseteq P(|X|)$ contains $\emptyset$ and $|X|$ and is closed under finite intersections and arbitrary unions. T is the collection of <u>open</u> subsets of X. $B(X)$, the collection of <u>Borel</u> subsets of X is the smallest collection which includes T and is closed under complementation and countable unions. The Borel hierarchy on X is defined recursively by the conditions

$$\utilde{\Sigma}^0_1(X) = T$$
$$\utilde{\Pi}^0_\alpha(X) = \{-A: A \in \utilde{\Sigma}^0_\alpha(X)\}$$
$$\utilde{\Pi}^0_{(\alpha)}(X) = \bigcup \{\utilde{\Pi}^0_\beta(X): \beta < \alpha\}$$
$$\utilde{\Sigma}^0_\alpha(X) = \{\bigcup \Phi: \Phi \in P_{(\omega_1)}(\utilde{\Pi}^0_{(\alpha)}(X))\}$$

$\utilde{\Pi}^0_2, \utilde{\Pi}^0_3$, etc. were classically known as G_δ, $F_{\sigma\delta}$, etc. $\utilde{\Delta}^0_\alpha(X) = \utilde{\Pi}^0_\alpha(X) \cap \utilde{\Sigma}^0_\alpha(X)$.

A function f on X to a topological space Y is Borel measurable, (respectively α-Borel), if $f^{-1}(A) \in B(X)$, $(\utilde{\Sigma}^0_\alpha(X))$, whenever $A \in \utilde{\Sigma}^0_1(Y)$. f is a Borel isomorphism, ((α,β)-generalized homeomorphism), if f is 1-1, onto, and both f and f^{-1} are Borel, (f is α-Borel, f^{-1} is β-Borel).

<u>Note</u>. Our $1+\alpha$-Borel maps are "measurable at level α" in the terminology of Kuratowski. That is because $\utilde{\Sigma}^0_1$ is the 0th additive level of his hierarchy.

ω^ω and 2^ω are the topological spaces formed on the sets ${}^\omega\omega$, ${}^\omega 2$ by taking the product topology over the discrete spaces ω and 2.

If I is any index set and G is a function on ${}^I P(X)$ to $P(X)$, we say G is an I-Boolean operation provided $s^{-1}(G(\langle A_i : i \in I\rangle) = G(\langle s^{-1}(A_i) : i \in I\rangle)$ whenever s is a function on X to X. Given a Boolean operation G and $\Gamma \subseteq P(X)$ we define $G(\Gamma) = \{G(A) : A \in {}^I\Gamma\}$ Clearly, each class $G(\underset{\sim}{\Sigma}^0_1(X))$ is closed under inverse continuous images. Working with W. Wadge's theory of reducibility by continuous functions, R. Steele and R. Van Wesep have recently showed that in a certain natural sense, for "almost all" $\Gamma \subseteq B(2^\omega)$ such that Γ closed under inverse continuous images, Γ has the form $\Gamma = G(\underset{\sim}{\Sigma}^0_1(X))$ where G is a ω-Boolean operation.

$A \subseteq X$ is <u>nowhere dense</u> if the closure of A includes no non-empty open subset. A is <u>meager</u> (of <u>first category</u>) if A is a countable union of nowhere dense sets. A is <u>almost open</u> (has the <u>Baire property</u>) if there exists an open set O such that the symmetric difference $O \Delta A$ is meager. X is a <u>Baire space</u> if no non-empty open subset of X is meager. X is <u>separable</u> if X includes a countable dense subset.

X is a <u>Polish space</u> if X is separable and admits a complete metrization. X is a <u>Suslin</u> (<u>analytic</u>) <u>space</u> if X is a metrizable continuous image of some Polish space. X is a <u>Lusin</u> (<u>absolutely Borel</u>) <u>space</u> if X is a metrizable, continuous one-one image of some Polish space.

Given a product space $X \times Y$ let $\pi_1 : X \times Y \to X$ be the projection mapping $(x,y) \mapsto x$. For any X, the projective hierarchy on X is defined by setting

$$\Sigma^1_1(X) = \{\pi_1(A): A \in \mathcal{B}(X \times Y) \text{ for some Polish } Y\}$$
$$\Pi^1_n(X) = \{\sim A: A \in \Sigma^1_n(X)\}$$
$$\Sigma^1_{n+1}(X) = \{\pi_1(A): A \in \Pi^1_n(X \times Y) \text{ for some Polish } Y\}$$
$$\Delta^1_n(X) = \Sigma^1_n(X) \cap \Pi^1_n(X).$$

Σ^1_1, Π^1_1, and Σ^1_2 subsets of Polish spaces were classically known as analytic, coanalytic, and PCA sets respectively.

There are many useful normal form results for Polish, Lusin and Suslin spaces. The following is a partial list (cf. Kuratowski [26]): Every Lusin space is a continuous one-one image of a closed subset of ω^ω and of a Π^0_2 subset of 2^ω. Every uncountable zero-dimensional Polish space is a union of a countable set and a set homeomorphic to ω^ω. All uncountable Lusin spaces are Borel isomorphic and every metrizable space which is Borel isomorphic to a Lusin space is Lusin. Every Π^0_2 subspace of a Polish space is Polish. If a subspace of a Polish space is Polish then it is Π^0_2. Every Borel subspace of a Lusin space is Lusin. If a subspace of a Lusin space is Lusin, then it is Borel. Every Σ^1_1 subset of a Suslin space is Suslin. If a subspace of a Suslin space is Suslin, it is Σ^1_1. A subset A of a Polish space X is Suslin if and only if it can be obtained by the operation (A) applied to Borel sets; that is, $A = \bigcup_{\xi \in {}^\omega\omega} \bigcap_n B_{\xi \restriction n}$ for some collection $\{B_s: s \in S_q\} \subseteq \mathcal{B}(X)$. $S_q = \bigcup\{{}^n\omega: n \in \omega\}$ is the set of finite sequences of natural numbers. One important effect of these normal forms is to reduce questions about Polish, Lusin, and Suslin spaces to questions about Borel and projective subsets of ω^ω or 2^ω.

Actions and Equivalence Relations

A topological group $G = (|G|, \mathcal{T}, \circ)$ is a triple such that

$(|G|,\mathcal{T})$ is a topological space, $(|G|,\circ)$ is a group, and the function $(g,h) \mapsto g \circ h^{-1}$ is a continuous map on the product space $G \times G$ to G. G is a Polish group, Lusin group, etc. if the space $(|G|,\mathcal{T})$ is Polish, Lusin, etc.

Given $J: G \times X \to X'$, we define $J^g: X \to X'$ for $g \in G$ and $J^x: G \to X'$ for $x \in X$ by the condition $J^g(x) = J^x(g) = J(g,x)$. If G is a group and the map $g \mapsto J^g$ is a homomorphism on G to the group of permutations of X, then $\mathcal{J} = (X,G,J)$ is an action. If X is a Polish space, G is a Polish group, and J is continuous, then $\mathcal{J}$ is a Polish action. When no confusion will arise, we write gh for $g \circ h$ and gx for $J(g,x)$. Given an action of G on X, we obtain an action of G on $\mathcal{P}(X)$ by setting $gA = \{gx: x \in A\}$.

Every action $\mathcal{J} = (X,G,J)$ induces an equivalence relation $E_{\mathcal{J}} = \{(x,y): (\exists g \in G)(gx = y)\}$. If X and G are Suslin and J is Borel (a fortiori, if $\mathcal{J}$ is a Polish action), then $E_{\mathcal{J}}$ is easily seen to be a $\underset{\sim}{\Sigma}{}^1_1$ subset of $X \times X$.

Given an equivalence relation E on a set X, we say that $A \subseteq X$ is E-invariant if $x \in A$ and yEx implies $y \in A$. For arbitrary $A \subseteq X$, $A^{+E} = \{y: (\exists x \in A)(yEx)\}$ and $A^{-E} = \{y: (\forall x)(xEy \Rightarrow x \in A\}$. When no confusion will result we write A^+ and A^- for A^{+E} and A^{-E}. A^+ and A^- are respectively the smallest invariant set including A and the largest invariant set included in A. For $x \in X$, $[x]_E = \{x\}^+$ is the E-equivalence class or orbit of x. If $B^{\#}$ is an invariant set such that $B^- \subseteq B^{\#} \subseteq B^+$ we say $B^{\#}$ is an E-invariantization of B.

X/E is the set of E-equivalence classes and π_E is the projection map $x \mapsto [x]_E$. When X is a topological space, X/E is topologized by giving it the strongest topology such that π_E is continuous. E is <u>lower semicontinuous</u> (respectively <u>upper semicontinuous</u>) if A^+ is open (closed) whenever A is open (closed). An equivalent condition is that π_E be an open (closed) mapping. Bourbaki [9] refers to lower (upper) semicontinuous equivalences as open (closed) equivalences. We have chosen our terminology, which agrees with that in Kuratowski [26] when equivalence classes are closed, to avoid confusion with equivalences whose graphs are open or closed.

When X is Suslin, we say E is a $\underset{\sim}{\Sigma}^1_n$ equivalence on X provided E is an equivalence on X and E is a $\underset{\sim}{\Sigma}^1_n$ subset of $X \times X$.

If E_1 and E_2 are equivalences on X and Y respectively, then $E_1 \times E_2 = \{((x_1,y_1),(x_2,y_2)): x_1E_1x_2 \ \& \ y_1E_2y_2\}$ is the <u>product equivalence</u> on $X \times Y$. $\underset{\sim}{1}$ will always be the identity equivalence. If J_1 and J_2 are actions of G on X and Y respectively, then the <u>product action</u> $J_1 \times J_2$ of G on $X \times Y$ is defined by setting $g(x,y) = (gx,gy)$. Note that $E_{J_1 \times J_2}$ is <u>not</u> generally the same as $E_{J_1} \times E_{J_2}$.

<u>Logic</u>

If $n \in \omega$ and R is any set then $\underline{R} = (1,(R,n))$ is an n-ary relation symbol and $n = n(\underline{R})$ is the arity of $\underline{R}$. For any set c, $\underline{c} = (1,(c,0))$ is a constant symbol.

A <u>similarity type</u> is a set of relation symbols and constant symbols. If ρ is a similarity type, then R_ρ is the set of relation symbols of ρ and C_ρ is the set of constant symbols of ρ.

For any set $A \neq \emptyset$, $|X_{\rho,A}|$ is the set

$$\prod_{\underline{R} \in R_\rho} {}^{A^{n(\underline{R})}}2 \times C_{\rho_A}$$

and $X_{\rho,A}$ is the corresponding topological product space formed over the discrete topologies on 2 and A. If $S \in X_{\rho,A}$ then (A,S) is a ρ-structure with universe A. V_ρ is the class of all ρ-structures and X_ρ, the canonical logic space of type ρ, is $X_{\rho,\omega}$.

Since we will be interested in questions of effectiveness for $L_{\omega_1\omega}$, we adopt a standard set theoretic "arithmetization" of the language as follows:

The set of symbols $L_{\omega_1\omega}(\rho)$ contains each symbol in ρ, plus variables $\underline{v}_n = (3,(n,0))$ for each $n \in \omega$, and logical connectives $\underset{\sim}{\equiv} = (0,(\emptyset,2))$, $\neg = 4$, $\underset{\sim}{\bigvee} = 5$, $\underset{\sim}{\exists} = 6$, $\underset{\sim}{\bigwedge} = 7$, $\underset{\sim}{\forall} = 8$.

Atomic formulas are $\tau_1 \underset{\sim}{\equiv} \tau_2 = (9,(\equiv,(\tau_1,\tau_2)))$ and $\underline{R}(\tau_1,\ldots,\tau_{n(\underline{R})}) = (9,(\underline{R},(\tau_1,\ldots,\tau_{n(\underline{R})}))$ where each τ_i is a variable or constant symbol and $\underline{R}$ is a relation symbol. $\neg\phi$ is $(4,\phi)$, $\underset{\sim}{\bigvee}\Theta$ is $(5,\Theta)$, $\underset{\sim}{\bigwedge}\Theta$ is $(7,\Theta)$, $\underset{\sim}{\exists}v\phi$ is $(6,(v,\phi))$, $\underset{\sim}{\forall}v\phi$ is $(8,(v,\phi))$.

If ϕ is an atomic formula, then ϕ, $\neg\phi$ are subbasic' formulas. A basic ' formula is a conjunction $\underset{\sim}{\bigwedge}\Theta$ where Θ is a finite set of subbasic' formulas. An open' $(\underset{\sim}{\Sigma}'^{0}_{1})$ formula is an arbitrary (possibly uncountable) disjunction $\underset{\sim}{\bigvee}\Theta$ where each $\theta \in \Theta$ is of the form $\underset{\sim}{\exists}v_1,\ldots,\underset{\sim}{\exists}v_n M$ where $v_1,\ldots,v_n$ are variables and M is basic'.

Here and below, disjunctions and conjunctions can only be formed which have finitely many free variables. The set $L_{\omega_1\omega}(\rho)$ of infinitary (first order) formulas of type ρ is the smallest set which includes the set of open' formulas and contains $\neg\phi$, $\underset{\sim}{\bigvee}\Theta$, $\underset{\sim}{\bigwedge}\Theta$, $\underset{\sim}{\exists} v\phi$, $\underset{\sim}{\forall} v\phi$ when it includes $\Theta \cup \{\phi\}$, v is a variable, and Θ is countable.

These definitions correspond to those in Vaught [46]. They differ from the usual definitions of $L_{\omega_1\omega}$ in allowing some uncountable disjunctions when ρ is uncountable. They have the virtue of being more natural from the topological standpoint. We obtain a hierarchy on $L_{\omega_1\omega}(\rho)$ which is analogous to the Borel hierarchy by defining the $\underset{\sim}{\Sigma}'^{o}_{1}$ formulas as above, and then recursively defining

$$\underset{\sim}{\Pi}'^{o}_{\alpha}(\rho) = \{\neg\phi : \phi \in \underset{\sim}{\Sigma}'^{o}_{\alpha}(\rho)\}$$
$$\underset{\sim}{\Pi}'^{o}_{(\alpha)}(\rho) = \bigcup\{\underset{\sim}{\Pi}'^{o}_{\beta}(\rho) : \beta < \alpha\}$$
$$\underset{\sim}{\Sigma}'^{o}_{\alpha}(\rho) = \{\underset{\sim}{\bigvee}\Theta : \Theta \text{ is countable and each } \theta \in \Theta \text{ is of the form}$$
$$\underset{\sim}{\exists} v_1 \ldots \underset{\sim}{\exists} v_k \psi \text{ where } k \in \omega, \text{ each } v_i \text{ is a variable, and } \psi \in \underset{\sim}{\Pi}'^{o}_{(\alpha)}(\rho)\}$$

A subset L of $L_{\omega_1\omega}(\rho)$ is a <u>fragment</u> if L contains every atomic formula and L is closed under subformulas, finite conjunctions and disjunctions, and negations.

The finitary predicate calculus $L_{\omega\omega}(\rho)$ is the fragment of $L_{\omega_1\omega}(\rho)$ obtained by restricting all disjunctions to be finite. The usual prefix hierarchy on $L_{\omega\omega}$ is defined by setting

$$\underset{\sim}{\forall}^{o}_{n}(\rho) = \underset{\sim}{\Pi}'^{o}_{n}(\rho) \cap L_{\omega\omega}(\rho)$$

and

$$\underset{\sim}{\exists}^{o}_{n}(\rho) = \underset{\sim}{\Sigma}'^{o}_{n}(\rho) \cap L_{\omega\omega}(\rho).$$

An n-<u>formula</u> is a formula with free variables among $\underline{v}_0, \ldots, \underline{v}_{n-1}$. A 0-formula is a <u>sentence</u>. A <u>propositional</u> <u>formula</u> is one with no variables at all. $\phi\binom{a}{b}$ is the result of replacing a by b in ϕ.

We will use standard abbreviations ($\phi \wedge \psi$ for $\underset{\sim}{\bigwedge}\{\phi,\psi\}$, $\phi \to \psi$ for $\neg(\phi \wedge \neg\psi)$, etc.) to simplify formal expressions.

$$(\underset{\sim}{\exists}!\underline{v}_i)(\phi) \text{ abbreviates } (\underset{\sim}{\exists}\underline{v}_i)(\phi \wedge (\underset{\sim}{\forall}\underline{v}_j)(\phi(\tfrac{\underline{v}_i}{\underline{v}_j}) \to \underline{v}_i \approx \underline{v}_j))$$

where j is the smallest such that $\underline{v}_j$ does not occur in ϕ.

$$(\underset{\sim}{\exists}^{\neq}v_1,\ldots,v_n)(\phi) \text{ abbreviates } (\underset{\sim}{\exists}v_1,\ldots,v_n)(\phi \wedge \bigwedge_{i<j\leq n} v_i \not\approx v_j).$$

Given similarity types ρ and ρ_1, $\rho + \rho_1$ is the result of adding the symbols of ρ_1 to ρ. It is defined by setting

$$\rho_1^{\rho} = \{(i,((\rho,a),n)\colon \ (i,(a,n)) \in \rho_1\}$$

and then defining $\rho+\rho_1 = \rho \cup \rho_1^{\rho}$.

A $\underset{\sim}{\Sigma}'^1_n$ (2nd order) <u>formula of type</u> ρ is an expression of the form

$$\underset{\sim}{\exists}\rho_1\underset{\sim}{\forall}\rho_2\underset{\sim}{\exists}\rho_3\cdots(\underset{\sim}{\exists} \text{ or } \underset{\sim}{\forall})\rho_n\phi$$

where $\rho_1,\ldots,\rho_n$ are countable similarity types and $\phi \in L_{\omega_1\omega}(\rho+\rho_1+\ldots+\rho_n)$.

In Vaught [44] $\underset{\sim}{\Sigma}'^1_n$ and $\underset{\sim}{\Pi}'^1_n$ were denoted as $\underset{\sim}{\exists}^1_n$ and $\underset{\sim}{\forall}^1_n$ respectively. We will use the expressions $\underset{\sim}{\forall}^1_n$, $\underset{\sim}{\exists}^1_n$ to denote the corresponding classes of <u>finitary</u> 2nd-order formulas obtained by restricting ϕ to belong to $L_{\omega\omega}(\rho+\rho_1+\ldots+\rho_n)$ in the definition of $\underset{\sim}{\Pi}'^1_n(\rho)$, $\underset{\sim}{\Sigma}'^1_n(\rho)$.

For $K \subseteq V_\rho$, $K^{(n)} \subseteq V_{\rho \cup \{\underline{0},\ldots,\underline{n-1}\}}$ is defined by the equation $K^{(n)} = \{(A,S,a_0,\ldots,a_{n-1}): a_0,\ldots,a_{n-1} \in A \ \& \ (A,S) \in K\}$. If ϕ is an n-formula, (A,S) is a structure, and $a_0,\ldots,a_{n-1}$ are elements of A which satisfy ϕ in (A,S) (in the obvious sense), we say $(A,S,a_0,\ldots,a_{n-1})$ is a model of ϕ and write $(A,S,a_0,\ldots,a_{n-1}) \in \mathrm{Mod}^{(n)}(\phi)$ or $(A,S,a_0,\ldots,a_{n-1}) \models \phi$. When $n = 0$ we drop the superscript. $\mathrm{Mod}^K(\phi) = \mathrm{Mod}(\phi) \cap K$. If Γ is a collection of ρ-formulas and $K \subseteq V_\rho$, we define $\Gamma(K^{(n)}) = \{\mathrm{Mod}^{(n)}(\phi) \cap K^{(n)}: \phi \text{ is an n-formula in } \Gamma\}$. We say K is $\underset{\sim}{\Sigma}'^0_\alpha$, $\underset{\sim}{\Sigma}'^1_n$, etc., if $K \in \underset{\sim}{\Sigma}'^0_\alpha(V_\rho)$, $\underset{\sim}{\Sigma}'^1_n(V_\rho)$, etc..

$\mathcal{J}_\rho = (X_\rho, \omega!, J_\rho)$, <u>the canonical logic action of type</u> ρ, is defined as follows:

$\omega!$ is the group of permutations of ω given the relative topology as a subspace of ω^ω.

For $g \in \omega!$, $S \in X_\rho$, $gS = J_\rho(g,S)$ is the usual isomorph of S under g. Thus, for each $\underline{R} \in \mathcal{R}$ and each $\underline{c} \in C_\rho$

$$(gS)_{\underline{R}}(i_1,\ldots,i_{n(\underline{R})}) = S_{\underline{R}}(g^{-1}(i_1),\ldots,g^{-1}(i_{n(\underline{R})}))$$

$$(gS)_{\underline{c}} = g(S_{\underline{c}}).$$

It is easily seen that $\mathcal{J}_\rho$ is a Polish action whenever ρ is countable. $I_\rho = E_{\mathcal{J}_\rho}$ is the usual isomorphism relation on X_ρ.

Let $\rho_N = \{\underline{n}: n \in \omega\}$. If ρ is disjoint from ρ_N, then Borel and projective subsets of X_ρ are naturally described by propositional formulas of type $\rho \cup \rho_N$ as follows. An <u>atomic name</u> is an expression $\underline{R}(\underline{i}_1,\ldots,\underline{i}_{n(\underline{R})})$ or $\underline{c} = \underline{i}$ where $\underline{R},\underline{c} \in \rho$ and $\underline{i},\underline{i}_1,\ldots,\underline{i}_{n(\underline{R})} \in \rho_N$. ϕ is a ρ-<u>name</u> if ϕ is propositional and every atomic subformula of ϕ is an atomic name. If $\phi \in L_{\omega_1\omega}(\rho \cup \rho_N)$, $\underset{\sim}{\Sigma}'^0_\alpha(\rho \cup \rho_N)$, $\underset{\sim}{\Sigma}'^1_n(\rho \cup \rho_N)$, etc., we say ϕ is a <u>Borel</u> ρ-<u>name</u>, $\underset{\sim}{\Sigma}^0_\alpha$-$\rho$-<u>name</u>, $\underset{\sim}{\Sigma}^1_n$-$\rho$-<u>name</u>, etc.. ϕ <u>names</u> the set $[\phi] = \{S: (\omega,S,0,1,\ldots) \models \phi\}$. Clearly, $B \subseteq X_\rho$ is Borel, $\underset{\sim}{\Sigma}^0_\alpha$, $\underset{\sim}{\Sigma}^1_n$, etc. if and only if B has a Borel-name, $\underset{\sim}{\Sigma}^0_\alpha$-name, $\underset{\sim}{\Sigma}^1_n$-name, etc.

<u>We assume throughout the dissertation that</u> ρ_N <u>is disjoint from all other similarity types which are mentioned</u>.

Subsets of X_ρ are also defined by arbitrary sentences of type ρ. Thus, if θ is a (first or second order) n-formula of type ρ, we identify $S \in X_\rho$ with (ω,S) and set $[\![\theta^{(n)}]\!] = \mathrm{Mod}^{(n)}(\theta) \cap X^{(n)}_\rho$. It is apparent that $[\![\theta^{(n)}]\!]$ is invariant under the canonical equivalence on $X^{(n)}_\rho = X_\rho \times \omega^n$. It is also apparent that $[\![\theta^{(n)}]\!]$ is $\underset{\sim}{\Sigma}^0_\alpha$, Borel,

$\underset{\sim}{\Sigma}^1_n$, etc. when θ is $\underset{\sim}{\Sigma}'^{o}_{\alpha}$, $L_{\omega_1\omega}$, $\underset{\sim}{\Sigma}'^1_n$, etc. (A $\underset{\sim}{\Sigma}^o_\alpha$-name etc. for $[\![\theta^{(n)}]\!]$ may be obtained by inductively replacing subformulas of the form $\underset{\sim}{\exists} v\phi$ by disjunctions $\underset{\sim}{\bigvee}\{\phi(\underset{\underline{i}}{v}): i \in \omega\}$). Thus, B is I_ρ-invariant $\underset{\sim}{\Sigma}^o_\alpha$, $\underset{\sim}{\Sigma}^1_n$, etc. whenever B is $\underset{\sim}{\Sigma}'^o_\alpha$, $\underset{\sim}{\Sigma}'^1_n$, etc.

For each of the Borel and projective classes the converse of the above holds and we have the identities "$\underset{\sim}{\Sigma}'^1_n(X_\rho)$ = invariant $\underset{\sim}{\Sigma}^1_n(X_\rho)$," "$L_{\omega_1\omega}(X_\rho)$ = invariant $\mathcal{B}(X_\rho)$," "$\underset{\sim}{\Sigma}'^o_\alpha(X_\rho)$ = invariant $\underset{\sim}{\Sigma}^o_\alpha(X_\rho)$." If θ is a $\underset{\sim}{\Sigma}^1_n$-$\rho$-name, then the equation

$$[\theta]^{+I_\rho} = [\![(\underset{\sim}{\exists}\rho_N)(\theta \wedge (\underset{\sim}{\forall}\underline{v}_o)(\underset{\sim}{\bigvee}\{\underline{v}_o = \underline{i}:\ i \in \omega\}) \wedge \underset{\sim}{\bigwedge}\{\underline{i} \neq \underline{j}:\ i < j < \omega\})]\!]$$

indicates that $[\theta]^+$ is $\underset{\sim}{\Sigma}'^1_n$. Since $B = B^+$ when B is invariant, the first identity follows. The second, for countable ρ, follows from the first and the Lopez-Escobar interpolation theorem: $\underset{\sim}{\Sigma}'^1_1(V_\rho) \cap \underset{\sim}{\Pi}'^1_1(V_\rho) = L_{\omega_1\omega}(V_\rho)$. The third identity is a recent result due to Vaught ([46]). It refines the second and extends it to arbitrary ρ. In chapter III we will add to the list of identities by proving "countable alternated union of $\underset{\sim}{\Pi}'^o_\alpha$ sets = invariant countable alternated union of $\underset{\sim}{\Pi}^o_\alpha$ sets." These sets coincide with the invariant

$\underset{\sim}{\Delta}^{o}_{\alpha+1}$ sets when ρ is countable.

Effectiveness

In chapter I and chapter III we will prove "effective" versions of topological and model theoretic results. For us it will be most convenient to formalize the concept of "effectiveness" in terms of hereditarily countable sets, admissible sets and primitive recursive set functions.

The canonical reference for primitive recursive set functions is Jenson-Karp [20]. We recall the basic definitions: A set function is primitive recursive (prim) if it can be obtained from the initial functions by substitution and recursion as follows:

Initial functions:

(i) $P_{n,i}(x_1,\ldots,x_n) = x_i; \quad 1 \leq n < \omega, \quad 1 \leq i \leq n$

(ii) $F(x,y) = x \cup \{y\}$

(iii) $C(x,y,u,v) = x$ if $u \in v$, y otherwise

Substitution:

(a) $F(\check{v},\check{w}) = G(\check{v},H(\check{v}),\check{w}); \quad \check{v} = (v_1,\ldots,v_m), \quad \check{w} = (w_1,\ldots,w_n)$

(b) $F(\check{v},\check{w}) = G(H(\check{v}),\check{w})$

Recursion:

$F(w,\check{v}) = G(\bigcup\{F(u,\check{v}) : u \in w\},w,\check{v})$

If $x_1,\ldots,x_k$ are sets, a function F is prim$(x_1,\ldots,x_k)$ if there exists a prim function G such that for all $v_1,\ldots,v_m$, $F(v_1,\ldots,v_m) = G(v_1,\ldots,v_m,x_1,\ldots,x_k)$. y is prim$(x_1,\ldots,x_k)$ if the constant function $F(v) = y$ is prim$(x_1,\ldots,x_k)$. It is apparent that y is prim$(x_1,\ldots,x_k)$ if and only if $y = G(x_1,\ldots,x_k)$ for some prim function G. The set of all such y is the prim-closure of $(x_1,\ldots,x_k)$.

A set A is transitive if $y \in x \in A$ implies $y \in A$. Given a set x, the transitive closure, $Tc(x)$, of x is the smallest transitive set A such that $x \in A$. x is hereditarily countable ($x \in HC$) if $\overline{\overline{Tc(x)}} \leq \omega$.

As an example, note that the definition $TC(y) = y \cup \bigcup\{TC(z): z \in y\}$ shows that the function $F(y) = TC(y)$ is primitive recursive.

If ρ is hereditarily countable then $X_\rho \subseteq HC$ and all of the first and second order formulas of type ρ are hereditarily countable. Moreover, all of the syntactical notions ("θ is a formula," "ϕ is a subformula of ψ," etc.) we will use are easily seen to be set theoretically primitive recursive (cf. Cutland [14] or Barwise [7]). In particular, the function $\theta \mapsto \theta^N$ which maps ρ-formulas to ρ-names by replacing variables by special constants is $\text{prim}(\rho)$ for $\rho \in HC$. -- Consider, for illustration the case $\rho = \{\underline{\varepsilon}\}$ where $\underline{\varepsilon}$ is binary. Then $\theta \mapsto \theta^N$ is defined by the recursive conditions:

$$(\underline{\varepsilon}(\underline{v}_i,\underline{v}_j))^N = \underline{\varepsilon}(\underline{i},\underline{j})$$
$$(\neg\phi)^N = \neg(\phi^N), \quad (\underset{\sim}{\bigvee}\Theta)^N = \underset{\sim}{\bigvee}\{\theta^N: \theta \in \Theta\}, \text{ dually for } \underset{\sim}{\bigwedge}$$
$$(\underset{\sim}{\exists}\underline{v}_i\phi)^N = \bigvee\{\phi^N(\tfrac{\underline{j}}{\underline{i}}): j \in \omega\}, \text{ dually for } \underset{\sim}{\forall}$$
$$(\underset{\sim}{\exists}\rho_1\phi)^N = \underset{\sim}{\exists}\rho_1\phi^N, \text{ dually for } \underset{\sim}{\forall}$$

The language of set theory is $L_{\omega\omega}(\{\underline{\varepsilon}\})$ where $\underline{\varepsilon}$ is binary. $\phi \in L_{\omega\omega}(\{\underline{\varepsilon}\})$ is Δ_o if every quantification in ϕ is restricted; that is, of the form $(\underset{\sim}{\forall}v)(v\underline{\varepsilon}w \rightarrow \ldots)$ or of the form $(\underset{\sim}{\exists}v)(v\underline{\varepsilon}w \wedge \ldots)$.

The set of Σ-formulas is the smallest which includes the Δ_o formulas and is closed under finite conjunctions and disjunctions, restricted universal quanification and arbitrary existential quantification.

A set A is admissible if A is transitive, prim-closed and satisfies the $\underset{\sim}{\Sigma}$-reflection principle:

<u>If</u> θ <u>is</u> Σ, $a_1,\ldots,a_n \in A$ <u>and</u> $(A,\varepsilon,a_1,\ldots,a_n) \models \theta$, <u>then</u> <u>for some transitive</u> $b \in A$, $a_1,\ldots,a_n \in b$ <u>and</u> $(b,\varepsilon,a_1,\ldots,a_n) \models \theta$.

A subset X of A is $\underset{\sim}{\Sigma}$-definable on A ($X \in \underset{\sim}{\Sigma}(A)$) if for some $n \in \omega$, some $\check{b} \in A^n$, and some $n+1$-formula $\phi \in \Sigma$, $X = \{a: (A,\varepsilon,\check{b},a) \models \phi\}$. X is $\underset{\sim}{\Delta}$-definable ($X \in \underset{\sim}{\Delta}(A)$) if both X and $A - X$ are $\underset{\sim}{\Sigma}$-definable.

All the facts we require about admissible sets may be found in [6] or [23].

For $\mathcal{A} \subseteq HC$, $\rho \in \mathcal{A}$, "Γ" = "$\underset{\sim}{\Sigma}^0_\alpha$", "$\underset{\sim}{\Sigma}^1_n$", etc., we define $\Gamma[\mathcal{A}](X_\rho) = \{[\phi]: \phi \in \mathcal{A}$ and ϕ is a Γ-ρ-name$\}$.

In I §2, we will require some standard results about the conventional "lightface" classes Σ^1_n, Π^1_n as found e.g. in [39]. It is known that if ρ is finite, $x \in X_\rho$, and $A = A_x$, the smallest admissible set containing x, then our classes $\underset{\sim}{\Sigma}^0_\alpha[A](X_\rho)$, $\underset{\sim}{\Sigma}^1_n[A](X_\rho)$ coincide with the lightface classes $\Sigma^0_\alpha(\text{Hyp } x)$, $\Sigma^1_n(x)$. Thus, the approach via prim-closed and admissible sets, subsumes and refines the lightface approach for our purposes.

The only results connecting "lightface" with "prim-closed" which are required for our arguments are the following obvious facts:

(i) $\Pi^1_1(2^{\omega\times\omega}) \subseteq \underset{\sim}{\Pi}^1_1[\mathcal{A}_\omega](2^{\omega\times\omega})$ where $\mathcal{A}_\omega$ is the prim-closure of $\{\omega\}$.

(ii) $\underset{\sim}{\forall}^1_1(2^{\omega\times\omega}) \subseteq \Pi^1_1(2^{\omega\times\omega})$.

<u>Numbered Items</u>

Certain statements in the body of the dissertation will be numbered. To assist the reader we have assigned these numbers in logical order rather than in the order of appearance in the text.

CHAPTER I: PROJECTIVE EQUIVALENCE RELATIONS AND INVARIANT PREWELLORDERINGS

Moschovakis [35] and Vaught [44] established invariant $\underset{\sim}{\Pi}^1_1$ and $\underset{\sim}{\Sigma}^1_1$ reduction theorems (and, implicitly, corresponding invariant prewellordering theorems) for canonical logic actions. In [46] Vaught did the same for arbitrary Polish actions. The main subject of this chapter (§1 and §2) is a proof that these results hold for arbitrary $\underset{\sim}{\Sigma}^1_1$ equivalence relations on arbitrary Suslin spaces. For suitable spaces X the results are obtained in a very effective "primitive recursive" version, (Theorem 2.1) -- as were the effective theorems of [44]. Assuming projective determinacy, (PD), the same arguments, (which are based on the ordinary Π^1_1 prewellordering theorem), extend without alteration to yield invariant reduction theorems for $\underset{\sim}{\Pi}^1_{2n+1}$ and $\underset{\sim}{\Sigma}^1_{2n+2}$, all $n \in \omega$.

The present chapter is the latest version of work begun jointly with John Burgess in [11]. That paper contained proofs of the invariant $\underset{\sim}{\Sigma}^1_2$ prewellordering and reduction theorems (due to the author) and of the invariant $\underset{\sim}{\Pi}^1_1$ reduction theorem for pairs (due to Burgess). At that time we conjectured the full invariant $\underset{\sim}{\Pi}^1_1$ prewellordering theorem, but could prove only a very special case of the theorem for $\underset{\sim}{\Pi}^1_{2n+1}$ $(n \geq 1)$ assuming PD ([11] 4.2). The conjecture (and, hence, full invariant $\underset{\sim}{\Pi}^1_1$ reduction) was established by R. Solovay based on an idea of the author (see our remark III, p. 40 for more details). This argument forms a central part of the proof of 2.1 below. Solovay's proof is sufficient to establish a corresponding "lightface" theorem -- as is a second argument (for invariant $\underset{\sim}{\Pi}^1_1$ prewellordering) due to Burgess [10]. The additional argument of §2 which establishes the stronger "primitive recursive" part of 2.1 is new. It is closely related to the methods of Vaught [44].

§ 3 contains a discussion of the invariant uniformization principle. Assuming $V = L$ we show that the principle holds for Σ^1_n $(n \geq 2)$. We prove a general result on counterexamples which is related to previous work of Dale Myers.

§1. Invariant Prewellorderings and the Invariant Reduction Principle

We begin by recalling some of the basic definitions. If $B = \langle B_i : i \in I \rangle$ and $A = \langle A_i : i \in I \rangle$ are sequences of subsets of X, we say that B reduces A provided that

(i) $B_i \subseteq A_i$ for each $i \in I$

(ii) $\bigcup\{B_i : i \in I\} = \bigcup\{A_i : i \in I\}$

(iii) $B_i \cap B_j = \emptyset$ whenever $i,j \in I$ and $i \neq j$

$\Gamma \subseteq P(X)$ has the *reduction property* if for every $A \in {}^{\omega}\Gamma$ there exists $B \in {}^{\omega}\Gamma$ such that B reduces A. The *reduction property for pairs* is obtained from the reduction property by replacing ω with 2.

If Γ has the reduction property for pairs, then $\breve{\Gamma} = \{-A : A \in \Gamma\}$ has the (weak) *first separation property*:

If A_0, A_1 are disjoint elements of $\breve{\Gamma}$, then there exists $B \in \breve{\Gamma} \cap \Gamma$ such that $A_0 \subseteq B \subseteq -A_1$, ($B$ *separates* A_0 from A_1).

To prove this consider B such that $(B, -B)$ reduces $(-A_1, -A_0)$.

A related concept is that of a uniformization. If B and A are subsets of a product space $X \times Y$, B is said to *uniformize* A provided

(i) $B \subseteq A$

(ii) If $(x,y) \in A$, then $(\exists y)((x,y) \in B)$. (domain (B) = domain (A)).

(iii) If (x,y_1), $(x,y_2) \in B$, then $y_1 = y_2$. (B is a function).

$\Gamma \subseteq X \times Y$ has the uniformization property if every member of Γ can be uniformized by a member of Γ. For $B, A \in {}^{\omega}P(X)$ it is easy to see that B reduces A if and only if $\hat{B} = \{(x,i): x \in B_i\}$ uniformizes $\hat{A} = \{(x,i): x \in A_i\}$. Thus, the reduction property for, say, $\Pi^1_1(X)$ is equivalent to the uniformization property for $\Pi^1_1(X \times \omega)$.

A relatively difficult theorem of descriptive set theory states that for arbitrary Polish spaces X, Y, the collections $\Pi^1_1(X \times Y)$ and $\Sigma^1_2(X \times Y)$ have the uniformization property. Assuming PD, the same is true for Π^1_{2n+1}, Σ^1_{2n+2} for each $n \in \omega$, cf. Kechris-Moschovakis [21]. On the other hand, if we assume the axiom of constructibility $(V = L)$, then $\Sigma^1_n(X \times Y)$ has the uniformization property whenever X, Y are Polish and $n \geq 2$, (Addison [1]). We will show that this last result has an invariant version while the other uniformization theorems do not.

Now suppose E is an equivalence relation on X and $\Gamma \subseteq P(X)$. Let E-inv(Γ) be the collection of E-invariant members of Γ. We are interested in E-invariant versions of the reduction and uniformization properties. The E-invariant Γ-reduction property is easy to formulate, viz. E-inv(Γ) has the reduction property. The E-invariant uniformization property is a bit more complicated. If $X = Y \times Z$ and $A \subseteq X$ is E-invariant, we say that B is an E-uniformization of A provided

(i) B is E-invariant

(ii) $B \subseteq A$

(iii) If $(y,z) \in A$, then $(\exists z)((y,z) \in B)$

(iv) If (y,z_1), $(y,z_2) \in B$, then $(y,z_1)\ E\ (y,z_2)$.

Condition (iv) says that B is as close to being a function as is consistent with E-invariance. Γ has the E-uniformization property if every E-invariant $A \in \Gamma$ has an E-uniformization which is a member of Γ.

Note that if $E = E' \times \underset{\sim}{1}$, where $\underset{\sim}{1}$ is the identity relation on Z, then an E-uniformization is just a uniformization which is E-invariant.

This definition of the E-uniformization property is essentially due to Vaught, see [44]. He formulated it for the special case $X = X_\rho \times X_{\rho'}$, $E = E_{J_\rho \times J_{\rho'}}$ and asked whether $\underset{\sim}{\Pi}^1_1(X)$ has the E-uniformization property in this case. D. Myers answered that question in the negative in [36] and [38].

To see the relation between the invariant reduction and uniformization properties, let $A \in {}^\omega P(X)$ be a sequence of E-invariant sets and let $B = \langle B_i : i \in \omega \rangle$ be a sequence which reduces A. It is apparent that each B_i is E-invariant just in case $\{(x,i): x \in B_i\}$ is an $E \times \underset{\sim}{1}$-uniformization of $\{(x,i): x \in A_i\}$. Thus, for "Γ" = "$\underset{\sim}{\Sigma}^1_n$" or "$\underset{\sim}{\Pi}^1_n$",

(1) E-inv($\Gamma(X)$) has the reduction property if and only if $\Gamma(X \times \omega)$ has the $E \times \underset{\sim}{1}$-uniformization property.

An important tool for our treatment of the $\underset{\sim}{\Pi}^1_1$ and $\underset{\sim}{\Sigma}^1_2$ reduction theorems is the notion of a prewellordering [5]. Given a set A, a prewellordering on A is a transitive, relexive, connected, well-founded relation on A. If $\preceq$ is a prewellordering on A, the associated norm $\phi_\preceq : A \to ON$ is obtained by defining $\phi_\preceq(a)$ to be the $\preceq$-rank of a. Conversely, every map $\phi: A \to ON$ induces a prewellordering

$\preceq_\phi$ on A by setting $a \preceq_\phi a'$ if and only if $\phi(a) \le \phi(a')$. Given $A \subseteq X$ and $\Gamma \subseteq \mathcal{P}(X^2)$, we define a Γ-prewellordering on A to be a triple $(\preceq, Q, Q')$ such that $\preceq$ is a prewellordering on A, $Q \in \Gamma$, $Q' \in \breve{\Gamma}$ and for every $a \in A$ and $x \in X$

$$(x \in A \;\&\; x \preceq a) \iff (x,a) \in Q \iff (x,a) \in Q' .$$

If "Γ" = "$\undertilde{\Sigma}^1_n$", "Σ^1_n", etc. and $A \subseteq X$, then a Γ-prewellordering on A is a $\Gamma(X^2)$-prewellordering on A. If $X = X_\rho$ and $\rho \in \mathcal{Q} \subseteq HC$, then a $\Gamma[\mathcal{Q}]$-prewellordering is a Γ-prewellordering $(\preceq, Q, Q')$ such that $Q \in \Gamma[\mathcal{Q}]$, $Q' \in \breve{\Gamma}[\mathcal{Q}]$.

Now suppose "Γ" = "$\undertilde{\Sigma}^1_n$" or "$\undertilde{\Pi}^1_n$". The utility of prewellorderings in proving reduction theorems stems from the fact:

(2) If $A \in \Gamma(X \times \omega)$ and $(\preceq, Q, Q')$ is a Γ-prewellordering on A, then the set

$B = \{(x,p)\colon (x,p) \in A$ and p is the smallest natural number which minimizes $\phi_\preceq(x,p)\}$

$= \{(x,p)\colon (x,p) \in A \;\&\; \bigwedge_{m \in \omega} [((x,m),(x,p)) \in Q' \Rightarrow ((x,p),(x,m)) \in Q]$
$\;\&\; \bigwedge_{n<p} [((x,n),(x,p)) \notin Q']\}$

is a member of $\Gamma(X \times \omega)$ which uniformizes A.

Thus, if every $A \in \Gamma(X \times \omega)$ has a $\Gamma((X \times \omega)^2)$-prewellordering, then $\Gamma(X \times \omega)$ has the uniformization property.

When $X = X_\rho$, $\rho \in HC$, the definition of B in (2) is uniform and effective. That is

(3) There is a prim(ω) function P such that if ϕ is a Γ-name for a subset of $X_\rho \times \omega$ and q,q' are Γ-names such that $([\phi]^2 \cap [q], [q], [\neg q'])$ is a Γ-prewellordering on $[\phi]$, then $P(\phi,q,q')$ is a Γ-name for a set which uniformizes $[\phi]$.

Suppose E is an equivalence relation on X, and that A, $(\preceq,Q,Q')$ and B are as in (2). Assume further that A is $E \times \underset{\sim}{1}$-invariant. It is clear from the definition of B that:

(4) If $E \times \underset{\sim}{1}$ is a congruence for $\preceq$ (i.e. if $(x,m) \preceq (x',m)$ whenever xEx'), then B is $E \times \underset{\sim}{1}$-invariant.

In a slightly more general context, suppose we are given a Suslin space X, an equivalence E on X, and a E-invariant set $A \subseteq X$. Then an E-<u>invariant</u> Γ-<u>prewellordering</u> on A is a Γ-prewellordering $(\preceq,Q,Q')$ on A such that $\preceq$ is an $E \times E$-invariant subset of X^2.

If every E-invariant $A \in \Gamma(X)$, (respectively, $\Gamma[\mathcal{A}](X_\rho)$), has an E-invariant Γ-prewellordering, ($\Gamma[\mathcal{A}]$-prewellordering), we say that X, (X_ρ), has the E-Γ-prewellordering property, (E-$\Gamma[\mathcal{A}]$-prewellordering property).

<u>Theorem 1.1</u>. Let "Γ" be "$\underset{\sim}{\Sigma}^1_n$" or "$\underset{\sim}{\Pi}^1_n$". Assume that X is a Suslin space with an equivalence E such that $X \times \omega$ has the $(E \times \underset{\sim}{1})$-$\Gamma$-prewellordering property. Let E_1 be an arbitrary equivalence on ω. Then

(a) $\Gamma(X\times\omega)$ has the $(E\times E_1)$-uniformization property.

(b) Suppose $X = X_\rho$, $\rho \in HC$. Then there is a prim(ω) function P_o such that if ϕ is a Γ-name for an $E\times E_1$ invariant subset of $X\times\omega$, ψ is a Γ-name for E_1, and q,q' are respectively $\Gamma,\breve{\Gamma}$-names which witness an $(E\times \underset{\sim}{1})$-invariant Γ-prewellordering on $[\phi]$, then $P_o(\phi,q,q',\psi)$ is a Γ-name for an $(E\times E_1)$-uniformization of $[\phi]$.

Proof.

First suppose $E_1 = \underset{\sim}{1}$. Then the conclusions (a) and (b) are immediate from (4) and (2) and (3) respectively. The general case is easily reduced to this case as follows.

Let $A \subseteq X\times\omega$ be $E\times E_1$-invariant and suppose $B \in \Gamma(X\times\omega)$ is an $E\times\underset{\sim}{1}$-uniformization of A. Let $B' = B^{+(E\times E_1)} =$ $\{(x,p)\colon \bigvee_{m\in\omega} ((x,m) \in B \ \&\ mE_1\, p)\}$. B' is obviously $E\times E_1$-invariant. Since A is $E\times E_1$-invariant and $B \subseteq A$, $B' \subseteq A$. $\mathrm{dom}(B') = \mathrm{dom}(B)$ so $\mathrm{dom}(B') = \mathrm{dom}(A)$. Finally, if $(y,m_1),(y,m_2) \in B$, then $m_1E_1m_2$ so $(y,m_1)E\times E_1(y,m_2)$. Thus, B' is an $E\times E_1$-uniformization of A. It is apparent from our definition of B' that when $X = X_\rho$, a Γ-name for B' can be obtained primitive recursively from ω and names for B and E_1. (b) then follows by (3). $\square$

§2. The Invariant $\underset{\sim}{\Pi}_1^1$ and $\underset{\sim}{\Sigma}_2^1$ Prewellordering Theorems.

The main result of this section is

Theorem 2.1. There exist prim(ω) functions P_1, P_2 with the following property. Assume $\rho \in HC$ is a similarity type and ψ is a $\underset{\sim}{\Sigma}_1^1$-name for an equivalence on X_ρ. Then

(a) If ϕ_1 is a $\underset{\sim}{\Pi}_1^1$-name for a $[\psi]$-invariant set, then $P_1(\rho,\psi,\phi_1)$ is an ordered pair of names which witnesses the existence of a $[\psi]$-invariant $\underset{\sim}{\Pi}_1^1$-prewellordering on $[\phi_1]$.

(b) If ϕ_2 is a $\underset{\sim}{\Sigma}_2^1$-name for a $[\psi]$-invariant set, then $P_2(\rho,\psi,\phi_2)$ is an ordered pair of names which witnesses the existence of a $[\psi]$-invariant $\underset{\sim}{\Sigma}_2^1$-prewellordering on $[\phi_2]$.

2.1 yields

Corollary 2.2.

(a) If X is Suslin, E is a $\underset{\sim}{\Sigma}_1^1$ equivalence on X, and "Γ" = "Π_1^1" or "$\underset{\sim}{\Sigma}_2^1$", then $X \times \omega$ has the $(E \times \underset{\sim}{1})$-$\Gamma$-prewellordering property.

(b) With X, E as in (a), both E-inv($\underset{\sim}{\Pi}_1^1(X)$) and E-inv($\underset{\sim}{\Sigma}_2^1(X)$) have the reduction property.

(c) The $\underset{\sim}{\Pi}_1^1[\mathcal{A}]$, (respectively $\underset{\sim}{\Sigma}_2^1[\mathcal{A}]$), subsets of $X_\rho \times \omega$ have the $E \times E_1$-uniformization property whenever $\mathcal{A} \subseteq HC$ is prim-closed, $\omega, \rho \in \mathcal{A}$, E is a $\underset{\sim}{\Sigma}_1^1[\mathcal{A}]$ equivalence on X_ρ, and E_1 is a $\underset{\sim}{\Pi}_1^1[\mathcal{A}]$, ($\underset{\sim}{\Sigma}_2^1[\mathcal{A}]$), equivalence on ω.

Proof of Corollary.

(c) is immediate from 2.1 and 1.1.

(b) is immediate from (a) and 1.1.

Since all uncountable Polish spaces are Borel isomorphic, (a) is immediate from 2.1 when X is Polish. For the general case, let X, E be as in (a) and let $A \in \Gamma(X)$ be invariant ("Γ" = "$\underset{\sim}{\Pi}^1_1$" or "$\underset{\sim}{\Sigma}^1_2$"). Let f be a Borel measurable function on 2^ω onto X. Defining E' by the equation $R_1 E' R_2 \Leftrightarrow f(R_1)\ E\ f(R_2)$, we see that E' is a Σ^1_1 equivalence and $f^{-1}(A) \in E'\text{-inv}(\Gamma(2^\omega))$. By 2.1 there is an E'-Γ-prewellordering $(\preceq, Q, Q')$ on $f^{-1}(A)$. Setting $Q_A =$ $\{(x,y): (\forall R_1, R_2)((f(R_1) = x\ \&\ f(R_2) = y) \Rightarrow (R_1, R_2) \in Q\}$, $Q_A' = \{(x,y): (\exists R_1, R_2)(f(R_1) = x\ \&\ f(R_2) = y\ \&\ (R_1,R_2) \in Q'\}$, $\preceq_A = Q \cap A^2$, it is easily seen that $(\preceq_A, Q_A, Q_A')$ is an E-Γ-prewellordering on A. $\square$

The Π^1_1 and Σ^1_2 cases of 2.1 will be treated separately. The argument for the Π^1_1 case is substantially longer than that for Σ^1_2. It is the only part of this dissertation which makes essential use of "lightface" notions and involves two separate parts (the first is due to Solovay -- see Remark III, p. 40). In part one we prove the lightface version of the theorem --

(8) If ρ is finite, $E \in \Sigma^1_1(X^2_\rho)$ is an equivalence on X_ρ, and $A \in \Pi^1_1(X_\rho)$ is E-invariant, then A has an E-invariant Π^1_1-prewellordering.

In part two we consider a particular "very universal" Π^1_1 set and Σ^1_1 equivalence and derive the general case of 2.1 by a process of taking "pseudo cross-sections".

The argument for the Σ^1_2 case is more direct -- we derive the invariant primitive recursive Σ^1_2 prewellordering theorem (2.1) from the non-invariant primitive recursive Π^1_1 prewellordering theorem.

Both our arguments are logically based on the ordinary Π^1_1 (lightface) prewellordering theorem so we have the following (see Remark I, p. 38 for details):

Corollary 2.3. (to the proof of 2.1) Assume $n \geq 1$ and every Π^1_n subset of 2^ω has a Π^1_n prewellordering. Then 2.1 and 2.2 hold with "$\underset{\sim}{\Pi}^2_1$", "$\underset{\sim}{\Sigma}^1_1$", and "$\underset{\sim}{\Sigma}^1_2$" respectively replaced by "$\underset{\sim}{\Pi}^1_n$", "$\underset{\sim}{\Sigma}^1_n$" and "$\underset{\sim}{\Sigma}^1_{n+1}$" throughout. In particular, the conclusion holds whenever n is odd and all $\underset{\sim}{\Delta}^1_{n-1}$ games are determined.

Proof of 2.1(a), part 1.

Let $X_o = 2^\omega$, $X_1 = \omega$. For $i,j \in \{0,1\}$ let $< >_{ij}$: $X_i \times X_j \to X_{i \cdot j}$ be a recursive bijection with recursive inverse s_{ij}: $w \mapsto (w^{ij}_o, w^{ij}_1)$. We will use $i,\ldots,n$ with subscripts to denote members of ω and $u,\ldots,z$ with subscripts to denote members of 2^ω. We will drop the subscripts on our pairing functions whenever possible.

Let $W \in \Pi^1_1(2^\omega \times \omega)$ be a universal Π^1_1 set. For each $i \in \omega$ let $W_i = \{x \in 2^\omega\colon\ (x,i) \in W\}$, (so $\Pi^1_1(2^\omega) = \{W_i\colon\ i \in \omega\}$). We further assume that W is "canonical" in that

(i) There exist recursive functions $f_i\colon \omega \to \omega$, $i = 1,\ldots,4$, such that for all n,m

$$W_{f_1(n)} = \{x\colon\ (\forall w)(<w,x> \in W_n)\}$$

$$W_{f_2(n)} = \{x\colon\ (\exists m)(<m,x> \in W_n)\}$$

$$W_{f_3(<n,m>)} = W_n \cup W_m$$

$$W_{f_4(<n,m>)} = W_n \cap W_m$$

(ii) For every recursive $h: 2^{\omega} \to 2^{\omega}$ there exists recursive $h^*: \omega \to \omega$ such that for every n, $W_{h^*(n)} = \{x: h(x) \in W_n\}$.

(iii) $W_o = \{\langle x,n \rangle: (x,n) \in W\}$.

Given $y \in 2^{\omega}$, $n \in \omega$ let $W_n^y = \{x: \langle y,x\rangle \in W_n\}$. The following well-known "uniform boundedness lemma" is the key to our proof of (8). It is originally due to Moschovakis. The reader should have no difficulty in transferring the proof of Lemma 9 in [34] to our context.

(5) Assume $(\leq, Q, Q')$ is a Π_1^1-prewellordering on W_o. There is a recursive function $b: 2^{\omega} \to 2^{\omega}$ such that for every $y \in 2^{\omega}$, $n \in \omega$, if $-W_n^y \subseteq W_o$ then $b(\langle y,n\rangle) \in W_o$ and $-W_n^y \subseteq \{z: b(\langle n,y\rangle) \not\leq z\}$.

The next lemma is the central part of the argument establishing (8).

(7) Suppose $E = \{(x,y): \langle x,y\rangle \notin W_{k_o}\}$ is a Σ_1^1 equivalence such that W_o is E-invariant. Then there exists an E-invariant Π_1^1 prewellordering on W_o.

<u>Proof of (7).</u>

Let $(\leq, Q, Q')$ be an ordinary Π_1^1 prewellordering on W_o and suppose $Q' = \{(x,y): \langle xy\rangle \notin W_{k_1}\}$. It follows from our uniformity assumptions (i-iii) on W that there is a recursive function $h: \omega \to \omega$ such that

$$-W_{h(0)} = \{\langle\alpha,y\rangle: (\exists z)(zEy \ \ \& \ \ (z,x) \in Q')\}$$
$$= \{\langle\alpha,y\rangle: (\exists z)(\langle z,y\rangle \notin W_{k_o} \ \ \& \ \ \langle z,x\rangle \notin W_{k_1})$$

and for all $i \in \omega$

$$-W_{h(i+1)} = \{<x,y>: \ (\exists z,w)(zEy \ \& \ (w,x) \in Q' \ \& \ \bigvee_{j<i} (z,b(\ h(j),w\ )) \in Q')\}$$

$$= \bigcup_{j\leq i} \{<x,y>: \ (\exists zw)(<z,y> \notin W_{k_o} \ \& \ <w,x> \notin W_{k_1}$$

$$\& \ <z,b(<h(j),w>)> \notin W_{k_1}\}.$$

Define $f: (\omega \times 2^{\omega}) \to 2^{\omega}$ by the recursive conditions:

$$f(0,x) = x, \qquad f(i+1,x) = b(<h(i),f(i,x)>).$$

Define

$$\preceq = \{(xy): \ (\exists i \in \omega)(x \leq f(i,y))\}$$

$$\mathcal{Q} = \{(x,y): \ (\exists i \in \omega)((x,f(i,y)) \in Q)\}$$

$$\mathcal{Q}' = \{(x,y): \ (\exists i \in \omega)((x,f(i,y)) \in Q')\}.$$

Since f is recursive, $\mathcal{Q}$ and $\mathcal{Q}'$ are respectively Π^1_1 and Σ^1_1. We claim that $(\preceq,\mathcal{Q},\mathcal{Q}')$ is an E-invariant Π^1_1-prewellordering:

Using the defining property of b, one easily verifies by induction that

(i) $(\forall i \in \omega)(\forall w \in 2^{\omega})(w \in W_o \Rightarrow b(h(i),w) \in W_o)$.

A second induction using (i) shows

(ii) $(\forall i \in \omega)(\forall x \in 2^{\omega})(x \in W_o \Rightarrow f(i,x) \in W_o)$.

Thus if $x \in W_o$ and $(y,x) \in \mathcal{Q}$ or $(y,x) \in \mathcal{Q}'$, then $y \in W_o$, and it follows that $\preceq = \mathcal{Q} \cap W_o^2 = \mathcal{Q}' \cap W_o^2$. Also, if $x \in W_o$ and yEx then $y \in -W^x_{h(0)}$, hence $y \leq f(1,x)$ and $y \preceq x$. Thus $\preceq$ is E-invariant. $x \leq y$ implies $x \preceq y$ so $\preceq$ is well-founded and connected.

It remains only to show that $\preceq$ is <u>transitive</u>. The transitivity of $\preceq$ will follow from

(6) Suppose $i < j < \omega$, $x,y \in W_o$ and $f(i,x) \leq f(j,y)$. Then $f(i+1,x) \leq f(j+1,y)$.

Proof of (6).

We must show $f(i+1,x) \leq f(j+1,y) = b(\langle h(j),f(j,y)\rangle)$. It suffices to show that $f(i+1,x) \in -W^{f(j,y)}_{h(j)}$, i.e. that there exist $k < j$, z, w such that

$$z \; E \; f(i+1,x) \text{ and } (w,f(j,y)) \in Q' \text{ and } (z,b(\langle h(k),w\rangle)) \in Q'.$$

This condition is satisfied if we choose $z = f(i+1,x)$, $w = f(i,x)$, $k = i$. (6) follows.

Now to verify that $\preceq$ is transitive, suppose $x \preceq y$ and $y \preceq z \in W_o$; say $x \leq f(i,y)$ and $y \leq f(j,z)$. Since $y \leq f(1,y)$ we may assume $j > 0$. By repeated application of (6) we obtain $f(i,y) \leq f(j+i,z)$. Then $x \leq f(j+i,z)$ so $x \preceq z$. This completes the proof of (7).

Proof of (8).

Suppose ρ is finite, $E \subseteq X^2_\rho$ is a Σ^1_1 equivalence, and $A \in \Pi^1_1(X_\rho)$ is E-invariant. Let $f\colon 2^\omega \to X_\rho$ be a recursive surjection. Then $f^{-1}(A) \in \Pi^1_1(2^\omega)$, say $f^{-1}(A) = W_n$. W_o is invariant under the equivalence $E' = \{(x,y)\colon x = y \text{ or } (\exists z,w)(x = \langle z,n\rangle \;\&\; y = \langle w,n\rangle \;\&\; f(z) \; E \; f(w)\}$. Applying (7), let $(\leq,Q,Q')$ be an E'-invariant Π^1_1-

prewellordering on W_o. Define

$\mathcal{Q} = \{(R,S) \in X_\rho^2: (\forall x,y \in 2^\omega)((f(x) = R \ \& \ f(y) = S) \Rightarrow (x,y) \in Q)\}$

$\mathcal{Q}' = \{(R,S) \in X_\rho^2: (\exists x,y \in 2^\omega)(f(x) = R \ \& \ f(y) = S \ \& \ (x,y) \in Q')\}$.

It is easily checked that $(\mathcal{Q} \cap A^2, \mathcal{Q}, \mathcal{Q}')$ is a Π_1^1 prewellordering on A. This completes part 1 of 2.1(a).

<u>Proof of</u> 2.1(a), <u>part</u> 2.

Let $\underline{F}$, $\underline{V}$ be the binary relation symbols (1,(0,2)) and (1,(1,2)) respectively and let $\rho_o = \{\underline{F},\underline{V}\}$. We will be concerned with members (F,V) of X_{ρ_o} such that for some $\rho \in HC$, F "codes" a pair of ρ-names and V "codes" a member of X_{ρ_o} via F. The material of this section is closely related to Vaught's "$\mathcal{L}$-logic" as found in [44]. The present situation is simpler than that of [44] in that we need only consider satisfaction for applied propositional logic (i.e. names). It is more complicated than that of [44] in that we must define not only a universal $\underset{\sim}{\Pi}_1^1$ set but a universal $\underset{\sim}{\Sigma}_1^1$ equivalence.

We first collect some helpful observations. When ρ is a similarity type and ϕ is a Borel ρ-name, at(ρ) and sub(ϕ) respectively denote the set of atomic ρ-names and the set of subnames (subformulas) of ϕ. Given $R \in X_\rho$ let V_R: at(ρ) $\to \{0,1\}$ be the characteristic function of R with respect to at(ρ). V is a ρ-valuation if $(\exists R \in X_\rho)(V = V_R)$. Note that V is a ρ-valuation if and only if V is a function on at(ρ) to 2 and for every $\underline{c} \in C_\rho$ there is a unique $i \in \omega$ such that $(\underline{c} \equiv i, 1) \in V$. Thus, it is apparent that

(9) There is a 1-formula Val $\varepsilon\ L_{\omega\omega}(\rho_o)$ such that for any similarity type ρ, if $\mathcal{A}$ is a transitive set which contains at(ρ) and $V \subseteq \mathcal{A}^2$ then $(\mathcal{A},\varepsilon\ V,at(\rho)) \models$ Val if and only if V is a ρ-valuation.

If V is a ρ-valuation let R_V be the unique $R\ \varepsilon\ X_\rho$ such that $V = V_R$. Let $\underline{B} = (1,(0,1))$. By considering the natural inductive definition of "$R\ \varepsilon\ [\phi]$" it is apparent that

(10) There is a 2-formula Sat $\varepsilon\ L_{\omega\omega}(\rho_o \cup \{\underline{B}\})$ such that for any similarity type ρ and Borel-ρ-name ϕ, if $\mathcal{A}$ is a transitive set which contains both at(ρ) and sub(ϕ), $V \subseteq \mathcal{A}^2$ is a ρ-valuation, and $B \subseteq \mathcal{A}$, then $(\mathcal{A},\varepsilon,V,B,at(\rho),sub(\phi)) \models$ Sat if and only if $B = \{\psi\ \varepsilon\ sub(\phi): R_V\ \varepsilon\ [\psi]\}$.

Given $a\ \varepsilon$ HC, suppose $F\ \varepsilon\ 2^{\omega\times\omega}$ is such that $(\omega,F) \cong (TC(\{a\}),\varepsilon)$. Then we say that F is a <u>representing relation</u> and that F <u>represents</u> a. We further specify that $TC(\{a\}) = i(F)$. Let $i_F: (\omega,F) \to (i(F),\varepsilon)$ be the unique isomorphism. For $b\ \varepsilon\ i(F)$, $c \subseteq i(F)$ we specify $b^F = i_F^{-1}(b)$, $c^{(F)} = \{ b^F : b\ \varepsilon\ c\}$.

It is not generally possible to effectively associate to each $a\ \varepsilon$ HC a specific structure which represents a. We do however have the following approximation:

(11) There is a prim(ω) function F_o such that for any $a\ \varepsilon$ HC, $F_o(a)$ is a Borel name for $\{F\ \varepsilon\ X_{\{\underline{F}\}}: F$ represents a$\}$.

Proof.

Consider the prim functions $a \mapsto \theta'_a$, $a \mapsto \theta''_a$ defined by the conditions:

$$\theta''_a(\underline{v}_o) = \bigwedge_{b\varepsilon a} (\exists ! \underline{v}_1)(\underline{F}(\underline{v}_1 \underline{v}_o) \wedge \theta''_b(\underline{v}_1)) \wedge (\forall \underline{v}_1)(\underline{F}(\underline{v}_1, \underline{v}_o) \to \bigvee_{b\varepsilon a} \theta''_b(\underline{v}_1))$$

$$\theta'_a = (\exists ! \underline{v}_o)(\forall \underline{v}_1)[(\underline{F}(\underline{v}_1 \underline{v}_o) \vee \underline{v}_1 \approx \underline{v}_o) \wedge \theta''_a(\underline{v}_o)].$$

Then for any set a, $\mathrm{Mod}(\theta'_a) = \{\mathfrak{A} : \mathfrak{A} = (TC(\{a\}), \varepsilon)\}$. Let $F_o(a)$ be $(\theta'_a)^N$ where $\theta \mapsto \theta^N$ is the prim(ω) function which replaces variables by numerical constants which was defined in the introduction. Clearly F_o has the required property. □

Given $b \in HC$, let $\theta_b(\underline{n})$ denote $(\theta''_b(\underline{v}_o))^N(\frac{o}{\underline{n}})$. Note that if F is a representing relation and $b \in i(F)$, then $F \in [\theta_b(\underline{n})]$ if and only if $n = b^F$.

Now suppose F is a representing structure, $at(\rho) \in i(F)$, $R \in X_\rho$. Define $V_{RF} = \{(\phi^F, i^F) : (\phi, i) \in V_R\}$. In this case we say $V = V_{RF}$ is a ρ-F-valuation and specify $R = R_{VF}$. R_{VF} is the unique $R \in X_\rho$ satisfying the condition

$$\bigwedge_{n,m\varepsilon\omega} [V \in [\underline{V}(\underline{n},\underline{m})] \Leftrightarrow \bigvee_{\phi\varepsilon at(\rho)} ((F \in [\theta_\phi(\underline{n}) \wedge \theta_1(\underline{m})] \wedge R \in [\phi])$$

$$\vee (F \in [\theta_\phi(\underline{n}) \wedge \theta_o(\underline{m})] \wedge R \in [\neg\phi]))].$$

From this expression and (9) it is apparent that

There is a prim(ω) function F_1 such that if F is a

(12) representing structure, $\rho \in HC$ and $at(\rho) \in i(F)$ then

$F_1(\rho)$ is a Borel $\rho_o + \rho$-name such that for $V \in 2^{\omega\times\omega}$,

$R \in X_\rho$, $(F,V,R) \in \lceil F_1(\rho) \rfloor$ if and only if V is a ρ-F-valuation and $R = R_{FV}$.

Also note

(13) If $(\omega,F,V) = (\omega,F',V')$, F is a representing structure and V is a ρ-F-valuation, then the same is true of F' and V', and $R_{VF} = R_{V'F'}$.

One final remark is needed for our construction of the universal equivalence. If $A \subseteq X^2_{\rho_o}$ is an arbitrary relation, then E_A, the smallest equivalence relation which includes A, may be obtained by setting

$$A' = A \cup \{(x,x)\colon x \in X_{\rho_o}\} \cup \{(x,y)\colon (y,x) \in A\}$$

and then defining

$$E_A = \{(x,y)\colon (\exists n \in \omega)(\exists x_1,\ldots,x_n)(x_1 = x \ \& \ x_n = y \ \& \ (\forall m < n)((x_m,x_{m+1}) \in A'))\}.$$

It is apparent that

(14) If $A \in \Sigma^1_1(X_{\rho_o})$ then $E_A \in \Sigma^1_1(X_{\rho_o})$.

Now we are in a position to define the universal equivalence.

Given ρ let $\hat{\rho} = \rho^\rho$ (so $\rho + \rho = \rho \cup \hat{\rho}$); given $\underline{S} = (i,(a,n)) \in \rho$, let $\hat{\underline{S}} = (i,((\rho,a),n))$ be the corresponding symbol in $\hat{\rho}$.

Let $\theta(v_1,\ldots,v_8) \in L_{\omega\omega}(\{\underline{F}\})$ be such that $\mathrm{Mod}(\theta) = \{(A,F,a_1,\ldots,a_8)\colon (A,F)$ is transitive and extensional and has a maximal element m such that $(A,F) \models m = (a_1,\ldots,a_8)\}$.

Let $\mathrm{ext}(\underline{W},\underline{V})$ be $(\underset{\sim}{\forall}\, \underline{v}_o\underline{v}_1)(\underline{V}(\underline{v}_o,\underline{v}_1) \rightarrow \underline{W}(\underline{v}_o,\underline{v}_1))$. Let

$$A = [\![(\underset{\sim}{\exists}\{\underline{B},\underline{W}\})(\underset{\sim}{\exists}v_1,\ldots,v_8)([\theta \wedge (\underset{\sim}{\forall} vw)(\underline{F}(v,w) \leftrightarrow \hat{\underline{F}}(v,w)) \wedge \mathrm{Val}(\underline{F},\underline{V},v_1) \wedge \mathrm{Val}(\hat{\underline{F}},\hat{\underline{V}},v_3)] \rightarrow [\mathrm{Val}(\underline{F},\underline{W},v_4) \wedge \mathrm{ext}(\underline{W},\underline{V}) \wedge \mathrm{ext}(\underline{W},\hat{\underline{V}}) \wedge \mathrm{Sat}(F,W,B,v_4,v_8) \wedge \underline{B}(v_6)]\!]$$

Then:

(15) If F represents $(\mathrm{at}(\rho),\mathrm{at}(\rho+\rho_1),\mathrm{at}(\hat{\rho}),\mathrm{at}((\rho \cup \hat{\rho})+\rho),\phi,\psi,$ $\mathrm{sub}(\phi), \mathrm{sub}(\psi))$ where ψ is a Borel $(\rho \cup \hat{\rho})+\rho_2$-name, V is a $\hat{\rho}$-F-valuation and V' is a ρ-F-valuation, then $(F,V,F,V') \in A$ if and only if $(R_{FV},R_{FV'}) \in [(\underset{\sim}{\exists}\rho_2)\psi]$.

Clearly $A \in \Sigma^1_1(X_{\rho_o+\rho_o})$. Let E_1 be the smallest equivalence which includes A, and let E be the smallest equivalence which contains $E_1 \cup \{(F,V,F',V'): (\omega,F,V) = (\omega,F',V')\}$. By (14), $E \in \Sigma^1_1(X_{\rho_o+\rho_o})$. Suppose F, ψ, V, V' are as in (15) and moreover that $[(\underset{\sim}{\exists}\rho_2)\psi]$ is an equivalence. Then $(F,V,F,V') \in E_1$ if and only if $(R_{FV},R_{FV'}) \in [(\underset{\sim}{\exists}\rho_2)\psi]$. Using this observation together with (13) we obtain

(16) Assume the hypothesis of (15) and additionally that $E_F = [(\underset{\sim}{\exists}\rho_2)\psi]$ is an equivalence. Then $(F,V,F,V') \in E$ if and only if $(R_{FV},R_{FV'}) \in E_F$.

Next we define the very universal set $\mathcal{U} \subseteq X_{\rho_o}$. Let $\mathcal{U}' = [\![(\underset{\sim}{\forall}\{\underline{W},\underline{B}\})(\underset{\sim}{\exists}v_1,\ldots,v_8)[\theta \wedge \mathrm{Val}(\underline{F},\underline{V},v_1) \wedge ((\mathrm{Val}(\underline{F},\underline{W},v_2) \wedge \mathrm{ext}(\underline{W},\underline{V}) \wedge \mathrm{Sat}(\underline{F},\underline{W},\underline{B},v_2,v_7)) \rightarrow \underline{B}(v_5))]\!]$. Let $\mathcal{U} = (\mathcal{U}')^{-E}$. It is apparent that

$\mathcal{U}'$ and hence $\mathcal{U}$ also, is $\Pi^1_1(X_\rho)$.

(17) Assume the hypothesis of (16) and additionally that ϕ is a Borel $\rho + \rho_1$ name and $[(\forall\rho_1)\phi] = A_F$ is E_F-invariant. Then $(F,V) \in \mathcal{U}$ if and only if $R_{FV} \in A_F$.

Proof.

By (9), (10) and the definition of $\mathcal{U}'$, $(F,V) \in \mathcal{U}'$ if and only if $R_{FV} \in A_F$. The conclusion of (17) follows by (16) and the assumption that A_F is E_F invariant. ◻

Now we can prove our main result on $\underset{\sim}{\Pi}^1_1$-names. For the reader's convenience we restate it:

(18) There is a prim(ω) function P_1 such that if $\rho \in HC$ is a similarity type, Ψ is a $\underset{\sim}{\Sigma}^1_1$-$\rho+\rho$-name for an equivalence on X_ρ and Φ is a $\underset{\sim}{\Pi}^1_1$-name for a $[\Psi]$-invariant set, then $\underset{\sim}{P}_1(\rho,\Psi,\Phi)$ is an ordered pair of names which witness a $[\Psi]$-invariant $\underset{\sim}{\Pi}^1_1$-prewellordering on $[\Phi]$.

Proof.

Applying (8) suppose $(\leq, Q, Q')$ is an E-invariant Π^1_1 prewellordering on $\mathcal{U}$.

Let F_2 be the prim(ω) map

$$F_2\colon (\rho, (\underset{\sim}{\exists}\rho_1)\psi, (\underset{\sim}{\forall}\rho_2)\phi)) \mapsto (at(\rho), at(\rho+\rho_1), at(\hat{\rho}), at(\rho \cup \hat{\rho}+\rho_2), \phi, \psi, sub(\phi), sub(\psi)).$$

Given ρ, $\Psi = (\underset{\sim}{\exists}\rho_1)\psi$, $\Phi = (\underset{\sim}{\forall}\rho_1)\psi$ we (uniformly) define

$$\mathcal{Q}_{(\rho,\Psi,\Phi)} = \{(R,R') \in X_{\rho+\rho} : (\forall F \in [F_o(F_2(\rho,\Psi,\Phi)])((F,V_{RF}),(F,V_{R'F})) \in Q\}$$

$$\mathcal{Q}'_{(\rho,\Psi,\Phi)} = \{(R,R') \in X_{\rho+\rho} : (\exists F \in [F_o(F_2(\rho,\Psi,\Phi))])((F,V_{RF}),(F,V_{R'F})) \in Q'\}$$

It is straightforward using (12) to define prim(ω) maps $P_1^{\,1}$, $P_1^{\,2}$ such that for all $s = (\rho,\Psi,\Phi)$ as in (18), $P_1^{\,1}(s)$ is a $\underset{\sim}{\Pi}^1_1$-name for $\mathcal{Q}_s$, $P_1^{\,2}(s)$ is a $\underset{\sim}{\Sigma}^1_1$-name for $\mathcal{Q}'_s$.

Fix $s = (\rho,\Psi,\Phi)$ as in (18). Let $\mathcal{Q} = \mathcal{Q}_s$, $\mathcal{Q}' = \mathcal{Q}'_s$, $\preceq\, = \mathcal{Q} \cap [\Phi]^2$. We claim that $(\preceq,\mathcal{Q},\mathcal{Q}')$ is a $\underset{\sim}{\Pi}^1_1$-prewellordering on $[\Phi]$.

From the corresponding properties of $(\leq,Q,Q')$, it is immediately apparent that $\mathcal{Q}' \cap [\Phi]^2$ is connected and reflexive, $\mathcal{Q} \cap [\Phi]^2$ is well founded and transitive, and $\mathcal{Q} \cap X_\rho \times [\Phi] \subseteq \mathcal{Q}' \cap X_\rho \times [\Phi]$. Also, if $R' \in [\Phi]$ and $(R,R') \in \mathcal{Q}'$ as witnessed by F, then $(F,V_{R'F}) \in \mathcal{U}$, $((F,V_{RF}),(F,V_{R'F})) \in Q'$, hence $(F,V_{RF}) \in \mathcal{U}$ and $R \in [\Phi]$. It remains only to show that $\mathcal{Q}' \cap [\Phi]^2 \subseteq \mathcal{Q} \cap [\Phi]^2$. Suppose $R_1, R_2 \in [\Phi]$ and $(R_1,R_2) \in \mathcal{Q}'$ as witnessed by F. If F' is any member of $[F_o(F_2(\rho,\Psi,\Phi))]$ then $(\omega,F') = (\omega,F)$, hence $(\omega,F,V_{R_iF}) = (\omega,F',V_{R_iF'})$, $i = 1,2$. Since $\leq$ is E-invariant,

$$(F',V_{R_1F'}) \leq (F,V_{R_1F}) \leq (F,V_{R_2F}) \leq (F',V_{R_2F'})$$

hence $((F',V_{R_1F'}),(F',V_{R_2F'})) \in Q'$. Thus, $(R_1,R_2) \in \mathcal{Q}$ as required.

Finally, suppose $(R_1,R_2) \in [\Psi] \cap [\Phi]^2$, and $F \in [F_o(F_2(\rho,\Psi,\Phi))]$. Then by (16), $((F,V_{R_1F},F,V_{R_2F})) \in E$ and, since $\leq$ is E-invariant, $((F,V_{R_1F}),(F,V_{R_2F})) \in Q'$ so $(R_1,R_2) \in \mathcal{Q}'$. Thus, $\preceq$ is $[\Psi]$-invariant and the prim function $P_1 = (P_1^{\,1},P_1^{\,2})$ satisfies the requirements of (18). The proof of 2.1(a) is complete. $\square$

In contrast to the above proof, the argument which establishes the $\underset{\sim}{\Sigma}_2^1$ case of 2.1 is quite direct. We simply "invariantize" the classical derivation of $\underset{\sim}{\Sigma}_2^1$ prewellordering from $\underset{\sim}{\Pi}_1^1$ prewellordering. This proof appeared in [11].

Proof of 2.1(b).

Suppose $E \in \underset{\sim}{\Sigma}_1^1(X_{\rho+\rho})$ is an equivalence relation, $B \in \underset{\sim}{\Pi}_1^1(X_{\rho+\rho_1})$, and $A = \{R: (\exists S \in X_{\rho_1})((R,S) \in B))\}$ is E-invariant. Given a $\underset{\sim}{\Pi}_1^1$ prewellordering $(\leq q, q')$ on B with associated norm $\xi: B \to ON$, we define an E-invariant $\underset{\sim}{\Sigma}_2^1$ prewellordering on A as follows:

$$\preceq\, = \{(R_1,R_2): \min\{\xi(R,S): R\, E\, R_1 \;\;\&\;\; (R,S) \in B\} \leq \min\{\xi(R,S): R\, E\, R_2 \;\;\&\;\; (R,S) \in B\}\}$$

$$Q = \{(R_1,R_2): (\exists R,S)[R\, E\, R_1 \;\;\&\;\; (R,S) \in B \;\;\&\;\; (\forall R',S')((R'\, E\, R_2 \;\;\&\;\; ((R',S'),(R,S)) \in q')) \to ((R,S),(R',S')) \in q)]\}$$

$$Q' = \{(R_1,R_2): (\forall R',S')[(R'\, E\, R_2 \;\;\&\;\; (R',S') \in B) \to (\exists R,S)(R\, E\, R_1 \;\;\&\;\; ((R,S),(R',S')) \in q')]\} .$$

It is apparent that Q is $\underset{\sim}{\Sigma}_2^1$, Q' is $\underset{\sim}{\Pi}_2^1$ and that appropriate names for Q and Q' can be (uniformly) primitive recursively obtained from names for E, B, q, and q'. Any $\underset{\sim}{\Sigma}_2^1$ name for A directly yields a $\underset{\sim}{\Pi}_1^1$-name for a suitable B, so the conclusion of 2.1 will follow from (18), (or 4.8 of Vaught [44]), once we show that $(\preceq,Q,Q')$ is an E-invariant $\underset{\sim}{\Sigma}_2^1$-prewellordering.

Suppose $R_1 \mathrel{E} R_2$, $R_2 \in A$. Then for every R,S,

$$(R \mathrel{E} R_1 \ \& \ (R,S) \in B) \Rightarrow (R \mathrel{E} R_2 \ \& \ (R,S) \in B)$$

hence $R_1 \preceq R_2$, so $\preceq$ is E-invariant.

Transitivity, connectedness, and well-foundedness for $\preceq$ are immediate from the definition and the fact that image(ξ) $\in$ ON. Finally, suppose $R_2 \in A$ and $(R_1,R_2) \in Q$ as witnessed by R, S. Then $R \in A$, hence $R_1 \in A$, and if R_o,S_o are such that $\xi(R_o,S_o) = \min\{\xi(R',S'): R' \mathrel{E} R_2 \ \& \ (R',S') \in B\}$, then

$$(\xi(R_o,S_o) \le \xi(R,S) \Rightarrow ((R_o,S_o),(R,S)) \in q').$$

It follows by the definition of Q that

$$(\xi(R_o,S_o)) \le \xi(R,S) \Rightarrow \xi(R,S) \le \xi(R_o,S_o).$$

Thus, $(R_2 \in A \ \& \ (R_1,R_2) \in Q) \Rightarrow (R_2 \in A \ \& \ R_1 \preceq R_2)$.

Similar calculations complete the proof that $\preceq = Q \cap X_\rho \times A = Q' \cap X_\rho \times A$. Thus, $(\preceq,Q,Q')$ is a $\underset{\sim}{\Sigma}^1_2$-prewellordering on A and the proof of 2.1 is complete. □

Remarks

I. On 2.3

Our proof of the $\underset{\sim}{\Pi}^1_1$ case of 2.1 (including (5)) depended on (i) The ordinary Π^1_1 prewellordering theorem, (ii) the existence of the "canonical" complete Π^1_1 set W_o and (iii) the construction of $\mathcal{U}$ and $\mathcal{E}$ carried out in Part 2. It is well-known that similar canonical complete Π^1_n sets exist for all $n \in \omega$. The construction

(iii) is easily modified to yield a "very universal" Π^1_n set and Σ^1_n equivalence for each n — one merely adds suitable alternating quantifications over valuations in the definitions. — For example, in the case $n = 2$, one would consider F's which represent sets of the form $(at(\rho), at(\rho+\rho_1), at(\rho+\rho_1+\rho_2), at(\hat{\rho}), at(\rho\cup\hat{\rho}+\rho_3), at(\rho\cup\hat{\rho}+\rho_3+\rho_4),$ $\phi, \psi, sub(\phi), sub(\psi))$ to discuss names of the form $\underset{\sim}{\forall}\rho_1\underset{\approx}{\exists}\rho_2\phi$, $\underset{\approx}{\exists}\rho_3\underset{\sim}{\forall}\rho_4\psi$. Since our proof of the $\underset{\sim}{\Sigma}^1_2$ case of 2.1 depended only on the $\underset{\sim}{\Pi}^1_1$ case, we can carry out the complete argument (for 2.1) for larger n, provided only that a suitable analog of (i) holds.

II. In [10] Burgess gave a second proof of 2.2(a). This argument also yields the "lightface" result (8). The two distinct arguments for (8) fall naturally into the pattern established by previous proofs of related results. Thus, Burgess' proof of (8) — like Vaught's proof of invariant separation and reduction theorems in [44] and our proof above of invariant $\underset{\sim}{\Sigma}^1_2$ prewellordering and reduction — proceeds by invariantizing a proof of the analogous classical theorem. Solovay's proof of (8) — like the Ryll-Nardzewski proof [46] of invariant (strong) 1st separation, and both proofs in [11] of $\underset{\sim}{\Pi}^1_1$ reduction for pairs — derives the invariant theorem directly from the classical result (as usual by an ω-sequence argument). Burgess' argument appears to be somewhat shorter. Our argument gives a single proof for both the $\underset{\sim}{\Pi}^1_1$ case and for the results on PD. Furthermore, assuming the possibility of, say, $\underset{\sim}{\Pi}^1_3$ reduction without $\underset{\sim}{\Delta}^1_2$ determinacy, our argument gives a slightly stronger result (Cor. 2.3).

III. It follows immediately from 2.2(a) that

(19) If X is a Suslin space, $E \subseteq X^2$ is a $\underset{\sim}{\Sigma}_1^1$ equivalence on X, and A is a E-invariant $\underset{\sim}{\Pi}_1^1$ set, then A is a union of ω_1 invariant Borel sets.

This fact has a simple proof from the ordinary boundedness theorem (cf. Kuratowski [26] 39 VIII) as follows:

Let A, X, E be as in (19). Let $(\leq, Q, Q')$ be a $\underset{\sim}{\Pi}_1^1$-prewellordering on A with associated norm ξ. The <u>constituents</u> B_α, $\alpha \in \omega_1$, of A are defined by setting $B_\alpha = \{x: \xi(x) < \alpha\}$. Each B_α is Borel and $A = \bigcup_{\alpha < \omega_1} B_\alpha$. It suffices to show that $\{\alpha: B_\alpha$ is invariant$\}$ is cofinal in ω_1. Let $\alpha_o \in \omega_1$. Since E is $\underset{\sim}{\Sigma}_1^1$, $B_{\alpha_o}^+$ is $\underset{\sim}{\Sigma}_1^1$ and by the boundedness theorem, $B_{\alpha_o}^+ \subseteq B_{\alpha_1}$ for some $\alpha_1 \in \omega_1$. Inductively chose α_i, $i \in \omega$, such that $B_{\alpha_i}^+ \subseteq B_{\alpha_{i+1}}$. Let $\alpha = \bigcup_{i\in\omega} \alpha_i$. Then $B_\alpha = \bigcup_{i \in \omega} B_{\alpha_i}^+$ is invariant and $\alpha \geq \alpha_o$. Since α_o was arbitrary, (19) is proved.

This proof contains the essential ω-chain construction which is central to the proof of (8). After proving (19) the author learned of the effective boundedness theorem and conjectured that it could be used to prove invariant $\underset{\sim}{\Pi}_1^1$-prewellordering. He discovered an argument for deriving 2.2(a) from a proposed "improvement" of the effective boundedness theorem. Solovay then showed that this "improvement" was untenable and gave a correct proof corresponding to our (7) and (8) above. A short time later, Burgess discovered the argument of [10] for the same result.

IV. In the classical theory one uses the ordinary analogue of (19) and the $\underset{\sim}{\Pi}_1^1$ uniformization theorem to show that every $\underset{\sim}{\Sigma}_2^1$ set is a union of ω_1 Borel sets. Although it is true for Polish actions (see Vaught [46]), the corresponding strengthening of (19) does not hold in general. To see this, let X be Polish, $A \in \underset{\sim}{\Sigma}_1^1(X) - \underset{\sim}{\Pi}_1^1(X)$, and define $E = \{(x,y): x = y \text{ or } x,y \in A\}$. Clearly, A is E-invariant $\underset{\sim}{\Sigma}_2^1$ but A is a single equivalence class and cannot be a union of invariant Borel sets. This example also shows that the invariant uniformization principle does not hold in general: If $A = \pi_1(B)$ for some $B \subseteq X \times Y$ then there is no $\underset{\sim}{\Pi}_1^1$ set B' which $E \times \underset{\sim}{1}$-uniformizes B. (If there were, we could apply (19) to write B' as a union of $E \times \underset{\sim}{1}$-invariant Borel sets and hence, to write A as a union of E-invariant Borel sets, which is impossible).

§ 3. Strong Well-orderings and the Invariant Uniformization Principle

In the preceding section we showed that invariant $\underset{\sim}{\Pi}^1_1$ and $\underset{\sim}{\Sigma}^1_2$ uniformization principles hold for certain product equivalences on Suslin spaces of the form $X \times \omega$. It is natural to ask whether these results can be extended to spaces of the form $X \times 2^\omega$ in analogy with the non-invariant theory, or to a larger class of equivalences on spaces $X \times \omega$ (such as the collection of equivalences induced by product actions).

As we will remark below, such extensions are impossible for $\underset{\sim}{\Pi}^1_1$ or for any projective class Γ such that every Γ-subset of 2^ω is almost open. If we assume the axiom of constructibility, however, we can obtain positive results for $\Gamma = \underset{\sim}{\Sigma}^1_n$, $n \geq 2$ in full analogy with the well-known theorem of Addison [1].

The main results of this section (3.1a,3.2,3.3) were obtained jointly with John Burgess and appeared in Burgess-Miller [11]. An unpublished result very close to 3.1 was presented at a Berkeley colloquium in 1972 by K. Kuratowski. He showed that the existence of a $\underset{\sim}{\Sigma}^1_k$ (not necessarily strong) well-ordering of 2^ω implies the existence of a $\underset{\sim}{\Sigma}^1_k$ selector for any $\underset{\sim}{\Sigma}^1_1$ equivalence relation such that every equivalence class is countable.

For $x \in 2^\omega$ and $i \in \omega$, we define $(x)_i \in 2^\omega$ by setting $(x_i)(m) = x(2^i(2m+1))$. We then define $((x)) = \{(x_i): i \in \omega\}$. A binary relation L on 2^ω is a $\underset{\sim}{\Sigma}^1_n$-__strong well-ordering__ provided L well orders 2^ω in type ω_1 and both L and $(L) = \{(x_o,x_1): ((x_1)) = \{x: x \mathrel{L} x_o\}\}$ are $\underset{\sim}{\Sigma}^1_n((2^\omega)^2)$.

The existence of a strong $\underset{\sim}{\Sigma}^1_2$ well-ordering of 2^ω follows from the axiom of constructibility $(V = L)$ by a theorem of Gödel and Addison (cf. Addison [11]). Silver has shown in [42] that the existence of a

Σ^1_3 strong well-ordering follows from the assumption that D is a normal ultrafilter on a measurable cardinal, and $V = L^D$. A recent theorem of Friedman and Mansfield states that if there exists a $\underset{\sim}{\Sigma}^1_2$ (not necessarily strong) well-ordering of 2^ω then $2^\omega \subseteq L[\alpha]$ for some $\alpha \in 2^\omega$ and hence, there exists a $\underset{\sim}{\Sigma}^1_2$-strong well-ordering.

If X is any set and E is an equivalence on X then a <u>selector</u> for E is a map $s: X \to X$ such that

(i) $(\forall x \in X)(s(x)\ E\ x)$

(ii) $(\forall x,y \in X)(x\ E\ y \Rightarrow s(x) = s(y))$.

<u>Theorem 3.1</u>. Assume that there is a $\underset{\sim}{\Sigma}^1_n$ strong well-ordering on 2^ω, $n \geq 2$. Let X be a Suslin space, $E \in \underset{\sim}{\Delta}^1_n(X^2)$ an equivalence on X. Then

(a) There exists $s \in \underset{\sim}{\Delta}^1_n(X^2)$ which is a selector for E.

(b) Let ϕ_1, ϕ_2 be $\underset{\sim}{\Sigma}^1_n$-names for L, (L) respectively. There is a $\text{prim}(\omega, \phi_1, \phi_2)$ map P such that if $\rho \in HC$ is a similarity type and ψ_1, ψ_2 are respectively a $\underset{\sim}{\Sigma}^1_n$-name and a $\underset{\sim}{\Pi}^1_n$-name for an equivalence E on X_ρ, then $P(\rho, \psi_1, \psi_2)$ is a $\underset{\sim}{\Sigma}^1_n$-name for a selector for E.

<u>Proof</u>.

(a) Let f be a Borel measurable function on 2^ω onto X. Given $x \in X$ define $s(x) = f(y)$, where y is the L-least element of $f^{-1}([x]_E)$. s is clearly a selector for E and since s has the explicit definition:

$$s = \{(x_1,x_2): (x_1,x_2) \in E \ \ \& \ \ (\exists y_1,y_2)(f(y_2) = x_2 \ \ \& \ \ (y_2,y_1) \in (L) \ \ \& \bigwedge_{m\in\omega} ((x_1, f((y_1)_m)) \notin E)\}$$

it is also clear that $s \in \underset{\sim}{\Sigma}^1_n(X^2)$. Since s is a function, $s \in \underset{\sim}{\Delta}^1_n(X^2)$.

(b) In view of the preceding argument it suffices to show that there is a prim(ω) function F such that $F(\rho)$ is a $\underset{\sim}{\Sigma}_1^1$ name for a function on 2^ω onto X_ρ whenever $\rho \in HC$ is a similarity type. We use the notation from §2. Let $g: x \mapsto (F_x, V_x)$ be a recursive bijection on 2^ω onto $X_{\rho_0} = 2^{\omega^2} \times 2^{\omega^2}$. Given $\rho \in HC$ let $R_{0\rho} \in X_\rho$ be the constant zero function. Let $f_\rho: 2^\omega \to X_\rho$ be defined by the equation

$$f_\rho(x) = \begin{cases} R_{V_x F_x} & \text{if } F_x \in [F_0(at(\rho))] \text{ and} \\ & (F_x^-, V_x, (at(\rho))^{F_x}) \in [\![Val]\!] \\ R_{0\rho} & \text{otherwise.} \end{cases}$$

It is apparent that each f_ρ maps 2^ω onto X_ρ. Using (9), (11), and (12) of §2, it is straightforward to define a prim(ω) function F such that for every ρ, $F(\rho)$ is a $\underset{\sim}{\Sigma}_1^1$-name for f_ρ. □

Corollary 3.2. Assume the hypothesis of 3.1 and $m \geq n$. Then

(a) X has the E-$\underset{\sim}{\Sigma}_m^1$-prewellordering property.

(b) The collection of E-invariant $\underset{\sim}{\Sigma}_m^1$ subsets of X has the reduction property.

Proof.

(b) follows from (a) and 1.1. To prove (a), let A be an invariant $\underset{\sim}{\Sigma}_m^1$ set and, (applying Addison's prewellordering theorem, cf. [21]), let $(\leq, Q, Q')$ be an ordinary $\underset{\sim}{\Sigma}_m^1$-prewellordering on A. Let s be the selector for E defined in 3.1. Define $(\precsim, q, q')$ by setting

$$\preceq = s^{-1}(\leq) = \{(x,y): s(x) \leq s(y)\}$$
$$q = s^{-1}(Q), \qquad q' = s^{-1}(Q').$$

It is easily seen that $(\preceq, q, q')$ is an E-invariant $\underset{\sim}{\Sigma}^1_m$-prewellordering on $s^{-1}(A) = A$. □

If $X = Y \times Z$ and E is an equivalence on X, then E is <u>coherent</u> provided that for all $y_o, y_1 \in Y$

$$(\exists z_o, z_1)((y_o,z_o)\ E\ (y_1,z_1)) \Rightarrow (\forall z_o)(\exists z_1)((y_o,z_o)\ E\ (y_1,z_1)).$$

Note that if E is coherent, then $E_1 = \{(y_o,y_1): (\exists z_o,z_1)((y_o,z_o) E (y_1,z_1))\}$ is an equivalence relation on Y. It is easily seen that every product equivalence and every equivalence which is induced by a product action is coherent.

<u>Corollary 3.3.</u>

Assume that there is a $\underset{\sim}{\Sigma}^1_{n+1}$ strong well-ordering on 2^ω, Y, Z are Suslin spaces, and E is a coherent $\underset{\sim}{\Sigma}^1_n$ equivalence relation on $Y \times Z$. Then for every $m > n$, $\underset{\sim}{\Sigma}^1_m(Y \times Z)$ has the E-uniformization property.

<u>Proof.</u>

Let $E' = \{(y_o,y_1): (\exists z_o,z_1)((y_o,z_o)\ E\ (y_1,z_1))\}$. Then $E' \in \underset{\sim}{\Sigma}^1_n(Y^2)$ and E' is an equivalence relation. Applying 3.1, let $s \in \underset{\sim}{\Sigma}^1_{n+1}(Y^2)$ be a selector for E'.

Let A be an E-invariant $\underset{\sim}{\Sigma}^1_m$ subset of $Y \times Z$. Applying Addison's uniformization theorem, let B' be an ordinary $\underset{\sim}{\Sigma}^1_m$ set which uniformizes A. Let

$B = \{(y,z): (\exists z_o)((y,z)\ E\ (s(y),z_o)\ \&\ (s(y),z_o) \in B'\}$.

Clearly B is $\underset{\sim}{\Sigma}^1_m$. If $(y,z) \in B$ and $(y,z)\ E\ (y',z')$, then $y\ E'\ y'$, so $s(y) = s(y')$. If $(y,z)\ E\ (s(y),z_o)$, then $(y',z')\ E\ (s(y'),z_o)$, so $(y',z') \in B$. Thus, B is E-invariant.

If $(y,z) \in B$ then $(y,z) \in (B')^{+E}$. $(B')^{+E} \subseteq A$, since $B' \subseteq A$ and A is E-invariant. Thus, $B \subseteq A$.

If $(y,z) \in A$, then, since E is coherent, $(s(y),z')\ E\ (y,z)$ for some z'. Since A is invariant, $(s(y),z') \in A$. Since B' uniformizes A, $(s(y),z_o) \in B'$ for some z_o. Again using the coherence of E, $(y,z_1)\ E\ (s(y),z_o)$ for some z_1. Then $(y,z_1) \in B$, so $\mathrm{Dom}(B) = \mathrm{Dom}(A)$.

Finally, if $(y,z), (y,z') \in B$, then for some z_o, $(y,z)\ E\ (s(y),z_o)$ and $(y,z')\ E\ (s(y),z_o)$. Thus, $(y,z)\ E\ (y,z')$ and B satisfies all the requirements of an E-uniformization. □

The reader should have no difficulty in extracting the obvious effective content of 3.2 and 3.3, as an application of 3.1(b).

V. Remarks and counterexamples.

In Remark III we gave an example of a $\underset{\sim}{\Sigma}^1_1$ product equivalence $E \times \underset{\sim}{1}$ on a Polish space $X \times Y$ and an invariant Borel set with no $\underset{\sim}{\Pi}^1_1$ $E \times \underset{\sim}{1}$-uniformization.

The invariant uniformization question was first raised by Vaught (cf. [44]) for the canonical logic spaces, and the first counterexamples to a general invariant $\underset{\sim}{\Pi}^1_1$ uniformization theorem for these spaces were given by Dale Myers in [36] and [38]. Myers' arguments were based on considerations of Baire category for the logic spaces, and it appears that measure and category are the key to one type of counter-example to invariant uniformization. In fact, the classical Vitalli construction of a non-measurable set of reals, which is based on a selector for the Borel equivalence $E_{\mathbb{Q}} = \{(x,x+q): x \in \mathbb{R},\ q \in \mathbb{Q}\}$, shows that the $E_{\mathbb{Q}} \times \underset{\sim}{1}$-invariant set $E_{\mathbb{Q}} \subseteq \mathbb{R}^2$ has no Lesbegue measurable $E_{\mathbb{Q}} \times \underset{\sim}{1}$-uniformization, (a fortiori, no $\underset{\sim}{\Pi}^1_1$ $E_{\mathbb{Q}} \times \underset{\sim}{1}$-uniformization).

By manipulating this example a bit, we will obtain a general method for constructing equivalence relations E on spaces of the form $Y = X \times \omega$, such that the set Y has no $\underset{\sim}{\Pi}^1_1$ (and assuming projective determinacy, no projective) E-uniformization.

We say that an action $\mathcal{J} = (X,G,J)$ is a <u>Vitalli action</u> if X is a Baire space, G is a countably infinite group, each $J^g: x \mapsto gx$ is continuous, and

(i) For every $x \in X$ and $g \in G$, if $g \neq \mathrm{id}$, the identity element of G, then $gx \neq x$.

(ii) For every non-empty open $U \subseteq X$, there exists a non-empty open set $V \subseteq U$ and $h \in G - \{\mathrm{id}\}$ such that $hV \subseteq U$.

If $\mathcal{J} = (X,G,J)$ is a Vitalli action, let $\mathcal{J}_V$ be the product action $\mathcal{J} \times \mathcal{T}$ of G on $X \times G$, where $\mathcal{T}$ is left translation.

If X is any non-meager topological group and G is a countable subgroup which is not discrete, then the action by left translation $(g,x) \mapsto gx$ is a Vitalli action. In particular, the Vitalli example,

$X = (\mathbb{R},+)$, $G = \mathbb{Q}$, is a Vitalli action. For an example closer to model theory, let $X = 2^{\omega}$, $G = S_q$, the set of finite sequences of 0's and 1's with addition as binary decimals reduced modulo 1, $J: (s,x) \mapsto s + x$ where addition is again as binary decimals reduced modulo 1. Then (X,G,J) is a Vitalli action.

Proposition 3.4. (compare Myers [38]). Suppose $\mathcal{J} = (X,G,J)$ is a Vitalli action. Then the set $X \times G$ has no $E_{\mathcal{J}_V}$-uniformization which is almost open in $X \times G$, G given the discrete topology.

Proof.

Let $E = E_{\mathcal{J}_V}$ and suppose A is an E-uniformization of $X \times G$. If $(x,g)\ E\ (x,h)$, then by (i), $h = g$, so A is a uniformization in the usual sense. A is $E_{\mathcal{J}_V}$-invariant, so $A = hA$ for all $h \in G$. For $g \in G$, define $A_g = \{x: (x,g) \in A\}$. It is apparent that $A_{hg} = hA_g$ for all $h,g \in G$.

Now assume for contradiction that A is almost open. Since G is discrete, every A_g is almost open in X. Fix $h \in G$.

If A_h is meager, then the same is true of each $A_g = A_{gh^{-1}h} = gh^{-1}A_h$. It follows that $X = \bigcup_{g \in G} A_g$ is meager, a contradiction.

If A_h is almost open and not meager, then $U - A_h$ is meager for some non-empty open set U. Choose a non-empty open set $V \subseteq U$ and $h' \neq \text{id}$ such that $h'V \subseteq U$. Then $h'V - A_h$ and $V - A_h$ are subsets of $U - A_h$, so both are meager, as is $h'(V - A_h) = h'V - A_{h'h}$. Since $h' \neq \text{id}$, $h'h \neq h$. Since A is a uniformization, $A_h \cap A_{h'h} = \emptyset$. Then $h'V \subseteq (h'V - A_h) \cup (hV - A_{h'h})$ is a meager open set, a second contradiction which proves the proposition. □

3.4 represents our candidate for the "urtheorem" underlying Myers' examples in [38]. Most (possibly all) of the examples in [38] can be represented as a Vitalli action where X is a subspace of a logic space and G is a quotient of the permutation group $\omega!$.

All these examples show that we cannot hope to have strong positive results about invariant uniformization without strong set theoretic hypotheses like those of 3.3. They further show that the invariant uniformization theorem for spaces $X \times \omega$ which we proved in 2.2(c) cannot be extended to arbitrary coherent equivalences or even to equivalences induced by a product of a pair of Polish actions on 2^ω and ω.

Chapter II: SPECIAL ACTIONS, SEMICONTINUOUS EQUIVALENCE RELATIONS AND THE *-TRANSFORM

We continue to study the various types of equivalence spaces which arise from consideration of the canonical logic actions. The first two sections deal primarily with Vaught's transform $B \mapsto B^* = \{x: \{g: J(g,x) \in B\}$ is comeager$\}$ which was introduced in [46]. B^* is defined whenever G is a topological space, X and X' are sets, J is a function on $G \times X$ to X', and $B \subseteq X'$. The transform appears to be most interesting when G is a non-meager topological group with a countable basis, $X = X'$ and $\mathcal{J} = (G,X,J)$ is an action. When, in addition, X is a topological space and J is continuous in each variable separately, we say $\mathcal{J}$ is a <u>special action</u>. Assuming a special action, Vaught showed

(1) For every $B \in \mathcal{B}(X)$, B^* is a Borel $E_{\mathcal{J}}$-invariantization of B.

In §1 we show that the same result holds under the weaker hypothesis, "X is a Borel space and J is Borel measurable." This result is partly due to Vaught -- see 1.2 below. It yields stronger versions of several of the main results in [46]. We also add to the list begun in [46] of the classes and properties preserved by the transform. We prove for example 1.5:

<u>Assume</u> $\mathcal{J}$ <u>is a special action</u>. <u>For every</u> $B \subseteq X$, <u>if</u> B <u>is almost open</u>, <u>then so is</u> B^*.

In §2 we are concerned with invariant separation for classes of Borel sets. We prove 2.3:

> If E is a lower semicontinuous equivalence on a completely metrizable space X, then the collection of E-invariant $\underset{\sim}{\Pi}^0_2$ sets has the first separation property.

This result is proved by invariantizing the strong version of the $\underset{\sim}{\Pi}^0_2$ separation theorem which involves the so-called "resolvable" sets. Our proof yields a construction principle for $\underset{\sim}{\Delta}^0_2$ sets in terms of invariant closed sets. Assuming X is Polish and E is induced by a special action, we use the *-transform to extend this invariant separation theorem to all the collections $\underset{\sim}{\Pi}^0_\alpha(X)$, $\alpha > 1$.

In §3 we leave the transform aside. We apply a theorem of Kuratowski and Ryll-Nardzewski to give a sufficient condition for the existence of a continuous selector for an equivalence on a Polish space. As we will show in chapter III §6, this result is closely related to the "Henkin method" of constructing a model from a complete theory.

§1. Some Remarks About the Transform $B \mapsto B^{*} = \{x : \{g : gx \in B\} \text{ is comeager}\}$

The following definitions and preliminary facts ((2)-(7)) are taken from Vaught [46]. 1.1 and 1.5 appeared in Burgess-Miller [11].

Throughout this section we assume that G is a Baire topological space, X and X' are sets, and J is a function on $G \times X$ to X'. Additional assumptions will be stated when they are required. The most important special case for us will be that of a special action. As we will see below however, consideration of other cases -- particularly the product case: $X' = G \times X$, J the identity function -- can aid in our study of the special actions.

For $B \subseteq X'$, $x \in X$, and $g \in G$, let $B^{x} = \{g \in G : J(g,x) \in B\}$, $B^{g} = \{x \in X : J(g,x) \in B\}$. If $U \neq \emptyset$ is open in G, we define

$$B^{*U} = \{x : B^{x} \cap U \text{ is comeager in } U\}, \quad B^{*} = B^{*G}$$
$$B^{\Delta U} = -(-B)^{*U} = \{x : B^{x} \cap U \text{ is not meager in } U\}$$
$$= \{x : B^{x} \cap U \text{ is not meager}\}, \quad B^{\Delta} = B^{\Delta G}.$$

When we wish to emphasize the dependence upon J we write B^{xJ}, $B^{*U,J}$, B^{*J}, etc.

The key fact relating the $*$-transform to action equivalences is an immediate consequence of the homogeneity of topological groups, the definition of an action, and the fact that G is a Baire space.

(2) Suppose G is a topological group, $X = X'$, $\mathcal{J} = (G,X,J)$ is an action, and $B \subseteq X$. Then B^{*} and B^{Δ} are $E_{\mathcal{J}}$-invariant and $B^{-E_{\mathcal{J}}} \subseteq B^{*} \subseteq B^{\Delta} \subseteq B^{+E_{\mathcal{J}}}$.

It follows from the closure of meager sets under countable unions that

(3) $$(\bigcap_{n\in\omega} B_n)^{*U} = \bigcap_{n\in\omega} B_n^{*U}$$

and

$$(\bigcup_{n\in\omega} B_n)^{\Delta U} = \bigcup_{n\in\omega} B_n^{\Delta U}.$$

A collection H of non-empty open sets is a weak basis for G provided every non-empty open set includes a member of H. <u>We henceforth assume that</u> "U" <u>and</u> "V" <u>range over members of a fixed weak basis</u> H <u>for</u> G. A set $B \subseteq X'$ is <u>normal</u> if for every $x \in X$, B^x is almost open in G. We may regard the normal sets as exactly the sets which are well-behaved with respect to * in view of the following:

<u>Proposition 1.1</u>. $B \subseteq X'$ is normal if and only if for every U, $B^{\Delta U} = \bigcup\{B^{*V}: V \subseteq U\}$, (and $B^{*U} = \bigcap\{B^{\Delta V}: V \subseteq U\}$).

<u>Proof</u>.

The "only if" part is 1.5 of Vaught [46]. It depends on the fact that G is a Baire space.

For the "if" part, suppose that for every U, $B^{\Delta U} = \bigcup\{B^{*V}: V \subseteq U\}$. Fixing $x \in X$, this implies that either $B^x \cap U$ is meager or $-B^x \cap V$ is meager for some $V \subseteq U$. Since H is a weak basis it follows that every non-empty open set contains a point where either B^x or $-B^x$ is meager. This proves that B^x is almost open (<u>cf</u>. Kuratowski [26] §11 IV). □

The "only if" part of 1.1 can be restated as

(4) If B is normal then $(-B)^{*U} = -\bigcup\{B^{*V}: V \subseteq U\}$.

Also note that the inclusion

$$\bigcup_{V \subseteq U} B^{*V} \subseteq B^{\Delta U}$$

holds for arbitrary (not necessarily normal) sets B.

The last algebraic formula we require deals with the behavior of $*$ and the operation (A).

(5) Assume G satisfies the countable chain condition (every disjoint collection of open subsets of G is countable) and that $\{A_s: s \in S_q = \bigcup_{n\in\omega} {}^n\omega\}$ is a collection of normal subsets of X'. Then for all U,

$$\Big(\bigcup_{\xi\in{}^\omega\omega} \bigcap_{n\in\omega} A_{\xi\restriction n}\Big)^{*U} =$$

$$\bigcap_{U_o \subseteq U} \bigcup_{V_o \subseteq U_o} \bigcup_{k_o\in\omega} \bigcap_{U_1 \subseteq V_o} \bigcup_{V_1 \subseteq U_1} \bigcup_{k_1\in\omega} \cdots \bigcap_{n\in\omega} A_{(k_o,\ldots,k_n)}^{*V_n}.$$

Formally, membership in the right hand side of (5) is defined in terms of the existence of a winning strategy for a certain infinite game (cf. Burgess [10]). The important feature is that when $\mathcal{H}$ is countable, the set indicated can be obtained by the operation (A) from a suitable indexing of $\{A_s^{*V}: s \in S_q, V \in \mathcal{H}\}$. In [10] Burgess derived analogous formulas for the behavior of $*$ under the more powerful "Kolmogorov operations" Γ^α, $\alpha \in \omega_1$. That part of Theorem 1.2 below which deals with operation (A) applies equally well to any of the Γ^{α}.

The next fact follows from the extra assumption and the hypothesis that G is a Baire space.

(6) Assume X and X' are topological spaces and J is continuous in each variable. If B is closed then for each U, $B^{*U} = \bigcap\{B^g : g \in U\}$, and B^* is closed.

It follows from (6), (4), and (3) that,

(7) Under the hypothesis of (6), if $B \in \underset{\sim}{\Pi}^0_{\alpha}(X')$, [respectively, $B \in \underset{\sim}{\Sigma}^0_{\alpha}(X')$], then $B^* \in \underset{\sim}{\Pi}^0_{\alpha}(X)$, $[B^{\Delta} \in \underset{\sim}{\Sigma}^0_{\alpha}(X)]$, $\alpha \geq 1$.

In particular, B^* and B^{Δ} are Borel if B is. As we will next prove, this last statement holds true with weaker assumptions on X, X' and J.

A Borel space is a set X with a σ-field $\mathcal{B}(X)$ of distinguished or Borel subsets (cf. Mackey [29a]). A function $f: X \to X'$ between Borel spaces is Borel measurable if $f^{-1}(B) \in \mathcal{B}(X)$ for all $B \in \mathcal{B}(X')$. The product Borel structure on $X \times X'$ is the σ-field generated by $\{X \times B : B \in \mathcal{B}(X')\} \cup \{B \times X' : B \in \mathcal{B}(X)\}$. It is the weakest structure which makes the canonical projection maps, $(g,x) \mapsto g$ and $(g,x) \mapsto x$, Borel measurable.

In analogy with the topological case, we say that $B \subseteq X$ is analytic if B can be obtained by operation (A) from Borel sets. The collection of C-sets (sieve sets) is the smallest collection containing the Borel sets and closed under complementation and the operation (A). A topological space is implicitly given the Borel structure generated by the open sets. A Borel space is standard if it is isomorphic to the Borel structure of a Polish space. Since two Polish spaces are Borel isomorphic if and only if they have the same cardinality, there are exactly two infinite standard Borel spaces, up to isomorphism.

Theorem 1.2 in its present form is due to Vaught. The author had earlier proved a version with the stronger assumption that X, X' were topological spaces (J still Borel). This was based directly on the "product case" of [46].

Theorem 1.2. Assume H is countable, X, X' are Borel spaces, J is Borel measurable on the product Borel space $G \times X$, $B \subseteq X'$. If B is respectively Borel, analytic, or $\mathcal{C}$, then the same is true of B^* and B^Δ.

Proof.

Let $I\colon G \times X \to G \times X$ be the identity function. Notice that for every x, $B^{xJ} = \{g\colon J(g,x) \in B\} = \{g\colon (g,x) \in J^{-1}(B)\} = (J^{-1}(B))^{xI}$. Thus $B^{*J} = (J^{-1}(B))^{*I}$. Since the Borel, analytic and $\mathcal{C}$-sets are each closed under inverse Borel images, it suffices to prove the theorem in the case $X' = G \times X$, $J = I$. Since the almost open sets are closed under complementation, countable union, and the operation (A), and since these operations commute with the passage $B \mapsto B^x$, they all preserve normality. $\mathcal{B}(G \times X)$ is generated by the collection

$\mathcal{G} = \{O \times X\colon O \in \underset{\sim}{\Sigma}^0_1(G)\} \cup \{G \times A\colon A \in \mathcal{B}(X)\}$ and each $B \in \mathcal{G}$ clearly has open cross-sections, so every $\mathcal{C}$-set is normal.

We first prove that $B^{*U} \in \mathcal{B}(X)$ whenever $B \in \mathcal{G}$ and $U \in \mathcal{H}$. Suppose $B = G \times A$ and $x \in X$; then $B^x = G$ if $x \in A$, and $B^x = \emptyset$ if $x \notin A$; so $B^{*U} = A$. Suppose $B = O \times X$ and $x \in X$; then $B^x = O$ and $B^{*U} = A$ if $O \cap -U$ is meager, $\emptyset$ otherwise. In any case, $B^{*U} \in \mathcal{B}(X)$ as claimed.

The conclusion of the theorem now follows by (3), (4), and (5) just as in [46]. In proving that B^{Δ} is analytic when B is, one uses the corresponding fact for B^* together with 1.1 and the fact that analytic sets are closed under countable unions. □

Remark I. This argument is particularly interesting when $\mathcal{J} = (G,X,J)$ is an action (J Borel). In order to see that B^* is a Borel invariantization when $B \in \mathcal{B}(X)$, we must consider both the "action case" J, (to see that B^* is invariant), and the corresponding "product case" I, (to see that B^* is Borel).

Since we have improved Vaught's original invariantization result, we get improved versions of its consequences. We state the most interesting one.

When $\mathcal{H}$ is countable, G is a topological group, $X = X'$ is a Borel space, $J\colon G \times X \to X$ is Borel measurable and $\mathcal{J} = (G,X,J)$ is an action, we say $\mathcal{J}$ is a special Borel action.

Corollary 1.3. Assume $\mathcal{J} = (G,X,J)$ is a special Borel action, then every $E_{\mathcal{J}}$-invariant analytic subset of X is a union of ω_1 invariant

Borel sets. If G is Polish and there is a separable metric topology which generates the Borel structure on X, then each orbit is Borel.

Proof.

The proofs of 2.5 and 2.6 in [46] suffice once we know that the Borel sets are closed under the transform $B \mapsto B^*$. That closure was proved in 1.2. □

Note: The separability assumption on X can be omitted. For this remark and for a stronger result on the measurability of orbits, see Miller [32].

Vaught's reduction theorems (2.7 of [46]) have similar extensions to Borel actions using 1.2. Note, however, that in any action $\mathcal{J} = (G,X,J)$ such that G and X are Polish and J is Borel, the induced equivalence $E_{\mathcal{J}}$ is $\underset{\sim}{\Sigma}^1_1$. Thus, the improved reduction theorems in this case can still be obtained by the methods of chapter I.

Our next result is a short proof of an invariant $\underset{\sim}{\Pi}^1_1$ reduction theorem for pairs, based on another type of preservation property of *. Since the equivalence relation $E_{\mathcal{J}}$ in part (b) may not be $\underset{\sim}{\Sigma}^1_1$, 1.4 (b) properly overlaps with theorem 2.7 (a) of chapter I. We have been unable, however, to construct an example where 1.4 (b) gives new information. For further discussion on this point, see 1.6 below or §4 of Burgess-Miller [11].

The statement and proof of 1.4 (a) corrects an error in 2.2 of [11].

For $\Gamma \subseteq P(X)$ recall that E-inv(Γ) is the collection of E-invariant members of Γ. When the context permits it, we will write inv(Γ) instead of E-inv(Γ).

Theorem 1.4. Assume $\mathcal{J} = (G,X,J)$ is an action.

(a) If $\Gamma \subseteq P(X)$ is closed under both $B \mapsto B^*$ and $B \mapsto B^\Delta$ and Γ has the reduction property for pairs, then so has $E_{\mathcal{J}}$-inv(Γ).

(b) If G has a countable basis, X is a Suslin space and J is Borel measurable, then $E_{\mathcal{J}}$-inv$(\underset{\sim}{\Pi}^1_1(X))$ has the reduction property for pairs.

Proof.

(a) Suppose (A_1,A_2) is a pair of $E_{\mathcal{J}}$-invariant members of . Let (B_1,B_2) be an arbitrary pair of Γ-sets which reduces (A_1,A_2). We claim that (B_1^*,B_2^Δ) also reduces (A_1,A_2). Since B_1^* and B_2^Δ are invariant, (a) follows from this claim. Since $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$, $B_1^* \subseteq A_1^* = A_1$ and $B_2^\Delta \subseteq A_2^\Delta = A_2$. $A_2 - A_1 \subseteq B_2$, so $(A_2 - A_1) = (A_2 - A_1)^\Delta \subseteq B_2^\Delta$. Similarly, $A_1 - A_2 \subseteq B_1^*$, so $B_1^* \cup B_2^\Delta = A_1 \cup A_2$. Finally, since $B_1 \subseteq -B_2$, $B_1^* \subseteq (-B_2)^* = -B_2^\Delta$ and $B_1^* \cap B_2^\Delta = \emptyset$. Thus, (B_1^*,B_2^Δ) reduces (A_1,A_2) as required, proving (a).

(b) If G has a countable basis, then it satisfies the countable chain condition. The fact that $\underset{\sim}{\Pi}^1_1(X)$ is closed under the transforms $B \mapsto B^*$ and $B \mapsto B^\Delta$ then follows by 1.2. Hence, (b) follows from (a) and the classical $\underset{\sim}{\Pi}^1_1$ reduction theorem. □

Now we turn to another preservation theorem which is particular to special actions.

Theorem 1.5. Assume (G,X,J) is a special action.

(a) If $B \subseteq X$ is meager, then so are B^Δ and B^*.

(b) If $B \subseteq X$ is almost open, then so are B^Δ and B^*.

Proof.

We may assume that $\mathcal{H}$ is a countable basis for G.

(a) Let $B \subseteq X$ be meager. Then $B \subseteq \bigcup_{n\in\omega} C_n$ for some collection $\{C_n : n \in \omega\}$ of closed nowhere dense sets, and $B^\Delta \subseteq (\bigcup_n C_n)^\Delta = \bigcup_n C_n^\Delta = \bigcup_n \bigcup_U C_n^{*U} = \bigcup_n \bigcup_U \bigcap_{g\in U} C_n^g$. Since each J^g is continuous and each C_n is nowhere dense, each $C_n^g = g^{-1}C_n$ is nowhere dense. It follows that $\bigcap_{g\in U} C_n^g = C_n^{*U}$ is nowhere dense for each n and U, and B^Δ is meager as required. Since $B^* \subseteq B^\Delta$, B^* is meager also.

(b) Now suppose B is almost open. Let $B = A \cup N$ where $A \in \mathbf{\Pi}_2^0(X)$ and N is meager. Then $B^\Delta = (A \cup N)^\Delta = A^\Delta \cup N^\Delta$. A^Δ is Borel by 1.2 and N^Δ is meager by part (a), so B^Δ is almost open. It follows from the closure of almost open sets under complementation that $B^* = -(-B)^\Delta$ is also almost open. □

Remark II. (On 1.9 of Vaught [46]).

Assume (G,X,J) is a special action. Let $B = \bigcup_{\xi\in{}^\omega\omega} \bigcap_n B_{\xi\restriction n}$. The classical approximations to B are defined by the conditions

$$B_s^0 = B_s, \quad B_s^{\alpha+1} = B_s^\alpha \cap \bigcup_{i\in\omega} B_{s^\frown i}^\alpha, \quad B_s^\lambda = \bigcap_{\alpha<\lambda} B_s^\alpha;$$

$$B_\alpha = B_\emptyset^\alpha, \qquad T_\alpha = \bigcup\{B_s - B_s^{\alpha+1} : s \in S_q\}.$$

$s^\frown i$ denotes $s \cup \{(n,i)\}$ where n is the domain of s. It is known, (see e.g. [26]), that $\bigcup\{B_\alpha - T_\alpha : \alpha < \omega_1\} = B = \bigcap\{B_\alpha : \alpha < \omega_1\}$. If X satisfies the countable chain condition and each B_s is almost open, then for some $\alpha < \omega_1$, T_α is meager. In this case, T_α^Δ is meager by 1.5, hence $(B_\alpha - T_\alpha)^* = B_\alpha^* - T_\alpha^\Delta$ is comeager in B^*. In 1.9 of [46] this conclusion is derived from the more restrictive assumption that each B_s is a C-set but without the assumption that J is an action. For another application of 1.5 (a), see Miller [32] Theorem 3.

In [10] Burgess considered certain Boolean operations called Borel game operations. For example, the $\underset{\sim}{\Pi}_2^0$-game operation operates on a collection of sets $\{A_{mn}^s : m,n \in \omega, s \in {}^n\omega\}$ to yield a set

$$A = \bigcap_{k_0} \bigcup_{k_1} \bigcap_{k_2} \bigcup_{k_3} \ldots \bigcap_m \bigcup_n A_{mn}^{k_0 \ldots k_{n-1}} .$$

The variant of operation $(\mathcal{A})$ found in statment (5) above is just the closed-game operation. These operations are quite powerful relative to the operation $(\mathcal{A})$ (see [10]). Let $\mathcal{G}$ be one of these operations and let $\mathcal{G}[X]$ be the smallest collection containing the Borel sets and closed under $\mathcal{G}$ and complementation. It is known (see [10]) that if X is a $\underset{\sim}{\Pi}_2^0$ subspace of ${}^\omega 2$ (equivalently, X is a zero-dimensional Polish space, cf. [26]), then every member of $\mathcal{G}[X]$ is "absolutely $\underset{\sim}{\Delta}_2^1$" (see [10]) and hence, almost open. In [10] Burgess showed that when G is a zero-dimensional Polish space, each operation $\mathcal{G}$ satisfies a condition analogous to (5). (He officially assumed an action but made no use of this hypothesis in his proof). It follows that, assuming X and X' are topological spaces, G is zero-dimensional Polish, and J is continuous in each variable, B^* is in $\mathcal{G}[X]$ whenever B is in $\mathcal{G}[X']$. In view of 1.2 this remains true if we assume only that X, X' are Borel spaces and J is Borel.

Every Polish space G is a union of a zero-dimensional $\underset{\sim}{\Pi}^0_2$ subspace $\hat{G}$ and a meager set (see e.g. the proof of 1.6 below). Suppose O is an non-empty open set in G and S is any subset of O. Then S is comeager in O if and only if $S \cap \hat{G}$ is comeager in $O \cap \hat{G}$. If we define $\hat{J} = J_{\restriction \hat{G}\times X}$, it follows that $B^{*OJ} = B^{*(O\cap \hat{G})\hat{J}}$ for every $B \in X'$. Since $\hat{G}$ is zero-dimensional and Polish, we conclude that $B^* \in G[X]$, when $B \in G[X]$ assuming only that G is Polish and J is Borel.

The next theorem shows that when X and X' are standard spaces we can drop the assumption that G is Polish. In particular, since the operation G preserves normal sets when G is zero-dimensional Polish (see [10]), it shows that each class $G[X]$ is closed under $B \to B^*$ whenever (G,X,J) is a special Borel action and X is standard.

<u>Theorem 1.6</u>. Assume X and X' are standard Borel spaces, J is Borel measurable and H is a countable <u>basis</u> for G. Then there exists a zero-dimensional Polish space $\hat{G}$ with basis $\hat{H}$ and a Borel measurable function $\hat{J}: \hat{G} \times X \to X'$ such that for $B \subseteq X'$,

$$B^{*\hat{J}} \subseteq B^{*J} \subseteq \bigcap\{B^{\Delta UJ}: U \in H\} \subseteq \bigcap\{B^{\Delta\hat{U}\hat{J}}: \hat{U} \in \hat{H}\}.$$

In particular, $B^{*J} = B^{*\hat{J}}$ if B is normal with respect to $\hat{J}$. If X and X' are Polish topological spaces, then $\hat{J}$ can be assumed measurable at the same level as J.

<u>Proof</u>.

We may assume that X and X' are Polish topological spaces.

Suppose $H = \{U_i : i \in \omega\}$. Let $G_1 = \bigcap\{U_i \cup -(\overline{U}_i) : i \in \omega\}$ where $\overline{U}_i$ is the closure of U_i. Define $V_{2i} = U_i \cap G_1$, $V_{2i+1} = -(\overline{U}_i) \cap G_1$. Note

(i) G_1 is comeager in G since each $U_i \cup -(\overline{U}_i)$ is.

(ii) $H' = \{V_i : i \in \omega\}$ is a countable basis for G_1 and each V_i is clopen in G_1.

Let $E = \{(g_1, g_2) \in G_1^2 : (\forall i \in \omega)(g_1 \in V_i \Leftrightarrow g_2 \in V_i)\}$. E is an equivalence relation; let $G_2 = G_1/E$ be the quotient topological space. Let $f: G_2 \to 2^\omega$ where $f(g)(i) = 1$ if and only if $g \in V_i$. It is easily seen that G_2 is Hausdorff and that f induces a homeomorphism on G_2 to a subspace of 2^ω. (cf. Kuratowski [26] §26.IV.2). To simplify notation we identify G_2 with $f(G_2) \subseteq 2^\omega$.

Let $g_1, g_2 \in G_1$, $x \in X$. If $J(g_1, x) \neq J(g_2, x)$, then, since X' is Hausdorff and J is Borel, there is a Borel set V in $G_1 \times X$ such that $(g_1, x) \in V$ and $(g_2, x) \notin V$. It follows that $(g_1, g_2) \notin E$. Thus, we can define a function $J_1: G_2 \times X \to X'$ by the equation $J_1([g]_E, x) = J_1(g, x)$.

Let $\overline{G}_2$ be the closure of G_2 in 2^ω. By a theorem of Lavrentiev and Kuratowski ([26] § 35 VI), there is a Borel set $H \subseteq \overline{G}_2 \times X$ and a function $J_2: H \to X'$ which is Borel measurable at the same level as J such that $J_2 \restriction G_2 \times X = J_1$. Let $G_3 =$

$\{g \in \overline{G}_2 \colon (\forall x \in X)((g,x) \in H)\}$. Then $G_2 \subseteq G_3$ and $G_3 \in \underset{\sim}{\Pi}^1_1(\overline{G}_2)$. Since $\underset{\sim}{\Pi}^1_1$ sets are almost open, there exists $\hat{G} = \underset{\sim}{\Pi}^0_2(\overline{G}_2)$ and a meager set $N \subseteq \overline{G}_2$ such that $G_3 = \hat{G} \cup N$. Let $\hat{H}$ be the canonical basis for $\hat{G}$ and let $\hat{J} = J\restriction_{\hat{G}\times X}$. We will show that $\hat{G}, \hat{H}, \hat{J}$ have the required property.

Since $\hat{G}$ is a $\underset{\sim}{\Pi}^0_2$ subset of 2^ω, $\hat{G}$ is zero-dimensional and Polish. Clearly, $\hat{J}$ is Borel at the same level as J. Since G_2 is dense in $\overline{G}_2$, $N \cap G_2$ is meager in G_2 (see [26] §10 IV 2) and $\hat{G} \cap G_2$ is comeager in G_2. The first inclusion of the theorem is proved by the following computation:

$$
\begin{aligned}
x \in B^{*\hat{J}} &\Leftrightarrow \{g \in \hat{G} \colon \hat{J}(g,x) \in B\} \text{ is comeager in } \hat{G} \\
&\Rightarrow \{g \in \hat{G} \cap G_2 \colon J_1(g,x) \in B\} \text{ is comeager in } G_2 \\
&\Rightarrow \{g \in G_2 \colon J_1(g,x) \in B\} \text{ is comeager in } G_2 \\
&\Rightarrow \{g \in G_1 \colon J(g,x) \in B\} \text{ is comeager in } G_1 \\
&\Rightarrow \{g \in G \colon J(g,x) \in B\} \text{ is comeager in } G \\
&\Leftrightarrow x \in B^{*J}.
\end{aligned}
$$

To prove the final inclusion of the theorem, note that f induces a correspondence between elements U of H and $\hat{U}$ of $\hat{H}$. The above computation is easily modified to show that

$(-B)^{*UJ} \subseteq (-B)^{*\hat{U}\hat{J}}$ for each $U \in H$. Then $B^{\Delta\hat{U}\hat{J}} \subseteq B^{\Delta UJ}$

and the final inclusion follows. □

Remark III: (Subactions of Polish actions).

In Burgess-Miller [11] (4.1) it was proved that if J is obtained by restricting a Polish action $\hat{J} = (\hat{G}, X, \hat{J})$ to a dense, non-meager subgroup G of $\hat{G}$, then E_J and $E_{\hat{J}}$ have the same invariant $\underset{\sim}{\Pi}^1_1$ sets. In fact, the connection between J and $\hat{J}$ is somewhat stronger than was indicated in [11] as shown by the following theorem.

Theorem 1.7: Suppose $\hat{J} = (\hat{G}, X, \hat{J})$ is a Polish action, G is a dense non-meager subgroup of $\hat{G}$, and $J = \hat{J}_{\restriction G\times X}$. Let $\hat{H}$ be a basis for G and let $H = \{\hat{U} \cap G\colon \hat{U} \in \hat{H}\}$. Then for $B \subseteq X$

$$B^{*\hat{J}} \subseteq B^{*J} \subseteq \bigcap\{B^{\Delta U,J}\colon U \in H\} \subseteq \bigcap\{B^{\Delta\hat{U},\hat{J}}\colon \hat{U} \in \hat{H}\}.$$

Hence, $B^{*\hat{J}} = B^{*J}$ if B is normal with respect to $\hat{J}$. In particular, $B^{*\hat{J}} = B^{*J}$ if B belongs to $\underset{\sim}{\Pi}^1_1(X)$ or even to $G[X]$ for any of the Borel game operations G.

Proof.

The theorem is proved by a short subcomputation of the computation in 1.5.

$$x \in B^{*\hat{J}} \Leftrightarrow \{g \in \hat{G}: gx \in B\} \text{ is comeager in } \hat{G}$$

$$\Rightarrow \{g \in G: gx \in B\} \text{ is comeager in } G$$

$$\Leftrightarrow x \in B^{*J}.$$

So $B^{*J} \subseteq B^{*\hat{J}}$.

$$x \in -B^{\Delta\hat{U},\hat{J}} \Leftrightarrow \{g \in \hat{U}: gx \in B\} \text{ is meager in } \hat{U}$$

$$\Rightarrow \{g \in \hat{U} \cap G: gx \in B\} \text{ is meager in } \hat{U} \cap G$$

$$\Leftrightarrow x \in -B^{\Delta\hat{U} \cap G, J}.$$

So $B^{\Delta\hat{U} \cap G, J} \subseteq B^{\Delta\hat{U},\hat{J}}$ and $\bigcap_{U \in \mathcal{H}} B^{\Delta U, J} \subseteq \bigcap_{U \in \hat{\mathcal{H}}} B^{\Delta\hat{U},\hat{J}}$. □

Remark IV. (Bases and weak bases for topological groups).

In Vaught [46], Burgess-Miller [11] and Burgess [10], a number of theorems are proved under the assumption, "G is a topological group with a countable weak basis." Assume G is such a group with a weak basis $\mathcal{H} = \{U_i: i \in \omega\}$. It is easily verified that $\mathcal{H}' = \{U_i U_i^{-1}: i \in \omega\}$ is a countable basis for the neighborhood system of the identity of G and hence, G is pseudometrizable (cf. Bourbaki [9] IX.3.1.1 and IX.1.4.2). Since G is separable, it follows that G has a countable basis. Thus,

A topological group has a countable weak basis if and only if it has a countable basis.

This fact was overlooked in the previous papers cited above. Of course it still may be useful to consider the inductive formulas for *, (such as (4)), with respect to particular weak bases for particular groups.

§2. The Invariant $\underset{\sim}{\Pi}^0_\alpha$ Separation Principle.

Let X be an arbitrary topological space in which every open set is $\underset{\sim}{\Sigma}^0_2$. Then every $\underset{\sim}{\Sigma}^0_\alpha$ subset of X is a countable union of disjoint $\underset{\sim}{\Delta}^0_\alpha$ sets, $(\alpha > 1)$. It follows easily that $\underset{\sim}{\Sigma}^0_\alpha(X)$ has the reduction property and (consequently) that $\underset{\sim}{\Pi}^0_\alpha(X)$ has the first separation property (cf. Kuratowski [26] §30. VII).

Given an equivalence E on X, it is natural to ask whether the E-invariant $\underset{\sim}{\Sigma}^0_\alpha$ sets have the reduction property. If there is a continuous selector for E, then X/E is homeomorphic to a closed subset of X (viz. the set of fixed points) and the E-invariant reduction and separation properties are immediate. This is the only positive result about invariant reduction for the Borel classes which we know. When E is the canonical equivalence on the logic space $2^{\omega\times\omega}$ it is not too difficult to see that invariant reduction fails at the first possible level:

Proposition 2.1: Let ρ consist of a single binary relation and let $I = I_\rho$ be the canonical equivalence on $X_\rho = 2^{\omega\times\omega}$. Let
$A_0 = \{R: (\exists n)(\forall m)(R(n,m) = 1)\}$, $A_1 = \{R: (\exists m)(\forall n)(R(n,m) = 1)\}$.
Then there is no pair of I-invariant $\underset{\sim}{\Sigma}^0_2$ sets which reduces (A_0, A_1).

Proof.

Choose R_0 so that (ω, R_0) is a dense linear order with left and right endpoints (i.e. an order of type $1 + \eta + 1$). Suppose B is an invariant $\underset{\sim}{\Pi}^0_2$ set which contains R_0. Then $B = [\![\theta]\!]$ for some $\underset{\sim}{\Pi}'^0_2$ sentence θ. Since $\underset{\sim}{\Pi}'^0_2$ classes are closed under unions of chains (cf. Weinstein [47]), B has members R_1 and R_2 which define orders of type $\eta + 1$ and $1 + \eta$ respectively; hence B cannot include either $-A_0 \cap A_1$ or $-A_1 \cap A_0$.

Suppose (B_0, B_1) reduces (A_0, A_1). Then, for $i = 0$ or 1, $R_0 \in -B_i$ and $-A_i \cap A_{1-i} \subseteq -B_i$. By the argument of the preceding paragraph, B_i is not invariant. □

The failure of invariant $\underset{\sim}{\Sigma}^0_2$ reduction does <u>not</u> entail the failure of invariant $\underset{\sim}{\Pi}^0_2$ separation.

Suppose $B \subseteq X$ and $B^{\#}$ is any E-invariantization of B. Then if B separates a pair of disjoint E-invariant sets, so does $B^{\#}$. Thus, the invariant $\underset{\sim}{\Pi}^0_2$ separation problem is connected to the $\underset{\sim}{\Delta}^0_2$ invariantization problem: "Given $B \in \underset{\sim}{\Delta}^0_2(X)$ find a $\underset{\sim}{\Delta}^0_2$ invariantization for B." Note that even when E is induced by a Polish action, the transform $B \mapsto B^*$ does not directly solve the invariantization problem -- if B is $\underset{\sim}{\Delta}^0_2$, then B^* is $\underset{\sim}{\Pi}^0_2$ and B^Δ is $\underset{\sim}{\Sigma}^0_2$, but neither is necessarily $\underset{\sim}{\Delta}^0_2$.

We will solve both the $\underset{\sim}{\Delta}^0_2$ invariantization problem and the invariant $\underset{\sim}{\Pi}^0_2$ separation problem for a wide class of equivalence spaces by considering a stronger version of the $\underset{\sim}{\Pi}^0_2$ separation theorem.

Assume that X is an arbitrary set.

Suppose Γ_1, Γ_2 are two subclasses of $P(X)$ such that $\Gamma_2 \subseteq \Gamma_1$ and Γ_2 is closed under complementation. We say that Γ_1 <u>has the strong separation property with respect to</u> Γ_2 provided that for $A_o, A_1 \in \Gamma_1$, if $A_o \cap A_1 = \emptyset$, then there exists $B \in \Gamma_2$ which separates A_o from A_1. An equivalent condition is that Γ_1 has the first separation property and $\Gamma_1 \cap \check{\Gamma}_1 = \Gamma_2$, (cf. Addison [2] for a discussion of this phenomenon).

Suppose $C = \langle C_\beta : \beta \leq \gamma \rangle$ is a sequence of subsets of X. C is decreasing if $C_\beta \subseteq C_{\beta'}$ whenever $\beta' < \beta \leq \gamma$. C is continuous if $C_\lambda = \bigcap_{\beta<\lambda} C_\beta$ whenever $\lambda \leq \gamma$ is a limit ordinal. $e(\gamma) = \{\beta \in \gamma : \beta \text{ is even}\}$. $D(C) = \bigcup\{C_\beta - C_{\beta+1} : \beta \in e(\gamma)\}$.

Let $\Gamma \subseteq P(X)$. C is suitable for $\mathcal{D}_\gamma(\Gamma)$ if $C \in {}^{\gamma+1}\Gamma$, C is decreasing and continuous, $C_o = X$ and $C_\gamma = \emptyset$. We define $\mathcal{D}_\gamma(\Gamma) = \{D(C) : C \text{ is suitable for } \mathcal{D}_\gamma(\Gamma)\}$, $\mathcal{D}_{(\gamma)}(\Gamma) = \bigcup\{\mathcal{D}_\nu(\Gamma) : \nu < \gamma\}$, $\mathcal{D}_{(\infty)}(\Gamma) = \bigcup\{\mathcal{D}_\nu(\Gamma) : \gamma \in ON\}$. $\mathcal{D}_{(\omega_1)}(\Gamma)$ is the collection of "countable alternated unions over Γ."

The important feature of alternated unions is their behavior under complementation. If C is suitable for $\mathcal{D}_\gamma(P(X))$ then it is easily seen (cf. [26]) that

(8) $$-D(C) = \bigcup\{C_{\beta-1} - C_\beta : \beta \in e(\gamma),\ \beta \text{ a successor}\}.$$

It follows that if Γ_1 is a class which includes $\Gamma \cup \check{\Gamma}$ and is closed under finite intersections and countable unions, then $\mathcal{D}_{(\omega_1)}(\Gamma) \subseteq \Gamma_1 \cap \check{\Gamma}_1$.

Now suppose X is a topological space. $\mathcal{D}_{(\infty)}(\underset{\sim}{\Pi}_1^0(X))$ was known classically as the collection of resolvable sets. A result of Montgomerey (cf. [26] §30.X) states that $\mathcal{D}_{(\infty)}(\underset{\sim}{\Pi}_1^0(X)) \subseteq \underset{\sim}{\Delta}_2^0(X)$ when X is metrizable, (when X is separable this is obvious). The basic $\underset{\sim}{\Pi}_2^0$ separation theorem (9) is due to Hausdorff, (cf. [26] §34 or reconstruct the argument by analogy with the proof of III.3.1 below).

(9) Assume X is completely metrizable. Then $\underset{\sim}{\Pi}_2^0(X)$ has the strong separation property with respect to $\mathcal{D}_{(\infty)}(\underset{\sim}{\Pi}_1^0(X))$.

When X is Polish, (9) can be extended to all higher levels of the Borel hierarchy.

(10) Assume X is Polish, $\alpha > 1$. Then $\underset{\sim}{\Pi}_\alpha^0(X)$ has the strong separation property with respect to $\mathcal{D}_{(\omega_1)}(\underset{\sim}{\Pi}_{(\alpha)}^0(X))$.

(10) is usually proved only for successor α (cf. [26]§37.III). For limit α the situation is simpler.-- One easily shows that $\underset{\sim}{\Delta}_\lambda^0(X) = \mathcal{D}_\omega(\underset{\sim}{\Pi}_{(\lambda)}^0(X))$ and (10) follows from the first separation property for $\underset{\sim}{\Delta}_\lambda^0$.

Now fix an equivalence E on X. In view of (9) we can solve the $\underset{\sim}{\Delta}_2^0$ invariantization problem for X, when X is completely metrizable, by solving each $\mathcal{D}_\gamma(\underset{\sim}{\Pi}_1^0)$ invariantization problem.

Given $C \in {}^\gamma(\mathcal{P}(X))$, let $\hat{C} = \langle C_\beta^{-E} : \beta < \gamma \rangle$.

Lemma 2.2. Assume C is suitable for $\mathcal{D}_\gamma(\mathcal{P}(X))$. Then $C^{\ominus}$ is suitable for $\mathcal{D}_\gamma(\mathrm{inv}(\mathcal{P}(X)))$ and $D(C^{\ominus})$ is an invariantization of $D(C)$.

Proof.

First note that for each $\beta < \gamma$,

$$C^-_\beta - C^-_{\beta+1} = C^-_\beta \cap (-C_{\beta+1})^+ \subseteq (C_\beta - C_{\beta+1})^+ .$$

It follows that

$$D(C^{\ominus}) = \bigcup\{C^-_\beta - C^-_{\beta+1}: \beta \in e(\gamma)\} \subseteq$$

$$\bigcup\{(C_\beta - C_{\beta+1})^+: \beta \in e(\gamma)\} = (\bigcup\{C_\beta - C_{\beta+1}: \beta \in e(\gamma)\})^+$$

$$= (D(C))^+ .$$

A similar calculation based on (8) shows that $-D(C^{\ominus}) \subseteq (-D(C))^+ = -(D(C)^-)$. $D(C^{\ominus})$ is clearly invariant, hence it is an invariantization of $D(C)$. Since the transform $C_\beta \mapsto C^-_\beta$ preserves inclusions and commutes with intersections, $C^{\ominus}$ is suitable for $\mathcal{D}_\gamma(\mathrm{inv}(\mathcal{P}(X)))$. □

Theorem 2.3. Assume X is a topological space and E is a lower semicontinuous equivalence on X.

(a) For every $\gamma \in \mathrm{ON}$, $\mathrm{inv}(\mathcal{D}_\gamma(\underset{\sim}{\Pi}^0_1(X)) = \mathcal{D}_\gamma(\mathrm{inv}(\underset{\sim}{\Pi}^0_1(X)))$.

(b) If X is completely metrizable, then $\mathrm{inv}(\underset{\sim}{\Pi}^0_2(X))$ has the strong separation property with respect to $\mathcal{D}_{(\infty)}(\mathrm{inv}(\underset{\sim}{\Pi}^0_1(X)))$.

Proof.

If B is closed and E is lower semicontinuous, then $B^{-} \varepsilon \operatorname{inv}(\underset{\sim}{\Pi}{}^{o}_{1}(X))$. Thus, $C^{\ominus} \varepsilon\ {}^{\gamma+1}((\operatorname{inv}\underset{\sim}{\Pi}{}^{o}_{1}(X)))$ when $C \varepsilon\ {}^{\gamma+1}(\underset{\sim}{\Pi}{}^{o}_{1}(X))$. (a) follows by 2.2. Now suppose X is completely metrizable, $A_{o}, A_{1} \varepsilon \operatorname{inv}(\underset{\sim}{\Pi}{}^{o}_{2}(X))$. Applying (9), let C be suitable for $\mathcal{D}_{(\omega_1)}(\underset{\sim}{\Pi}{}^{o}_{1}(X))$, $A_{o} \subseteq D(C) \subseteq -A_{1}$. Then $D(C^{\ominus}) \varepsilon \mathcal{D}_{(\omega_1)}(\operatorname{inv}(\underset{\sim}{\Pi}{}^{o}_{1}(X)))$ and $A_{o} \subseteq D(C^{\ominus}) \subseteq -A_{1}$. □

Assuming a special action we can replace "-" with "*" to invariantize (10).

Suppose G, X, X', J satisfy the basic hypothesis of §1 and let $C \varepsilon\ {}^{\gamma}\mathcal{P}(X')$. Define $C^{\circledast} = < C^{*}_{\beta} : \beta < \gamma >$.

Lemma 2.4. Assume $\gamma \varepsilon \omega_{1}$ and C is suitable for $\mathcal{D}_{\gamma}(\mathcal{P}(X'))$. Then $C^{\circledast}$ is suitable for $\mathcal{D}_{\gamma}(\mathcal{P}(X))$ and $(D(C))^{*} \subseteq D(C^{\circledast}) \subseteq (D(C))^{\Delta}$.

Proof.

Since the intersection of a comeager subset of G with a non-meager set is non-meager, we have for each $\beta < \gamma$,

$$C^{*}_{\beta} - C^{*}_{\beta+1} = C^{*}_{\beta} \cap (-C_{\beta+1})^{\Delta} \subseteq (C_{\beta} - C_{\beta+1})^{\Delta}.$$

Since the transform $B \mapsto B^{\Delta}$ commutes with countable unions and the transform $B \mapsto B^{*}$ commutes with countable intersections and preserves inclusions, we may substitute "*" for "-", "Δ" for "+" in the proof of 2.2 to obtain a proof of 2.4. □

Theorem 2.5. Assume that $\mathcal{J} = (G,X,J)$ is a special action, $1 < \alpha < \omega_1$, $\gamma < \omega_1$.

(a) If C is suitable for $\mathcal{D}_\gamma(\underset{\sim}{\Pi}^0_{(\alpha)}(X))$, then $C^{\circledast}$ is suitable for $\mathcal{D}_\gamma(\mathrm{inv}(\Pi^0_{(\alpha)}(X))$ and $D(C^{\circledast})$ is an invariantization of $D(C)$.

(b) $E_{\mathcal{J}}\text{-inv}(\mathcal{D}_\gamma(\underset{\sim}{\Pi}^0_{(\alpha)}(X))) \quad = \quad \mathcal{D}_\gamma(E_{\mathcal{J}}\text{-inv}(\underset{\sim}{\Pi}^0_{(\alpha)}(X)))$.

(c) If X is Polish then $E_{\mathcal{J}}\text{-inv}(\underset{\sim}{\Pi}^0_{\alpha}(X))$ has the strong separation property with respect to $\mathcal{D}_{(\omega_1)}(E_{\mathcal{J}}\text{-inv}(\underset{\sim}{\Pi}^0_{(\alpha)}(X)))$.

Proof.

(a) follows from 2.4, (2) and (7). (b) follows from (a). (c) follows from (a) and (10). □

§3. On Continuous Cross-Sections.

Let E be an equivalence relation on a set X. $s\colon X/E \to X$ is a cross-section for E if $\pi \circ s$ is the identity on X/E, where π is the canonical projection. An equivalent condition is that $\check{s} = s \circ \pi$ is a selector for E, (as defined in I §3). Note also that every selector s induces a cross-section $\hat{s}\colon [x]_E \mapsto s(x)$. When X is a topological space and X/E has the quotient topological structure, it is apparent that s is a continuous cross-section if and only if $\check{s}$ is a continuous selector.

(11) Suppose s is a continuous selector. If T is the collection of fixed points of s, then T is closed and $\pi \restriction T$ is a homeomorphism with inverse $\hat{s}$. If P is any property which is hereditary with respect to closed subspaces, (e.g. "complete", "Polish"), then X/E satisfies P when X does. If B is any set, then $B^- \subseteq s^{-1}(B) \subseteq B^+$; so $B \mapsto s^{-1}(B)$ solves the Γ-invariantization problem for any collection $\Gamma \subseteq \mathcal{P}(X)$ which is closed under the operation of taking inverse continuous images. In particular this is true whenever $\Gamma = \mathcal{G}(\utilde{\Sigma}_1^0)$, where $\mathcal{G}$ is any Boolean operation.

Given a sequence $A = \langle A_i\colon i \in I\rangle \in {}^I\mathcal{P}(X)$ and a function $s\colon X \to X$, let $s^{-1}(A) = \langle s^{-1}(A_i)\colon i \in I\rangle$. If A and B are sequences such that B reduces A, then $s^{-1}(B)$ reduces $s^{-1}(A)$. Suppose s is a continuous selector and Γ is a class which has the reduction property and is closed under inverse continuous images. Then $\text{inv}(\Gamma)$ has the reduction property. To see this let

$A \in {}^{\omega}(\mathrm{inv}(\Gamma))$ and let $B \in {}^{\omega}\Gamma$ reduce A; then $s^{-1}(B) \in {}^{\omega}(\mathrm{inv}(\Gamma))$ reduces $s^{-1}(A) = A$.

If Γ is a collection of sequences, let $\mathrm{inv}(\Gamma) = \{A \in \Gamma: (\forall i \in \mathrm{dom}(A))(A_i \text{ is invariant})\}$. A property $P(A_1,\ldots,A_n)$ of sequences is <u>Boolean</u> if $P(A_1,\ldots,A_n)$ implies $P(s^{-1}(A_1),\ldots,s^{-1}(A_n))$ for any function s. The argument of the preceding paragraph is easily generalized to show:

(12) Suppose X is a topological space, $P(A_1,\ldots,A_n)$ is a Boolean property, and $Q(\Gamma_1,\ldots,\Gamma_n)$ is defined by the equation

$$Q(\Gamma_1,\ldots,\Gamma_n) \equiv (\forall A_1 \in \Gamma_1)(\exists A_2 \in \Gamma_2)(\forall A_3 \in \Gamma_3)\ldots(({}^{\forall}_{\exists})A_n \in \Gamma_n(P(A_1,\ldots,A_n)).$$

Suppose for $j = 1,\ldots,n$, Γ_j is a collection of sequences which is closed under $A \mapsto s^{-1}(A)$ whenever s is continuous. If there exists a continuous selector for E, then $Q(\Gamma_1,\ldots,\Gamma_n)$ implies $Q(\mathrm{inv}(\Gamma_1),\ldots,\mathrm{inv}(\Gamma_n))$.

In view of these strong consequences, it is important to determine just which equivalence spaces admit continuous cross-sections.

We will apply the following result (13), due to Kuratowski and Ryll-Nardzewski (see [27]) to obtain a sufficient condition for the existence of a continuous cross-section for X/E when X is Polish.

(13) Let X be a Polish space, Y an arbitrary set, and $\mathcal{L}$ a field of subsets of Y. Let $\mathcal{L}_\sigma$ be the closure of $\mathcal{L}$ under

countable unions. Suppose F is a function on Y to the collection of closed subsets of X such that for every open set $G \subseteq X$, $\{y: F(y) \cap G \neq \emptyset\} \in \mathcal{L}_\sigma$. Then there exists a function $f: Y \to X$ such that

(i) $f(y) \in F(y)$ whenever $y \in Y$;

(ii) $f^{-1}(G) \in \mathcal{L}_\sigma$ whenever G is open in X.

Theorem 3.1. Let E be a lower semicontinuous (l.s.c.) equivalence on a Polish space X. If X/E is T_1 (points are closed) and zero-dimensional, then there exists a continuous cross-section for E.

Proof.

Let $Y = X/E$, $\mathcal{L} = \{O \subseteq X/E: O \text{ is clopen}\}$, F = the identity map $[x] \mapsto [x]$. X/E is T_1 just when each equivalence class is closed in X. Since E is lower semicontinuous, $\pi(G^+) = \{y: F(y) \cap G \neq \emptyset\}$ is open for every open set $G \subseteq X$. The function f given by (13) is a continuous cross-section for E. □

Remarks.

Assume X is Polish.

V. If X has a basis $\mathcal{H}$ of clopen sets such that B^{+E} is clopen for every $B \in \mathcal{H}$, then E is l.s.c. and X/E is zero-dimensional. This is the case which relates to model theory (see III §6 below).

VI. If X is zero-dimensional and E is both lower and upper semicontinuous, then the hypothesis of remark (V) is fulfilled. In this case the identity function on X/E to the space 2^X of closed subsets of X (with the exponential topology) is continuous (see Kuratowski [26] §19.IV). The continuous section may then be obtained from a theorem of Čoban (*cf.* Engelking, Heath, Michael [15]). This may account for the absence of 3.1 from the extensive literature on the Kuratowski-Ryll-Nardzewski selector theorem.

VII. If a 0-dimensional T_1 space has a countable basis, then it is metrizable (*cf.* [26] §22.II.1). Thus, assuming the hypothesis of 3.1, the conclusion "X/E is Polish" may be derived from a classical theorem of ~~Banach~~ Hausdorff *viz.*: *If f is a continuous open map from a Polish space to a metrizable space, then the image of f is Polish* (*cf.* Sierpinski [41] p. 197).

Chapter III: SOME APPLICATIONS OF TOPOLOGICAL METHODS TO MODEL THEORY

In this chapter we will apply some of the theory developed in chapter II and in the work of previous authors to the canonical logic actions. This will yield results in the model theory of the language $L_{\omega_1\omega}$ and its fragments (including $L_{\omega\omega}$).

The first four sections are concerned primarily with the $\underset{\sim}{\Pi}'^{o}_{\alpha}$ separation theorem and its consequences. Sections five and six contain two additional applications of Vaught's transform method.-- In § 5 we apply the transform to derive a recent "Global Definability Theorem" of M. Makkai [30] from a classical theorem of Lusin; in §6 we characterize "invariant α-Borel measurable functions" between logic spaces as the "$\underset{\sim}{\Delta}'^{o}_{\alpha}$-definable functions", thereby extending results of Craig [13] and Lopez-Escobar [28]. In §7 we discuss consequences of the selector theorem of II §3 and some related material.

This is a convenient point to collect some new notations and facts which we will use throughout the chapter.

Some set algebraic definitions, (e.g. of "A reduces B", "K has the first separation property," "$\langle C_\beta : \beta < \gamma \rangle$ is decreasing"), apply without modification to proper classes and will be used in this way. All of these definitions can be easily formalized, say, in Morse-Kelly set theory (cf. [24]) or translated into statements about predicates of Zermelo-Frankel set theory in the standard fashion.

Let ρ be an arbitrary fixed similarity type.

Except when the contrary is explicitly stated, all equivalence-theoretic terms refer to the canonical equivalence I_ρ when applied to subsets of X_ρ and all action-theoretic terms refer to the canonical logic action. In particular, all uses of Vaught's *-transform refer to this action.

We will make extensive use of the definability results obtained in Vaught [46]. For many applications the basic result --

(1) $\mathrm{inv}(\underset{\sim}{\Pi}^{o}_{\alpha}(X)) = \underset{\sim}{\Pi}'^{o}_{\alpha}(X)$ for all $\alpha \geq 1$ and all invariant $X \subseteq X_\rho$ --

will suffice. In some contexts, notably in sections four and five, the stronger result (2) from which (1) is derived will be applied.

$\underline{^n\omega}$ is the collection of all one-one functions on n to ω. For $s \in \underline{^n\omega}$, $[s] \subseteq \omega!$ is the set of permutations which extend s. For $n \in \omega$, $B \subseteq X_\rho$,

$$B^{*n} = \{(R,s)\colon\ s \in \underline{^n\omega}\ \ \&\ \ R \in B^{*[s]}\}.$$

$B^{\Delta n}$ is defined dually.

Vaught ([46] 3.1) proved

For all $\alpha \geq 1$, $n \in \omega$, and all invariant $X \subseteq X_\rho$, if

(2) $B \in \underset{\sim}{\Pi}^{o}_{\alpha}(X)$, (respectively $\underset{\sim}{\Sigma}^{o}_{\alpha}(X)$), then B^{*n}, $(B^{\Delta n})$, belongs to $\underset{\sim}{\Pi}'^{o}_{\alpha}(X^{(n)})$, $(\underset{\sim}{\Sigma}'^{o}_{\alpha}(X^{(n)}))$.

We will at one point have use for the effective version, ([46] 5.1), of (2).

(3) There is a prim(ω,ρ)-function $\theta \mapsto \langle\theta^{*n}: n \in \omega\rangle$ such that if θ is a $\underset{\sim}{\Pi}^0_\alpha$-$\rho$-name, then for every n, $\theta^{*n} \in \Pi'^0_\alpha(\rho)$ is an n-formula and $[\theta]^{*n} = [\![\theta^{*n}(n)]\!]$.

Given a similarity type ρ, let $\hat{\rho}$ be the result of replacing each constant symbol $\underline{c} \in \rho$ by a unary predicate $\underline{R}_{\underline{c}} = (1,((\rho,\underline{c}),1))$. Each ρ-structure $\mathfrak{A}$ becomes a $\hat{\rho}$ structure $\hat{\mathfrak{A}}$ by replacing each $\underline{c}^{\mathfrak{A}}$ with $\{\underline{c}^{\mathfrak{A}}\}$. The map $\mathfrak{A} \mapsto \hat{\mathfrak{A}}$ carries V_ρ onto the class $\hat{V}_\rho = \mathrm{Mod}(\bigwedge_{\underline{c}\in C_\rho} (\underset{\sim}{\exists}! \underline{v}_0)\underline{R}_{\underline{c}}(\underline{v}_0)) \subseteq V_{\hat{\rho}}$. Note that $\hat{V}_\rho$ is $\underset{\sim}{\Pi}'^0_2$ if C_ρ is countable. $\hat{\rho}$, $\hat{\mathfrak{A}}$ are the <u>relationalizations</u> of ρ and $\mathfrak{A}$. It is easily seen that

(4) There exist prim functions $\phi \mapsto \hat{\phi}$, $\psi \mapsto \check{\psi}$ such that for every type ρ and $\phi \in L_{\omega_1\omega}(\rho)$, $\psi \in L_{\omega_1\omega}(\hat{\rho})$

(i) $\mathrm{Mod}(\hat{\phi}) = \{\hat{\mathfrak{A}}: \mathfrak{A} \in \mathrm{Mod}(\phi)\}$, $\mathrm{Mod}(\check{\psi}) = \{\mathfrak{A}: \hat{\mathfrak{A}} \in \mathrm{Mod}(\psi)\}$

(ii) $(\forall\alpha \geq 1)[\psi \in \underset{\sim}{\Pi}'^0_\alpha(\hat{\rho})$ (resp. $\underset{\sim}{\Sigma}'^0_\alpha(\hat{\rho})$) $\Rightarrow$

$$\check{\psi} \in \underset{\sim}{\Pi}'^0_\alpha(\rho) \quad (\underset{\sim}{\Sigma}'^0_\alpha(\rho))]$$

(iii) $(\forall\alpha \geq 2)[(C_\rho$ countable & $\phi \in \underset{\sim}{\Pi}'^0_\alpha) \Rightarrow \hat{\phi} \in \underset{\sim}{\Pi}'^0_\alpha]$

A formula ϕ is in <u>negation normal form</u> if the symbol $\neg$ occurs only in subformulas of the form $\neg\theta$ when θ is atomic. From the infinitary DeMorgan laws one obtains

(5) There is a prim function $\phi \mapsto \phi^{\neg}$ such that $(\forall \rho)(\forall \phi \in L_{\omega_1\omega}(\rho))(\mathrm{Mod}(\phi) = \mathrm{Mod}(\phi^{\neg})$ & $\phi^{\neg}$ is in negation normal form).

If ρ is countable, then so is the set of finite ρ-structures. It follows that every collection of finite ρ-structures is $\utilde{\Sigma}'^{0}_{2}(V_\rho)$. This fact, together with the Löwenheim-Skolem theorem is aften sufficient to extend definability results for X_ρ to corresponding results over all models. For some purposes -- notably in dealing with $\utilde{\Pi}'^{0}_{2}$ or with questions of effectiveness -- this ad hoc approach breaks down and we need to accommodate finite models in a variant of the usual logic space.

The (familiar) trick is to treat equality as a non-logical symbol so that an infinite set of natural numbers can represent a single element of a finite structure.

Assume ρ has no operation symbols. Let $\approx$ be a binary relation symbol and let $\bar{\rho} = \rho + \approx$. Let $\bar{X}_\rho \subseteq X_{\bar{\rho}}$ be the collection of all $(S,\sim)$ such that $\sim$ is a congruence on ω for each relation in S and each congruence class is infinite. Since each equality axiom is $\utilde{\Pi}'^{0}_{1}$, $\bar{X}_\rho$ is $\utilde{\Pi}'^{0}_{2}$ in $X_{\bar{\rho}}$ when ρ is countable. Given $(S,\sim) \in \bar{X}_\rho$, the natural quotient structure $(S,\sim)/\sim$ is a ρ-structure and it is apparent that every finite or infinite countable ρ-structure can be obtained as such a quotient.

Given $\phi \in L_{\omega_1\omega}(\rho)$ let $\overline{\phi} = \phi(\underset{\sim}{\overline{=}})$ be the result of substituting $\approx$ for the equality symbol $\underset{\sim}{=}$ throughout ϕ. Clearly $\overline{\phi}$ has the same position in the Borel' hierarchy on $\overline{\rho}$ that ϕ has in the hierarchy on ρ. Furthermore, if ϕ is an n-formula, then

$$[\![\overline{\phi}^{(n)}]\!] \cap \overline{X}_\rho^{(n)} = \{(S,\sim,i_1,\ldots,i_n): ((\omega,S,\sim)/\sim,[i_1],\ldots,[i_n]) \models \phi\}.$$

Here, $[i] = [i]_\sim$ is the $\sim$-orbit of i.

$[\![\overline{\phi}^{(n)}]\!] \cap \overline{X}_\rho^{(n)}$ will be denoted $(\!|\phi^{(n)}|\!)$. As usual we drop the superscript when $n = 0$. Given any class Γ of ρ-formulas, we let $\overline{\Gamma} = \{\overline{\phi}: \phi \in \Gamma\}$.

Since each $\sim$ is a congruence, any isomorphism between structures $(S,\sim)$, $(S',\sim') \in \overline{X}_\rho$ induces an isomorphism between the corresponding quotients. It follows that each class $(\!|\phi|\!)$ is an I_ρ-invariant subset of $\overline{X}_\rho$, and we have for each $n \in \omega$,

$$\text{(6)} \qquad \overline{\underset{\sim}{\Pi}'^{o}_{\alpha}(\overline{X}_\rho^{(n)})} \subseteq \text{inv}(\underset{\sim}{\Pi}^{o}_{\alpha}(\overline{X}_\rho^{(n)})).$$

Since all congruence classes have the same cardinality, any isomorphism between quotient structures $(S,\sim)/\sim$ and $(S',\sim')/\sim'$ can be lifted to an isomorphism between the structures $(S,\sim)$ and $(S',\sim')$. Thus, I_ρ is the natural equivalence on $\overline{X}_\rho$ to study for applications to logic.

With a slight modification of the proof, Vaught's main definability results (1)-(3) go over to the new situation. The key remark which allows this modification is

(7) Assume $C_\rho = \emptyset$. If ψ is an m-formula of $L_{\omega_1\omega}(\overline{\rho})$ such that the symbol $\equiv$ does not occur in ψ, and $n \leq m$, then

$$[\![(\exists^{\neq} \underline{v}_n \cdots \underline{v}_{m-1})(\psi)^{(n)}]\!] = [\![(\exists \underline{v}_n \cdots \underline{v}_{m-1})(\psi)^{(n)}]\!].$$

The inclusion from left to right in (7) is trivial. For the reverse inclusion let $(S,\sim) \in \overline{X}_\rho$ and suppose $(\omega,S,\sim,i_o,\ldots,i_m) \models \psi$. Since each congruence class is infinite, there exist distinct numbers $i'_n,\ldots,i'_m$ such that $i_j \sim i'_j$ for $j = n,\ldots,m$. Since $\sim$ is a congruence, $(\omega,S,\sim,i_o,\ldots,i_{n-1},i'_n,\ldots,i'_m) \models \psi$ and $(S,\sim,i_o,\ldots,i_{n-1}) \in [\![(\exists^{\neq} \underline{v}_n \cdots \underline{v}_m)(\psi)^{(n)}]\!]$ as required, establishing (7).

<u>Proposition 0.1</u>. Assume $C = \emptyset$.

(a) Assume $n \in \omega$, $\alpha \geq 1$. If $B \in \underset{\sim}{\Pi}^o_\alpha(X_{\overline{\rho}})$, then $B^{*n} \cap \overline{X}^{(n)}_\rho \in \overline{\underset{\sim}{\Pi}'^o_\alpha}(\overline{X}^{(n)}_\rho)$. If $B \in \underset{\sim}{\Sigma}^o_\alpha(X_{\overline{\rho}})$, then $B^{\Delta n} \cap \overline{X}^{(n)}_\rho \in \overline{\underset{\sim}{\Sigma}'^o_\alpha}(\overline{X}^{(n)}_\rho)$.

(b) There is a $\mathrm{prim}(\omega,\rho)$ function $\theta \mapsto \langle\theta^{(*)n}\ n\epsilon\omega\rangle$ such that if θ is a $\underset{\sim}{\Pi}^o_\alpha$-$\overline{\rho}$-name, then for every n, $\theta^{(*)n} \in \underset{\sim}{\Pi}'^o_\alpha(\rho)$ and $[\theta]^{*n} \cap \overline{X}^{(n)}_\rho = (\theta^{(*)n(n)})$.

<u>Proof</u>.

(a) Vaught's proof of (2) is easily modified using (7) to establish (a). We prove by induction that for $B \in \mathcal{B}(X_{\overline{\rho}})$, each $B^{*n} \cap \overline{X}^{(n)}_\rho$, $B^{\Delta n} \cap \overline{X}^{(n)}_\rho$ has the form $[\![\psi^{(n)}]\!] \cap \overline{X}^{(n)}_\rho$ where $\psi \in L_{\omega_1\omega}(\overline{\rho})$ is of the proper form and <u>does not contain the equality symbol</u>. (a) then follows since $\overline{\psi(\underset{\equiv}{\sim})} = \psi(\underset{\equiv}{\approx})(\underset{\sim}{\equiv}) = \psi$ if $\equiv$ does not appear in ψ.

Consider the initial step. Let $\mathcal{B}$ be a basic clopen set in $X_{\overline{\rho}}$. Then $B = [\psi(\underline{0},\ldots,\underline{m})]$ for some basic name ψ which does not involve the equality symbol. We know (e.g. from [46]) that $B^{\Delta n} = [\![(\underset{\sim}{\exists}^{\neq}\underline{v}_n\ldots\underline{v}_m)(\psi)^{(n)}]\!]$. By (7),

$$[\![(\underset{\sim}{\exists}^{\neq}\underline{v}_n\ldots\underline{v}_m)(\psi)^{(n)}]\!] \cap \overline{X}_{\rho}^{(n)} \quad = \quad [\![(\underset{\sim}{\exists}\underline{v}_n\ldots\underline{v}_m)(\psi)^{(n)}]\!] \cap \overline{X}_{\rho}^{(n)} .$$

The remaining steps are similar. At each stage we carry the additional hypothesis that the formulas defined previously do not contain $\underline{=}$; we use the argument from [46] to construct a new formula; then we use (7) to eliminate the equality symbol from that new formula.

(b) Just as in [46], the proof of (a) is uniform and establishes the effective result (b). □

<u>Corollary 0.2</u>. Assume $C_{\rho} = \emptyset$. For all $\alpha \geq 1$, $n \in \omega$,

$$\mathrm{inv}(\underset{\sim}{\Pi}^{0}_{\alpha}(\overline{X}_{\rho}^{(n)})) \quad = \quad \overline{\underset{\sim}{\Pi}'^{0}_{\alpha}}(\overline{X}_{\rho}^{(n)})$$

and

$$\mathrm{inv}(\underset{\sim}{\Sigma}^{0}_{\alpha}(\overline{X}_{\rho}^{(n)})) \quad = \quad \overline{\underset{\sim}{\Sigma}'^{0}_{\alpha}}(\overline{X}_{\rho}^{(n)}) .$$

<u>Proof</u>.

In each case the inclusion from right to left was noted in (7). The reverse inclusions follow immediately from 0.1(a). □

§1. The $\underset{\sim}{\Pi}'^{0}_{\alpha}$ Separation Theorem

Consider the $\underset{\sim}{\Sigma}^1_1$ separation theorem in topology and logic. The basic theorem -- "disjoint $\underset{\sim}{\Sigma}^1_1$ subsets of a Polish space can be separated by a Borel set" -- was obtained by Lusin in 1927 (cf. [26]). In 1957, W. Craig proved an analogous fact in logic -- "Disjoint $\underset{\sim}{\exists}^1_1(\rho)$ classes can be separated by an $L_{\omega\omega}(\rho)$ elementary class." Several years later, noting the analogy between these results, D. Scott conjectured that a similar result held for $L_{\omega_1\omega}$. This was established by E. G. K. Lopez-Escobar in 1965 ([28]). At about the same time, J. Keisler [22] developed a theory of finitary approximations to infinitary formulas (which will be summarized below) which allows one to dervie Craig's theorem from Lopez-Escobar's. Finally, Vaught showed in [46] how to obtain Lopez-Escobar's theorem from Lusin's classical result.

Thus, we can derive Craig's theorem from that of Lusin as follows: Given mutually inconsistent $\underset{\sim}{\exists}^1_1$ sentences θ_1, θ_2 of type ρ (ρ necessarily countable), note that $[\![\theta_1]\!]$ and $[\![\theta_2]\!]$ are disjoint $\underset{\sim}{\Sigma}^1_1$ subsets of the Polish space X_ρ. By Lusin's theorem, there is a Borel set B which separates them. By Vaught's results, the Borel set B^* also separates them and $B^* = [\![\phi]\!]$ for some $\phi \in L_{\omega_1\omega}(\rho)$. By the Löwenheim-Skolem theorem, $\mathrm{Mod}(\phi)$ separates $\mathrm{Mod}(\theta_1)$ from $\mathrm{Mod}(\theta_2)$ over infinite models. Since every collection of finite models is definable in $L_{\omega_1\omega}(\rho)$, there is a variant $\phi' \in L_{\omega_1\omega}(\rho)$ such that $\mathrm{Mod}(\phi')$ separates $\mathrm{Mod}(\theta_1)$ from $\mathrm{Mod}(\theta_2)$ over all models. By Keisler's results, the same is true of some $\sigma \in L_{\omega\omega}(\rho)$ which "approximates" ϕ'.

In this section we will see that the Shoenfield $\underset{\sim}{\forall}^{o}_{n}$ separation theorem has an analogous relation to the Hausdorff-Kuratowski $\underset{\sim}{\Pi}^{o}_{\alpha}$ separation theorem.

The principle "$\underset{\sim}{\forall}^{o}_{n}(V_{\rho})$ $(n > 1)$ has the first separation property" was conjectured by Addison (see [2]) based on the analogy between logic and descriptive set theory. This conjecture was established in a strong form by Shoenfield, (cf. [2]).

Theorem 1.2 below is the intermediate step in a derivation of Shoenfields's theorem from Hausdorff's.

Remark I. In his dissertation [37], Myers proved a separation theorem for multiplicative classes in the $L_{\omega_1\omega}$ hierarchy based on quantifier depth (without regard to infinite conjunction and disjunction). Myers' result also yields Shoenfield's via the approximation theory, but it is much less natural topologically. We do not know a topological theorem about logic spaces from which Myers' result can be obtained.

Let K be a collection of ρ-structures and suppose $\phi =$ $\langle\phi_\beta\colon \beta \le \gamma\rangle$ is a sequence of ρ-sentences. We say ϕ is decreasing, (respectively continuous), over K provided $\langle Mod(\phi_\beta) \cap K\colon \beta \le \gamma\rangle$ is decreasing (continuous). $D(\phi) = \underset{\sim}{\bigvee}\{\phi_\beta \wedge \neg\phi_{\beta+1}\colon \beta \in e(\gamma)\}$. Let $\Omega \subseteq L_{\omega_1\omega}(\rho)$. ϕ is suitable for $\mathcal{D}^{K}_{\gamma}(\Omega)$ if $\phi \in {}^{\gamma+1}\Omega$, ϕ is decreasing and continuous over K, $\mathrm{Mod}(\phi_o) \cap K = K$ and $\mathrm{Mod}(\phi_\gamma) \cap K = \emptyset$. We define $\mathcal{D}^K_\gamma(\Omega) = \{D(\Phi)\colon\ \Phi$ is suitable for $\mathcal{D}^K_\gamma(\Omega)\}$, $\mathcal{D}^K_{(\gamma)}(\Omega) = \bigcup\{\mathcal{D}^K_\beta(\Omega)\colon\ \beta < \gamma\}$. When $K = V_\rho$ we omit the superscript.

Note that under over convention "$\Omega(K) = \{\mathrm{Mod}(\phi) \cap K\colon \phi \in \Omega\}$ "

$$\mathcal{D}^K_\gamma(\Omega)(K) = \mathcal{D}_\gamma(\Omega(K))$$

(where $\mathcal{D}_\gamma(\Omega(K))$ is interpreted with respect to the "universe" K). In view of this identity we could state most of our results without defining the syntactical classes $\mathcal{D}^K_\gamma(\Omega)$. This would have the effect of making some results (e.g. 1.1) appear to have less syntactical content. When discussing syntactical notions, such as effectiveness and finite approximations, the syntactical classes $\mathcal{D}_\gamma(\Omega)$ seem to be indispensible.

In chapter II we solved the invariantization problem for each class $\mathcal{D}_\gamma(\underset{\sim}{\Pi}^o_\alpha(X))$ in any Polish action. In the canonical logic actions we can combine this with Vaught's characterization (1) of the invariant $\underset{\sim}{\Pi}^o_\alpha$ sets to obtain an analogous result for the "small Borel classes" $\mathcal{D}_\gamma(\underset{\sim}{\Pi}^o_\alpha(X_\rho))$.

<u>Theorem 1.1</u>. Assume ρ is an arbitary similarity type, $\gamma < \omega_1$, and $1 < \alpha < \omega_1$.

(a) If X is any invariant subspace of X_ρ, then

$$\mathrm{inv}(\mathcal{D}_\gamma(\Pi^o_{(\alpha)}(X)) = \mathcal{D}^X_\gamma(\underset{\sim}{\Pi}'^o_\alpha)(X).$$

(b) If ρ is countable, then

$$\mathrm{inv}(\underset{\sim}{\Delta}^o_\alpha(X_\rho)) = \mathcal{D}^{X_\rho}_{(\omega_1)}(\underset{\sim}{\Pi}'^o_\alpha)(X_\rho) .$$

Assume $C_\rho = \emptyset$. Then

(c) $$\mathrm{inv}(\mathcal{D}_\gamma(\underset{\sim}{\Pi}^o_{(\alpha)}(\overline{X}_\rho))) = \overline{\mathcal{D}_\gamma(\underset{\sim}{\Pi}'^o_{(\alpha)})}(\overline{X}_\rho) .$$

(d) If ρ is countable, then

$$\mathrm{inv}(\underset{\sim}{\Delta}^o_\alpha(\overline{X}_\rho)) = \overline{\mathcal{D}_{(\omega_1)}(\underset{\sim}{\Pi}'^o_{(\alpha)})}(\overline{X}_\rho).$$

<u>Proof</u>.

(a) By II.2.5(a), $\mathrm{inv}\mathcal{D}_\gamma(\Pi^o_{(\alpha)}(X)) = \mathcal{D}_\gamma(\mathrm{inv}(\Pi^o_{(\alpha)}(X)))$. By (1), $\mathrm{inv}(\Pi^o_{(\alpha)}(X)) = \underset{\sim}{\Pi}'^o_{(\alpha)}(X)$, so $\mathcal{D}_\gamma(\mathrm{inv}(\underset{\sim}{\Pi}^o_{(\alpha)}(X)) = \mathcal{D}_\gamma(\underset{\sim}{\Pi}'^o_{(\alpha)}(X)) = \mathcal{D}^X_\gamma(\underset{\sim}{\Pi}'^o_{(\alpha)})(X)$.

(b) If ρ is countable, then $\underset{\sim}{\Delta}^o_\alpha(X_\rho) = \mathcal{D}_{(\omega_1)}(\underset{\sim}{\Pi}^o_{(\alpha)})(X_\rho))$, so (b) follows from (a).

(c) (c) follows from 0.2, II.2.5(a) and the fact that a sequence $\langle\phi_\beta\colon \beta \le \gamma\rangle$ of $L_{\omega_1\omega}(\rho)$ sentences is decreasing or continuous over V_ρ if and only if $\langle\{\phi_\beta\}\colon \beta \le \gamma\rangle$ is decreasing or continuous.

(d) (d) follows from (c) and the invariant separation theorem II.2.5(c). Note that when ρ is countable, $\overline{X}_\rho$ is a $\underset{\sim}{\Pi}_2^0$ subspace of the Polish space $X_{\overline{\rho}}$ and the restriction of the canonical action to $\overline{X}_\rho$ is still Polish. □

Theorem 1.2. Let ρ be a countable similarity type and let $\alpha \geq 2$. Then the collection $\underset{\sim}{\Pi}'^0_\alpha(V_\rho)$ has the strong separation property with respect to $\mathcal{D}_{(\omega_1)}(\underset{\sim}{\Pi}'^0_{(\alpha)})(V_\rho)$.

Proof.

First assume ρ contains no operation symbols. Let $\mathrm{Mod}(\theta_1)$, $\mathrm{Mod}(\theta_2)$ be disjoint Π'^0_α classes. Then $\{\theta_1\}$, $\{\theta_2\}$ are disjoint invariant $\underset{\sim}{\Pi}^0_\alpha$ subsets of $X_{\overline{\rho}}$. By II.2.5 there is a set $D \in \mathcal{D}_{(\omega_1)}(\mathrm{inv}\underset{\sim}{\Pi}^0_{(\alpha)}(X_{\overline{\rho}}))$ which separates $\{\theta_1\}$ from $\{\theta_2\}$. By 1.1(c), $D \cap \overline{X}_\rho = \{\phi\}$ for some $\phi \in \mathcal{D}_{(\omega_1)}(\underset{\sim}{\Pi}'^0_{(\alpha)}(\rho))$.

Clearly $\mathrm{Mod}(\phi)$ separates $\mathrm{Mod}(\theta_1)$ from $\mathrm{Mod}(\theta_2)$ over countable models and by the Löwenheim-Skolem theorem for $L_{\omega_1\omega}(\rho)$, $\mathrm{Mod}(\phi)$ separates $\mathrm{Mod}(\theta_1)$ from $\mathrm{Mod}(\theta_2)$ over all models.

Now let ρ be arbitrary. Given θ_1, θ_2 as above, pass to the relationalizations $\hat{\theta}_1, \hat{\theta}_2$. Since $\alpha \geq 2$, $\hat{\theta}_1, \hat{\theta}_2 \in \underset{\sim}{\Pi}'^0_\alpha(\hat{\rho})$. Choose $\phi \in \mathcal{D}_{(\omega_1)}(\underset{\sim}{\Pi}'^0_{(\alpha)}(\hat{\rho}))$ as in the previous paragraph. Then $\check{\phi} \in \mathcal{D}_{(\omega_1)}(\underset{\sim}{\Pi}'^0_{(\alpha)}(\rho))$ is easily seen to have the required property. □

Next we will apply Keisler's theory of finite approximations to derive Shoenfield's separation theorem from 1.3 (and by the transitivity of "derive", from Hausdorff's separation theorem).

The approximation theory applies to a wide class of languages (see [22]). We summarize that part which we will apply. Fixing ρ, the set $A(\phi)$ of finite approximations to ϕ is defined, for every formula ϕ of $L_{\omega_1\omega}(\rho)$ which is in negation normal form, by the recursive conditions:

(8) (i) If $\phi \in L_{\omega\omega}$, then $A(\phi) = \{\phi\}$

(ii) If $\phi = \bigwedge \Theta$, then $A(\phi) = \{\bigwedge\{\sigma_o,\ldots,\sigma_n\}\colon n \in \omega$ and for some distinct $\theta_o,\ldots,\theta_n \in \Theta$, for every $i \leq n$, $\sigma_i \in A(\theta_i)\}$.

(iii) If $\phi = (\forall v)(\psi)$, then $A(\phi) = \{(\forall v)(\bigwedge\{\theta_o,\ldots,\theta_n\})\colon n \in \omega,$ $\theta_o,\ldots,\theta_n \in A(\psi)\}$.

(iv) If $\phi = \bigvee \Theta$ or $(\exists v)(\psi)$ then $A(\phi)$ is obtained by the dual condition to (ii) or (iii) (replace $\bigwedge$ by $\bigvee$ and $\forall$ by $\exists$).

$A^c(\phi)$ is the closure of $A(\phi)$ under finite conjunction and disjunction.

In [22] (Cor. 3.4) Keisler showed

(9) Suppose $\theta_1,\theta_2 \in L_{\omega\omega}(\rho)$, $\phi \in L_{\omega_1\omega}(\rho)$, ϕ is in negation normal form, and $\mathrm{Mod}(\phi)$ separates $\mathrm{Mod}(\theta_1)$ from $\mathrm{Mod}(\theta_2)$. Then there is an approximation $\sigma \in A^c(\phi)$ such that $\mathrm{Mod}(\sigma)$ also separates $\mathrm{Mod}(\theta_1)$ from $\mathrm{Mod}(\theta_2)$.

$B^o_n(\rho)$ is the closure of $\underset{\sim}{\forall}^o_n(\rho) \cup \underset{\sim}{\exists}^o_n(\rho)$ under finite conjunctions and disjunctions. Induction on subformulas shows

(10) If $\phi \in \underset{\sim}{\Pi}'^o_n$, (resp. $\underset{\sim}{\Sigma}'^o_n$), and $\psi \in A^c(\ulcorner\phi\urcorner)$ then $\mathrm{Mod}(\psi) \in \underset{\sim}{\forall}^o_n(V_\rho)$, $(\underset{\sim}{\exists}^o_n(V_\rho))$.

It follows immediately from (9) and the definition (7) that

(11) If $\phi \in \mathcal{D}_{(\omega_1)}(\underset{\sim}{\Pi}'^o_n)$ and $\psi \in A^c(\ulcorner\psi\urcorner)$, then $\mathrm{Mod}(\psi) \in B^o_n(V_\rho)$.

Corollary 1.3. Assume ρ is countable and $n \geq 2$.

(a) $\underset{\sim}{\Delta}^o_n(\overline{X}_\rho) \cap \overline{L}_{\omega\omega}(\overline{X}_\rho) = = \overline{B}^o_{n-1}(\overline{X}_\rho)$.

(b) The collection $\underset{\sim}{\forall}^o_n(V_\rho)$ has the strong separation property with respect to $B^o_{n-1}(V_\rho)$.

Proof.

(a) The inclusion in (a) from right to left is trivial. For the reverse inclusion, suppose $B \in \underset{\sim}{\Delta}^o_n(\overline{X}_\rho) \cap \overline{L}_{\omega\omega}(\overline{X}_\rho)$, say $B = \langle\theta\rangle$ for $\theta \in L_{\omega\omega}(\rho)$. By 1.1(d), $B = \langle\phi\rangle$ for some $\phi \in \mathcal{D}_{(\omega_1)}(\underset{\sim}{\Pi}'^o_{n-1}(\rho))$. It follows from the Löwenheim-Skolem theorem that $\mathrm{Mod}(\phi) = \mathrm{Mod}(\theta)$. By (9), $\mathrm{Mod}(\theta) = \mathrm{Mod}(\sigma)$ for some $\sigma \in A^c(\ulcorner\phi\urcorner)$. By (11), $\langle\sigma\rangle \in \overline{B}^o_n(\overline{X}_\rho)$.

(b) Let θ_1, θ_2 be mutually inconsistent members of $\underset{\sim}{\forall}^o_n(\rho)$. By 1.2, there exists $\phi \in \mathcal{D}_{(\omega_1)}(\underset{\sim}{\Pi}'^o_{n-1}(\rho))$ such that $\mathrm{Mod}(\phi)$ separates $\mathrm{Mod}(\theta_1)$ from $\mathrm{Mod}(\theta_2)$. By (9), the same is true of some $\sigma \in A^c(\ulcorner\phi\urcorner)$. By (11), $\mathrm{Mod}(\sigma) \in B^o_n(V_\rho)$. □

Remarks.

II. After proving 1.2 we learned from Myers that, (at least for successor α and over infinite models), it was an unpublished result of G. E. Reyes. He apparently derived the case $\alpha = 2$ from Hausdorff's proof and the fact that the closure of any invariant subset of X_ρ is closed', and then translated the result to other successor α using Skolem predicates (presumably by the argument of remark IV, compare also our proof of §3.1 below).

III. The utility of Keisler's approximations for results like 1.3 was pointed out to the author by W. Wadge. Wadge had proved the identities $\underset{\sim}{\Pi}^{o}_{n}(X_\rho) \cap L_{\omega\omega}(X_\rho) = \underset{\sim}{\forall}^{o}_{n}(X_\rho)$, $n \geq 1$. Upon learning of Vaught's result (1) he remarked that his result followed from (1) via (9).

IV. The $\underset{\sim}{\Pi}'^{o}_{\alpha}$ separation theorem for successor $\alpha > 2$ can be reduced to the case $\alpha = 2$ by the following method. The method seems to be essential for the effective theorem of §2. It shows that the *-transform can be avoided in deriving 1.3 for successor α (though apparently not for limit α, nor for definability results such as 1.1).

Let ρ be countable and suppose $K_o, K_1 \in \underset{\sim}{\Pi}'^{o}_{\beta+1}(V_\rho)$ are disjoint, $\beta \geq 2$. For $i = 0,1$ choose $\theta_i = \bigwedge_n \underset{\sim}{\forall} \underline{v}_o \cdots \underline{v}_{k^i_n} \bigvee_m \underset{\sim}{\exists} \underline{v}_{k^i_n+1} \cdots \underline{v}_{k^i_{nm}} \theta^i_{nm}$ such that each $\theta^i_{nm} \in \underset{\sim}{\Pi}'^{o}_{(\beta)}(\rho)$, $K_i = \mathrm{Mod}(\theta_i)$. Let L be the smallest fragment of $L_{\omega_1\omega}(\rho)$ which contains each θ^i_{nm}. Let $\rho^\# = \rho^{\#L}$ be the similarity type which contains an n-ary predicate $\underline{R}_{\phi,n}$ for each

$n \in \omega$ and each n-formula $\phi \in L$. Given $\mathfrak{A} \in V_\rho^{(n)}$, $\psi \in L_{\omega_1\omega}(\rho)$, let $\mathfrak{A}^{\#} \in V_{\rho^{\#}}^{(n)}$ be the canonical expansion of $\mathfrak{A}$, and let $\mathrm{Mod}^{\#(n)}(\psi) = \{\mathfrak{A}^{\#}: \mathfrak{A} \in \mathrm{Mod}^{(n)}(\psi)\}$.

Let $V_\rho^{\#} = \{\mathfrak{A}^{\#}: \mathfrak{A} \in V_\rho\}$. Note that

(12) Each $\mathrm{Mod}^{\#(k^i_{nm}+1)}(\theta^i_{nm}) \in \underset{\sim}{\Pi}'^{\,0}_1(V_\rho^{\#(k^i_{nm}+1)})$, hence each $\mathrm{Mod}^{\#}(\theta_i) \in \underset{\sim}{\Pi}'^{\,0}_2(V_\rho^{\#})$.

(13) If $\phi \in \underset{\sim}{\Pi}'^{\,0}_1(\rho^{\#})$, then $\mathrm{Mod}(\phi) \cap V_\rho^{\#} = \mathrm{Mod}^{\#}(\psi)$ for some $\psi \in \underset{\sim}{\Pi}'^{\,0}_\beta(\rho)$.

By (12) and the $\underset{\sim}{\Pi}'^{\,0}_2$ separation theorem for $\rho^{\#}$, there exists $\phi \in \mathcal{D}_{(\omega_1)}(\underset{\sim}{\Pi}'^{\,0}_1(\rho^{\#}))$ such that $\mathrm{Mod}(\phi)$ separates $\mathrm{Mod}^{\#}(\theta_1)$ from $\mathrm{Mod}^{\#}(\theta_0)$. By (13), $\mathrm{Mod}(\phi) \cap V_\rho^{\#} = \mathrm{Mod}^{\#}(\psi)$ for some $\psi \in \mathcal{D}_{(\omega_1)}(\underset{\sim}{\Pi}'^{\,0}_\beta(\rho))$. Then $\mathrm{Mod}(\phi)$ separates $\mathrm{Mod}(\theta_1)$ from $\mathrm{Mod}(\theta_0)$.

§2. $\underset{\sim}{\Pi}'^{0}_{\alpha}$ Separation and the Problem of Effectiveness.

The main result of this section, (2.2), is an "admissible" version of the $\underset{\sim}{\Pi}^{0}_{\alpha}(X_\rho)$ separation theorem for $\rho \in HC$ and $\alpha \geq 2$, a successor ordinal. Since the construction used in 1.2 is highly effective, we will obtain a corresponding $\underset{\sim}{\Pi}'^{0}_{\alpha}$ separation theorem for certain admissible languages as a corollary.

The following lemma is an effective version of the classical method of generalized homeomorphisms (i.e. the classical method of Skolem predicates). It will be used to reduce the general case of the $\underset{\sim}{\Pi}^{0}_{\alpha+1}$ separation theorem to the case $\alpha = 1$.

Given ρ, α, ϕ, let $\underset{\sim}{\Pi}^{0}_{\alpha}(\rho)$ be the set of $\underset{\sim}{\Pi}^{0}_{\alpha}$-$\rho$-names and recall that $at(\rho)$ is the set of atomic ρ-names, $sub(\phi)$ is the set of subnames of ϕ. Let $\underset{\sim}{\Pi}^{0}_{(\alpha)}(\rho) = \bigcup\{\underset{\sim}{\Pi}^{0}_{\beta}(\rho)\colon\ \beta < \alpha\}$.

Lemma 2.1. Let $\mathcal{A} \subseteq HC$ be prim-closed, $\omega, \rho \in \mathcal{A}$, $1 < \alpha < \omega_1$. Suppose $\Theta \in \mathcal{A}$, $\Theta \subseteq \underset{\sim}{\Pi}^{0}_{(\alpha)}(\rho)$. Then there exist $\rho_1, \Psi, F_o, F_1 \in \mathcal{A}$, $g\colon X_\rho \to X_{\rho_1}$ such that

(i) ρ_1 contains only 0-ary relation symbols (i.e., ρ_1 is propositional).

(ii) Ψ is a $\underset{\sim}{\Pi}^{0}_{2}$-$\rho_1$-name and g is a $(1,\alpha)$-generalized homeomorphism on X_ρ onto $[\Psi]$.

(iii) $F_o\colon at(\rho_1) \to \underset{\sim}{\Sigma}^{0}_{\alpha}(\rho)$, $F_1\colon \Theta \to at(\rho_1)$ are functions such that for $\psi \in at(\rho_1)$, $\theta \in \Theta$, $[F_o(\psi)] = g^{-1}([\psi])$, and $[F_1(\theta)] \cap [\Psi] = g([\theta])$.

Proof.

Let $L = \{at(\rho)\} \cup \{sub(\theta): \theta \in \Theta\}$. Let ρ_1 be the similarity type with a 0-ary relation symbol $\underline{P}_\phi = (1,((L,\phi),0))$ for each $\phi \in L$. Let $\Psi \in A$ be a $\underset{\sim}{\Pi}_2^0$-ρ_1-name for $B_1 \cap B_2$ where

$$B_1 = \bigcap_{\underline{c} \in C_\rho} ((\bigcup_{i \in \omega} [\underline{P}_{\underline{c}=\underline{i}}]) \cap \bigcap_{i \neq j} [\underline{P}_{\underline{c}=\underline{i}} \to \neg \underline{P}_{\underline{c}=\underline{j}}])$$

$$B_2 = \bigcap_{\neg\phi \in L} [\underline{P}_{\neg\phi} \leftrightarrow \neg \underline{P}_\phi] \cap \bigcap_{\underset{\sim}{\bigvee}\Gamma \in L} [\underline{P}_{\underset{\sim}{\bigvee}\Gamma} \leftrightarrow \bigvee_{\gamma \in \Gamma} \underline{P}_\gamma]$$

$$\cap \bigcap_{\underset{\sim}{\bigwedge}\Gamma \in L} [\underline{P}_{\underset{\sim}{\bigwedge}\Gamma} \leftrightarrow \bigwedge_{\gamma \in \Gamma} \underline{P}_\gamma]$$

Let $L: L \to \underset{\sim}{\Sigma}_\alpha^0(\rho)$ be a prim function such that for every $\phi \in L$, $[L(\phi)] = [\phi]$. Define $F_o = \{(P_\phi, L(\phi)): \phi \in L\}$, $F_1 = \{(\theta, P_\theta): \theta \in \Theta\}$. For $R \in X_\rho$, set $g(R)(\underline{P}_\phi) = 1$ if $R \in [\phi]$, 0 otherwise. It is easily checked that ρ_1, Ψ, g, F_o, F_1 have the required properties. □

Given a sequence $\Phi = \langle\phi_\beta: \beta < \gamma\rangle$ of ρ-names, let $[\Phi] = \langle[\phi_\beta]: \beta < \gamma\rangle$.

Theorem 2.2. Assume $\mathcal{A} \subseteq HC$ is admissible, $\omega \in \mathcal{A}$, $1 \leq \mu < \omega_1$. Suppose that $\theta_1, \theta_2 \in \mathcal{A}$ are $\underset{\sim}{\Pi}_{\mu+1}^0$-$\rho$-names for disjoint subsets of X_ρ and that $\mathcal{A}$ contains a well-ordering of $TC(\theta_1, \theta_2, \rho)$. Then there exists $\delta \in \omega_1$, and a δ-sequence Φ of $\underset{\sim}{\Pi}_\mu^0$-$\rho$-names, such that $\Phi \in \mathcal{A}$, $[\Phi]$ is suitable for $\mathcal{D}_\delta(\underset{\sim}{\Pi}_\mu^0(X_\rho))$, and

$$[\theta_1] \subseteq D([\Phi]) \subseteq -[\theta_2].$$

Proof.

We consider two cases.

Case 1. $\mu = 1$ and ρ is propositional.

Since $\mathcal{A}$ is prim-closed and contains a well-ordering of $TC(\theta_1, \theta_2, \rho)$, there is some $\gamma \in \mathcal{A} \cap \omega_1$ and a sequence $\theta = \langle \theta_{\alpha\beta} : (\alpha,\beta) \in \gamma^2 \rangle \in \mathcal{A}$ of Basic ρ-names such that

$$[\theta_1] = [\underset{\sim}{\bigwedge}\{\underset{\sim}{\bigvee}\{\theta_{\alpha\beta}: \beta \in \gamma\}: \alpha \in o(\gamma)\}]$$

$$[\theta_2] = [\underset{\sim}{\bigwedge}\{\underset{\sim}{\bigvee}\{\theta_{\alpha\beta}: \beta \in \gamma\}: \alpha \in e(\gamma)\}].$$

$o(\gamma)$ and $e(\gamma)$ are respectively the sets of odd and even members of γ. We may further assume that for some $\gamma' \leq \gamma$, there is an enumeration $P = \langle \underset{\sim}{P}_\alpha : \alpha \in \gamma' \rangle \in \mathcal{A}$ such that $\rho = \{\underset{\sim}{P}_\alpha : \alpha \in \gamma'\}$. Extend P to $\overline{P}: \gamma \to \rho$ by setting $\underset{\sim}{P}_\alpha = \underset{\sim}{P}_0$ for $\alpha \geq \gamma'$.

Since $[\theta_1] \cap [\theta_2] = \emptyset$, $-[\theta_1] \cup -[\theta_2] = X_\rho$ and we have

$$(14) \qquad (\forall x \in X_\rho)(\forall f \in {}^\gamma\gamma)(\exists \alpha)(x \notin [\theta_{\alpha f(\alpha)}]).$$

Let d range over $P_{(\omega)}(\gamma)$. Let σ, τ range over the set T of finite functions with domain, range included in γ (i.e. "partial Skolem functions"). Let s, t range over the collection Γ of finite sets of subbasic ρ-names (i.e. "partial elements of X_ρ").

Each $\theta_{\alpha\beta}$ is a finite conjunction of subbasic names, say $\theta_{\alpha\beta} = \underset{\sim}{\bigwedge} t_{\alpha\beta}$. Given $\sigma \in T$, let $t^\sigma = \bigcup\{t_{\alpha\sigma(\alpha)} : \alpha \in \mathrm{dom}(\sigma)\}$, $\theta^\sigma = \underset{\sim}{\bigwedge} t^\sigma$, $[\theta^\sigma] = \bigcap_{\alpha \in \mathrm{dom}(\sigma)} [\theta_{\alpha\sigma(\alpha)}]$.

Choose a set $\infty \notin \mathrm{ON} \cup \mathcal{Q}$. Define $\mathrm{Rk}: \Gamma \times T \to \mathrm{On} \cup \{\infty\}$ by the conditions:

(15) $\mathrm{Rk}(s,\sigma) \geq 1$ if $[\underset{\sim}{\wedge} s] \cap [\theta^{\sigma}] \neq \emptyset$

$\mathrm{Rk}(s,\sigma) \geq \alpha+1$ if $(\forall \beta \in \gamma)(\exists (t,\tau) \in \Gamma \times T)$

$[s \subseteq t \;\; \& \;\; \sigma \subseteq \tau \;\; \& \;\; (\underline{P}_{\beta} \in t \text{ or } \neg \underline{P}_{\beta} \in t) \;\; \&$
$\beta \in \mathrm{dom}(\tau) \;\; \& \;\; \mathrm{Rk}(t,\tau) \geq \alpha]$

$\mathrm{Rk}(s,\sigma) \geq \lambda$ if $\mathrm{Rk}(s,\sigma) \geq \beta$ for every $\beta < \lambda$

$$\mathrm{Rk}(s,\sigma) = \begin{cases} \text{the smallest } \alpha \text{ such that } \mathrm{Rk}(s,\sigma) \not\geq \alpha+1 \text{ if such exists,} \\ \infty \quad \text{otherwise.} \end{cases}$$

Note $[\underset{\sim}{\wedge} s] \cap [\theta^{\sigma}] = \emptyset$ if and only if

$$(\exists \beta \in \gamma)(\exists \alpha \in \mathrm{dom}(\sigma))(\underline{P}_{\beta}, \neg \underline{P}_{\beta} \in s \cup t_{\alpha\sigma(\alpha)}).$$

Thus, the relation on s,σ: "$\mathrm{Rk}(s,\sigma) \geq 1$" is definable by a Δ_0 formula in the parameters Γ, T, γ, $\overline{P} \in \mathcal{Q}$. It follows from the form of (15) that the relation on s,σ,α: "$\mathrm{Rk}(s,\sigma) \geq \alpha$" is primitive recursive in parameters Γ, T, γ, $\overline{P}$, hence $\mathrm{Rk} \cap \mathcal{Q}^3$ is $\underset{\sim}{\Delta}$-definable on $\mathcal{Q}$. We claim

(18) $$\mathrm{Rk} \in \mathcal{Q}.$$

Let us postpone verification of (18) and proceed. Let $\xi(\alpha,\beta,s,\sigma)$ be the relation:

$$(\forall (t,\tau) \in \Gamma \times T)[(s \subseteq t \;\; \& \;\; \sigma \subseteq \tau \;\; \& \;\; (\underline{P}_{\beta} \in t \text{ or } \neg \underline{P}_{\beta} \in t) \;\; \& \;\; \beta \in \mathrm{dom}(\tau)) \Rightarrow \mathrm{Rk}(t,\tau) < \alpha].$$

Then $Rk(s,\sigma) < \alpha+1$ implies $(\exists\beta \in \gamma)(\xi(\alpha,\beta,s,\sigma))$.

Let $\alpha_0 = \text{image}(Rk)$ and let $<_\ell$ be the lexicographic order on $\alpha_0 \times \gamma$. Let δ be the ordinal of $<_\ell$ and let $i\colon (\alpha_0 \times \gamma, <_\ell) \to (\delta, \varepsilon)$ be the unique isomorphism. Define $R\colon \Gamma \times T \to \delta$ by the equation

$$R(s,\sigma) = \min\{\eta \in \delta\colon (\exists(\alpha,\beta) \in \alpha_0 \times \gamma)(Rk(s,\sigma) = \alpha \;\&\; \xi(\alpha,\beta,s,\sigma) \;\&\; \eta = i(\alpha,\beta)\}.$$

Note R is primitive recursive in parameters from $\mathcal{A}$ and $\text{dom}(R) \in \mathcal{A}$, so $R \in \mathcal{A}$.

For $s \in \Gamma$, let $\check{s} = \{\neg\underline{P}\colon \underline{P} \in s\} \cup \{\underline{P}\colon \neg\underline{P} \in s\}$.
For $\eta \leq \delta$ define $p_\eta = \{\underset{\sim}{\bigvee}(\check{t}^\sigma \cup \check{s})\colon (s,\sigma) \in \Gamma \times T \;\&\; R(s,\sigma) < \eta\}$, $\phi_\eta = \underset{\sim}{\bigwedge} p_\eta$. Let $\Phi = <\phi_\eta\colon \eta \leq \delta>$. Φ is primitive recursive in parameters from $\mathcal{A}$, $\text{dom}(\Phi) \in \mathcal{A}$, so $\Phi \in \mathcal{A}$. The sequence $<p_\eta\colon \eta \in \delta>$ is increasing, so $[\Phi]$ is decreasing. Clearly, $[\Phi]$ is continuous. $p_0 = \emptyset$ so $[\phi_0] = [\underset{\sim}{\bigwedge}\emptyset] = X_\rho$. Since $R(\emptyset,\emptyset) < \delta$, $\underset{\sim}{\bigvee}\emptyset \in p_\delta$ and $[\phi_\delta] = \emptyset$. Thus, $[\Phi]$ is suitable for $\mathcal{D}_\delta(\underset{\sim}{\Pi}^0_1(X_\rho))$. We claim

(19) $$[\theta_1] \subseteq D([\Phi]) \subseteq -[\theta_2].$$

To establish (19), suppose $x \in X_\rho$. Let $\eta_x = \min\{R(s,\sigma)\colon x \in [\underset{\sim}{\bigwedge} s] \cap [\theta^\sigma]\}$, and choose s,σ such that $x \in [\underset{\sim}{\bigwedge} s] \cap [\theta^\sigma]$ and $\eta_x = R(s,\sigma)$. Suppose $\eta_x = i(\alpha,\beta)$ and let

$$s' = \begin{cases} s \cup \{\underline{P}_\beta\} & \text{if } x \in [\underline{P}_\beta] \\ s \cup \{\neg\underline{P}_\beta\} & \text{otherwise}. \end{cases}$$

Since $R(s,\sigma) = i(\alpha,\beta)$, $\xi(s,\sigma,\alpha,\beta)$ holds and we have

$$(\forall \tau \supseteq \sigma)(\beta \in \mathrm{dom}(\tau) \Rightarrow \mathrm{Rk}(s',\tau) < \alpha).$$

By the minimality property of η_x, it follows that $x \notin \bigcup\{[\theta_{\beta\zeta}]: \zeta \in \gamma\}$. If $x \in [\theta_1]$, then β must be even, hence η_x is even; if $x \in [\theta_2]$, then η_x is odd. Using the minimality property again, if $R(s',\sigma') < \eta_x$, then $x \in -([\bigwedge s'] \cap [\theta^{\sigma'}]) = [\bigvee(\breve{s}' \cup \breve{t}^{\sigma'})]$. Thus $x \in [\phi_{\eta_x}] - [\phi_{\eta_x+1}]$. (19) follows.

It remains to prove (18). We first show

(17) $$\mathrm{Image}(\mathrm{Rk}) \subseteq \mathrm{ON}.$$

Since $\Gamma \times T$ is countable, there exists $\alpha < \omega_1$ such that

$$(\forall s,\sigma)[\mathrm{Rk}(s,\sigma) \geq \alpha \Rightarrow \mathrm{Rk}(s,\sigma) \geq \alpha+1].$$

Let $j: \omega \to \gamma$ be a bijection. If $\mathrm{Rk}(\emptyset,\emptyset) \geq \alpha$, then also $\mathrm{Rk}(\emptyset,\emptyset) \geq \alpha+1$ and for some t_o,τ_o

$$(\underline{P}_{i(o)} \in t_o \text{ or } \neg\underline{P}_{i(o)} \in t_o) \ \& \ i(o) \in \mathrm{dom}(\tau_o) \ \& \ \mathrm{Rk}(t_o,\tau_o) \geq \alpha.$$

We may proceed inductively to define t_n,τ_n for each $n \in \omega$ such that

(16) $$\forall n \in \omega[(\forall m < n)(t_m \subseteq t_n \ \& \ \tau_m \subseteq \tau_n \ \& \ (\underline{P}_{i(n)} \in t_n \text{ or } \neg\underline{P}_{i(n)} \in t_n) \ \& \ i(n) \in \mathrm{dom}(\tau_n) \ \& \ \mathrm{Rk}(t_n,\tau_n) \geq \alpha].$$

Let x be the unique member of $[\bigwedge \bigcup\{t_n: n \in \omega\}]$. If $x \notin [\theta^{\tau_n}]$, then since $-[\theta^{\tau_n}]$ is open, for some $m > n$

$$[\bigwedge t_m] \subseteq -[\theta^{\tau_n}] \subseteq -[\theta^{\tau_m}],$$

and hence, $Rk(t_m, \tau_m) = 0$. This contradicts (16) and shows $x \in \bigcap_n [\theta_{j(n)\tau_n(j(n))}]$. This in turn contradicts (14), so $Rk(\emptyset,\emptyset) < \alpha$ and (17) follows.

If $Rk(\emptyset,\emptyset) \notin \mathcal{A}$, then for some s,σ, $R(s,\sigma) = \omega_1 \cap \mathcal{A}$ and

$$(\mathcal{A},t) \models (\underset{\sim}{\forall}(t,\tau) \in \Gamma \times T)[(s \subseteq t \wedge \sigma \subseteq \tau) \rightarrow (\underset{\sim}{\exists}\alpha)(\alpha \in ON \wedge Rk(t,\tau) < \alpha)].$$

Applying $\underset{\sim}{\Sigma}$-reflection, we obtain $\omega_1 \cap \mathcal{A} \in \mathcal{A}$, a contradiction which establishes (18) and completes the proof of case 1.

Case 2. $\mu \geq 1$, ρ arbitrary.

Let θ_1, θ_2 be as in 2.2 and suppose $\theta_i = \bigwedge_{j \in J_i} \bigvee_{k \in K_i} \theta_{jk}$, $i = 1,2$ where each $\theta_{jk} \in \underset{\sim}{\Pi}^0_{(\mu)}(\rho)$. Let $\Theta = \{\theta_{jk}: (j,k) \in J_1 \times K_1 \cup J_2 \times K_2\}$ and choose $\rho_1, \Psi, F_o, F_1 \in \mathcal{A}$ as given by 2.1. Let $\theta_j' = \Psi \wedge \bigwedge_{j \in J_i} \bigvee_{k \in K_i} F_1(\theta_{jk})$, $i = 1,2$. Then $\theta_1', \theta_2', \rho_1$ satisfy the hypothesis of case 1 of 2.2. Let $\Phi' = \langle \phi_\beta': \beta < \delta \rangle \in \mathcal{A}$ be a sequence of $\underset{\sim}{\Pi}^0_1$-ρ_1-names given by case 1 such that $[\theta_1'] \subseteq D([\Phi']) \subseteq -[\theta_2']$. Let $\hat{F}_o(\phi_\beta')$ be the result of replacing in ϕ_β each $\underline{P} \in \rho_1$ by $F_o(\underline{P})$. Let $\Phi = \langle \hat{F}_o(\phi_\beta'): \beta < \delta \rangle$. As in Remark IV, it is easily checked that Φ satisfies the requirements of 2.2. The proof of 2.2 is complete. □

$\mathcal{A} \subseteq HC$ satisfies the <u>axiom of choice</u> if for every $x \in \mathcal{A}$, $\mathcal{A}$ contains a well-ordering of x. Let $\mathcal{D}_{[\mathcal{A}]}(\underset{\sim}{\Pi}^0_\alpha(X_\rho)) = \{D([\Phi]): (\exists\gamma)(\Phi \in \mathcal{A} \cap {}^\gamma\underset{\sim}{\Pi}^0_\alpha(\rho)$ & $[\Phi]$ is suitable for $\mathcal{D}_\gamma(\underset{\sim}{\Pi}^0_\alpha(X_\rho)))\}$.

$\mathcal{D}_{[\mathcal{A}]}(\underset{\sim}{\Pi}'^{o}_{\alpha}(V_{\rho}))$ has the obvious analogous definition.

Corollary 2.3. Assume $\mathcal{A} \subseteq HC$ is admissible and satisfies the axiom of choice, $\omega, \rho \in \mathcal{A}$, $1 \leq \alpha < \omega_1$. Then

(a) $\underset{\sim}{\Pi}^{o}_{\alpha+1}[\mathcal{A}](X_{\rho})$ has the strong separation property with respect to $\mathcal{D}_{[\mathcal{A}]}(\underset{\sim}{\Pi}^{o}_{\alpha}(X_{\rho}))$.

(b) $\underset{\sim}{\Pi}'^{o}_{\alpha+1}[\mathcal{A}](V_{\rho})$ has the strong separation property with respect to $\mathcal{D}_{[\mathcal{A}]}(\underset{\sim}{\Pi}'^{o}_{\alpha}(V_{\rho}))$.

Proof.

It is apparent from the form of II (8) that

$$\mathcal{D}_{[\mathcal{A}]}(\underset{\sim}{\Pi}^{o}_{(\gamma)}(X_{\rho})) \subseteq \underset{\sim}{\Pi}^{o}_{\gamma}[\mathcal{A}](X_{\rho}) \cap \underset{\sim}{\Sigma}^{o}_{\gamma}[\mathcal{A}](X_{\rho})$$

whenever $\mathcal{A} \subseteq HC$ is prim-closed. Thus, (a) is immediate from 2.2. Using 0.1(b), our proof of 1.2 is easily made effective, giving (b) as a consequence of (a).

Remarks .

V. The rank function used in the proof of 2.2 is based on a similar rank function used by D. A. Martin in [31] to prove the ordinary (boldface) $\underset{\sim}{\Pi}^{o}_{n}$ separation theorems for 2^{ω}. The classical argument could not be used here because we have no effective way of obtaining from a $\underset{\sim}{\Pi}^{o}_{2}$-name ϕ, a $\underset{\sim}{\Pi}^{o}_{1}$-name for the closure of $[\phi]$.

VI. Every admissible set of the form $L_{\alpha}[x]$ (the αth level of the constructive hierarchy built over x), $x \in X_{\rho}$, ρ finite , satisfies the axiom of choice. A much stronger hypothesis is that of local countability. $\mathcal{A}$ is locally countable if for every $x \in \mathcal{A}$, $\mathcal{A}$ contains a map of ω onto x. If $\mathcal{A}$ is locally countable and prim(ω)-closed, then the standard proof of $\underset{\sim}{\Sigma}^0_\mu$-reduction shows that $\underset{\sim}{\Sigma}^0_\mu[\mathcal{A}](X_\rho)$ has the reduction property and hence $\underset{\sim}{\Pi}^0_\mu[\mathcal{A}](X_\rho)$ has the weak first separation property ($\mu > 1$, $\rho \in \mathcal{A}$). We doubt that $\underset{\sim}{\Sigma}^0_\mu[\mathcal{A}]$-reduction holds when $\mathcal{A}$ is not locally countable.

VII. Before we obtained 2.2 Richard Haas considered a variant (call it Γ) of the difference hierarchy on Π^0_1 (lightface) and proved that $\Pi^0_2({}^\omega\omega)$ has the strong separation property with respect to Γ. If his result can be shown to relativize to arbitrary parameters or to extend to higher levels of the hyperarithmetical hierarchy it would improve the result one obtains from 2.2 in these cases (where $\mathcal{A} = L_{\omega_1^x}[x]$, $x \in 2^\omega$) by avoiding the introduction of hyperarithmetic parameters.

§3. Hausdorff' and Sierpinski': Proofs Derived from Topology

In this section we carry Addison's "method of analogies" and Vaught's "topology prime" notation to their natural extreme and construct two proofs by the following recipe:

"Take a theorem and its proof from classical descriptive set theory. Give model theoretic interpretation to all the terms used in the proof in such a way that the arguments remain valid. The result is a theorem of model theory."

We will apply the recipe to the Hausdorff proof of the $\underset{\sim}{\Pi}^0_2$ separation theorem, and to the Sierpinski proof of the theorem "Operation (A) preserves the Baire property in separable spaces." The results are new proofs of the $\underset{\sim}{\Pi}'^0_2$ separation theorem, and of the fact "The game quantifier preserves the Baire' property for countable ρ." Vaught (see [46]) had earlier applied the recipe to derive this second fact (for arbitrary ρ) from the Kuratowski proof that "Operation (A) preserves the Baire property in arbitrary spaces." Sierpinski's argument is considerably shorter than Kuratowski's for the cases it covers, and our proof is similarly shorter than Vaught's original argument.

The following definitions are mainly from Vaught [46]. Let ρ be arbitrary and $K_1 \subseteq K^{(n)} \subseteq V_\rho^{(n)}$. K_1 is <u>meager'</u> (relative to $K^{(n)}$) provided $K_1 \subseteq \mathrm{Mod}^K(\bigvee_m \underset{\sim}{\exists} v_n \ldots v_{i_m} \phi_m)$ where each $\phi_m \in \underset{\sim}{\Pi}'^0_1$ and every $\underset{\sim}{\Sigma}'^0_1(K^{(i_m)})$ subclass of $\mathrm{Mod}^{(i_m)}(\phi_m) \cap K^{(i_m)}$ is empty. K_1 is <u>almost' open'</u> if the symmetric difference $K_1 \Delta \mathrm{Mod}^{K^{(n)}}(\phi)$ is meager' for some $\phi \in \Sigma'^{0}_{1}(\rho)$. The Tarski closure $c(K_1)$ of K_1 is the intersection of all the closed' classes which include K_1. Note that

(20) K_1 is meager' (resp. almost' open') in K_2 if and only if K_1 is meager' (almost' open') in $c(K_2)$.

K_1 is <u>dense</u>' in K_2 if $K_2 \subseteq c(K_1)$. Also note that $c(K) \in \Pi'^{0}_{1}(V_\rho)$ for any K.

<u>Assume for the remainder of §3 that ρ is countable.</u>

(21) and (22) list the model theoretic translation of the basic topological facts used by Hausdorff and Sierpinski.

(21) Let $K \subseteq V^{(n)}_{\rho}$, $n \in \omega$.

(i) Every disjoint collection of $\Sigma'^{0}_{1}(K)$ classes is countable.

(ii) Every strictly decreasing collection of $\Pi'^{0}_{1}(K)$ classes is countable.

(iii) If K is non-empty and $\Pi'^{0}_{2}(V\ )$, then K is not meager' in itself.

<u>Proof</u>.

(i) and (ii) are immediate from the countability of the set of basic' ρ-formulas. (iii) is a variant of the well-known omitting types theorem and has a long history (<u>cf</u>. [46]). It is most efficiently derived from the Löwenheim-Skolem theorem and the Baire category theorem, (see [46] but substitute $\overline{X_{\hat{\rho}}}$ for X_ρ to get an argument which is valid when K contains no infinite models). □

In complete analogy with the topological situation (cf. [26] §24), 21(i) and (iii) and (20) imply

(22) Let $K \subseteq V_\rho^{(n)}$, $n \in \omega$. Every disjoint collection of Baire', non-meager' (relative to K) classes is countable.

Theorem 3.1. (1.2 revisited) Assume ρ is countable. The collection $\underset{\sim}{\Pi}'^{o}_2(V_\rho)$ has the strong first separation property with respect to $\mathcal{P}_{\omega_1}(\underset{\sim}{\Pi}'^{o}_1)(V_\rho)$.

Proof.

Let $K_1, K_2 \in \underset{\sim}{\Pi}'^{o}_2(V_\rho)$ be disjoint. Recursively define $\underset{\sim}{\Pi}'^{o}_1$ classes C_α, $\alpha \in \omega_1$ by the conditions:

$$C_o = c(K_1)$$
$$C_\lambda = \bigcap_{\beta<\lambda} C_\beta$$
$$C_{\lambda+2n+1} = c(K_2 \cap C_{\lambda+2n}), \qquad C_{\lambda+2n+2} = c(K_1 \cap C_{\lambda+2n+1}).$$

It follows from 21(ii) that for some $\gamma \in \omega_1$, $C_\gamma = C_\eta$ for all $\eta \geq \delta$. Then $C_\delta = c(C_\delta \cap K_1) = c(C_\delta \cap K_2)$. Thus, $C_\delta \cap K_1$, $C_\delta \cap K_2$ are disjoint $\underset{\sim}{\Pi}'^{o}_2$ classes, each of which is dense' in C_δ. It follows that C_δ is meager' in itself, hence empty. Then

$$K_1 \subseteq \bigcup\{C_\alpha - C_{\alpha+1}: \alpha \in e(\gamma)\} \subseteq -K_2$$

and the proof is complete. □

For finite ρ it is well-known that $c(K) = \{\mathfrak{A}:$ every finite substructure of $\mathfrak{A}$ can be embedded in a member of $K\}$. We can combine this observation with a variant of the last proof to give an algebraic characterization of the $\underset{\sim}{\Pi}'^{o}_{2}$ classes which contain only finite models.

The use of a substructure chain argument in place of the Baire category theorem is not new — see e.g. Addison [3].

Theorem 3.2. Assume ρ is finite and K is a collection of finite ρ-structures. The following are equivalent.

(i) $K \in \mathcal{D}_{(\omega_1)}(\underset{\sim}{\Pi}'^{o}_{1}(V_\rho))$

(ii) $K \in \underset{\sim}{\Pi}'^{o}_{2}(V_\rho)$

(iii) K includes no infinite substructure chain.

Proof.

(i) implies (ii) trivially. Since $\underset{\sim}{\Pi}'^{o}_{2}$ classes are closed under unions of chains, (ii) implies (iii).

Now assume (iii). Define C_α for $\alpha \in \omega_1$ as in the proof of 3.1 with $K_o = K$, $K_1 = V_\rho - K$, and let $\delta \in \omega_1$ be such that $C_\delta = C_{\delta+1}$. Just as in 3.1, it suffices to show C_δ is empty.

Suppose it is not.

Let $\mathfrak{A} \in C_\delta$. Since $C_\delta = c(C_\delta \cap K)$, there exists $\mathfrak{A}_o \in C_\delta \cap K$. Since $\mathfrak{A}_o$ is finite and $C_\delta = c(C_\delta - K)$, $\mathfrak{A}$ has an extension (necessarily a proper extension) $\mathfrak{B}_o \in C_\delta - K$. Let $b \in |\mathfrak{B}_o| - |\mathfrak{A}_o|$ and let $\mathfrak{C}_o$ be the substructure of $\mathfrak{B}_o$ generated by $\{b\} \cup |\mathfrak{A}|$. Since $\underset{\sim}{\Pi}'^{o}_{1}$ classes are closed under substructures, $\mathfrak{C}_o \in C_\delta$ and since $C_\delta = c(C_\delta \cap K)$, $\mathfrak{C}_o$ has an extension $\mathfrak{A}_1 \in C_\delta \cap K$. $\mathfrak{A}_1$ is a proper extension of $\mathfrak{A}_o$, and the process may be continued to inductively define an infinite substructure chain $\{\mathfrak{A}_i: i \in \omega\} \subseteq C_\delta \cap K \subseteq K$ in contradiction to (iii). $\square$

Next we give model theoretic interpretation to Sierpinski's theory of approximation to the operation (A).

The following definitions are due to Vaught. Let ρ be arbitrary. Let S be the set of all finite sequences from ω of even length, and suppose we are given

(23) $K = \{K_s: s \in S\}$, with each $K_{i_1 \ldots i_{2n}} \in V_\rho^{(2n)}$.

The class $G(K) \subseteq V_\rho$ is defined by the condition

$$\mathfrak{A} = (A,R) \in G(K) \iff (\forall i_1 \in \omega)(\forall a_1 \in A)(\exists i_2 \in \omega)(\exists a_2 \in A)$$

$$(\forall i_3 \in \omega)(\forall a_3 \in A)\ldots \bigwedge_{n \in \omega} (\mathfrak{A}, a_1, \ldots, a_{2n}) \in K_{i_1 \ldots i_{2n}}.$$

The set $\{(\delta_s^\alpha, \tau_s^\alpha): \alpha \in ON, s \in S\}$ of approximations for $G(K)$ is defined recursively by the conditions

(i) $(\mathfrak{A}, a_1, \ldots, a_{2n}) \in \delta^o_{i_1 \ldots i_{2n}} \iff (\forall m \le n)((\mathfrak{A}, a_1, \ldots, a_{2m}) \in K_{i_1 \ldots i_{2m}})$

(ii) $(\mathfrak{A}, a_1, \ldots, a_{2n}) \in \delta^{\alpha+1}_{i_1 \ldots i_{2n}} \iff$

$$(\forall i_{n+1} \in \omega)(\forall a_{n+1} \in A)(\exists i_{n+2} \in \omega)(\exists a_{n+2} \in A)[(\mathfrak{A}, a_1, \ldots, a_{2n+2}) \in \delta^\alpha_{i_o \ldots i_{2n+2}}]$$

(iii) $(\mathcal{A}, a_1, \ldots, a_{2n}) \in \delta^{\lambda}_{i_1 \ldots i_{2n}} \iff (\forall \alpha < \lambda)((\mathcal{A}, a_1, \ldots, a_{2n}) \in \delta^{\alpha}_{i_1 \ldots i_{2n}})$

(iv) $\tau^{\alpha}_s = \delta^{\alpha}_s - \delta^{\alpha+1}_s$.

We set $\delta^{\alpha} = \delta^{\alpha}_{\emptyset}$ and define τ^{α} by the condition

$$\mathcal{A} \in \tau^{\alpha} \iff (\exists n)(\exists s \in {}^{2n}\omega)(\exists \check{a} \in {}^{2n}A)((\mathcal{A}, \check{a}) \in \tau^{\alpha}_s).$$

It is known (cf. Vaught [44]) that

(24) $\qquad (\forall \alpha \in ON) \qquad \delta^{\alpha} - \tau^{\alpha} \subseteq G(K) \subseteq \delta^{\alpha}$.

Theorem 3.4. Assume ρ is countable and $K \subseteq V_{\rho}$ is arbitrary. Suppose $\mathcal{K}$ is a collection as in (23), such that each $K_s \subseteq K$ and is almost' open' in $K^{(\text{length}(s))}$. Then

(a) For some $\alpha_o < \omega_1$, τ^{α_o} is meager' in K

(b) (Vaught) $G(K)$ is almost' open' in K.

Proof.

(a) Fix $s \in {}^{2n}\omega$. Then $\{\tau^{\alpha}_s\colon \alpha < \omega_1\}$ is a disjoint collection of almost' open' subclasses of $K^{(2n)}$. By (22) we can find $\alpha(s) < \omega_1$ such that for every $\alpha \geq \alpha(s)$, τ^{α}_s is meager', say $\tau^{\alpha}_s \subseteq B^{\alpha}_s$ where $B^{\alpha}_s = \mathrm{Mod}^{K^{(2n)}}(\phi^{\alpha}_s(\underline{v}_o, \ldots, \underline{v}_{2n-1}))$ is meager', $\Sigma'^{o}_2(K^{(2n)})$. Let $\alpha_o = \bigcup\{\alpha(s)\colon s \in S\}$. Then $\tau^{\alpha_o} \subseteq \mathrm{Mod}^K(\bigvee_{n \in \omega} \bigvee_{s \in {}^{2n}\omega} (\exists \underline{v}_o, \ldots, \underline{v}_{2n-1})(\phi^{\alpha_o}_s))$ hence τ^{α_o} is meager' in K.

(b) It is easily seen that the collection of almost' open' classes is closed under complementation, countable union, cylindrification $(K \mapsto K^{(n)})$, and projection $(K^{(n)} \mapsto K)$. Hence τ^{α} and δ^{α} are almost' open' in K for each $\alpha < \omega_1$. Since $\delta^{\alpha_0} - G(K)$ is meager' by (a), $G(K)$ is almost' open'. □

§4. Remarks on Orbits

Let ρ be a fixed countable similarity type.

Given $n \in \omega$ and an n-formula $\phi = \phi(\underline{v}_0,\ldots,\underline{v}_{n-1}) \in L_{\omega_1\omega}(\rho)$, define $\ulcorner\phi\urcorner = \{R \in X_\rho : (\omega,R,0,\ldots,n-1) \models \phi\}$. Let L be a countable fragment of $L_{\omega_1\omega}(\rho)$ which is closed under quantification and let X^L be the topological space formed on the set $|X_\rho|$ by taking $\{\ulcorner\phi\urcorner : \phi \in L\}$ as a basis.

Given $R \in X^L$, identify R with (ω,R) and let [R] be the orbit of R under the canonical action. Then [R] is Borel, and in general, there will be orbits of arbitrarily high Borel rank. In [8] M. Benda proved a result relating a model theoretic condition on R to the topological complexity of [R] in $X^{L_{\omega\omega}}$; viz.

(25) If R is saturated and Th(R) is not ω-categorical, then [R] is not Σ_2^0 in $X^{L_{\omega\omega}}$.

Topological questions about orbits in $X^{L_{\omega\omega}}$ were also considered briefly by Suzuki in [43].

In this section we will obtain further results of this kind, mainly as an application of the invariant Π_α^0 separation theorem. In particular, both 4.2 and 4.5 will improve (25). For definitions and basic results about elementary types, etc. see [12].

Let $\rho^{\#L}$ be the similarity type with a Skolem predicate $\underline{P}_\phi$ for each formula $\phi \in L$. Then the canonical embedding $J: R \mapsto R^{\#L}$

of X^L into $X_{\rho\#L}$ defines a homeomorphism of X^L with an invariant $\underset{\sim}{\Pi}^0_2$ subset of $X_{\rho\#L}$. It follows that X^L is Polish. Moreover, since the canonical embedding commutes with the canonical actions on X_ρ and $X_{\rho\#L}$, Vaught's result (1) can be translated into a definability result for X^L. The definition of the classes $L\text{-}\underset{\sim}{\Sigma}'^0_\alpha$, $L\text{-}\underset{\sim}{\Pi}'^0_\alpha$, (read "$\underset{\sim}{\Sigma}'^0_\alpha$ over L", etc.), is obtained from the definition of $\underset{\sim}{\Sigma}'^0_\alpha$, $\underset{\sim}{\Pi}'^0_\alpha$, by replacing the condition "each M_n is basic'" in the definition of $\underset{\sim}{\Sigma}'^0_1$, by the condition "each $M_n \varepsilon L$", to define $L\text{-}\underset{\sim}{\Sigma}'^0_1$; and then proceeding as before.

We have

(26) For $\alpha \geq 1$, invariant $\underset{\sim}{\Sigma}^0_\alpha(X^L) = L\text{-}\Sigma'^0_\alpha(X_\rho)$.

<u>Proof</u>.

Inclusion from right to left is trivial. To go from left to right assume $B \varepsilon \text{inv}(\underset{\sim}{\Sigma}^0_\alpha(X^L))$; then $J(B) \varepsilon \text{inv}(\underset{\sim}{\Sigma}^0_\alpha(J(X^L)))$. By (1) $J(B) = [\![\phi]\!] \cap J(X^L)$ for some $\theta \varepsilon \underset{\sim}{\Sigma}'^0_\alpha(\rho^{\#L})$. Let ψ be the result of replacing each atomic subformula of θ by the corresponding formula of L. Then ψ is $L\text{-}\underset{\sim}{\Sigma}'^0_\alpha$ and $B = [\![\psi]\!]$. □

Our first result provides the second half of the "inverse" to Suzuki's observation ([43] Thm. 2) that the orbit of a prime model R is a comeager $\underset{\sim}{\Pi}^0_2$ subset of $[\![\underset{\sim}{\bigwedge}\text{Th}(R)]\!] \subseteq X^L$. ([43] Thm. 3 is the first half. Suzuki worked with $L = L_{\omega\omega}$ but his arguments work in the general context considered here.)

<u>Proposition 4.1</u>. If $[R] \in \underset{\sim}{\Pi}^0_2(X^L)$, then (ω,R) is L-atomic (every finite sequence from ω realizes a principle L-type in (ω,R)).

<u>Proof</u>.

If $[R] \in \underset{\sim}{\Pi}^0_2$, then $[R] = [\![\bigwedge_n \underset{\sim}{\forall} \underline{v}_0 \cdots \underline{v}_{i_n - 1} \bigvee_m \underset{\sim}{\exists} \underline{v}_{i_n} \cdots \underline{v}_{i_{nm}-1} \phi_{nm}]\!]$ where each ϕ_{nm} is an n-formula of L. Let $\Delta_n =$ $\{ \underset{\sim}{\neg\exists} \underline{v}_{i_n} \cdots \underline{v}_{i_{nm}-1} \phi_{nm} \colon m \in \omega\}$; then $[R] = \{S\colon S$ omits each type Δ_n, $n \in \omega\}$. If R realized a non-principle type Σ, we could find S which omits $\{\Sigma\} \cup \{\Delta_n\colon n \in \omega\}$. But then $S \in [R]$ and $S \not\cong R$, a contradiction. □

Note that $[\![\underset{\sim}{\bigwedge} Th(R)]\!]$ is the closure of $[R]$ in X^L, hence

(27) $[R]$ is closed if and only if $Th(R)$ is ω-categorical.

In view of the intrinsic invariance of the Borel classes (<u>cf</u>. Kuratowski [26] §35), for every α, $[R]$ is a $\underset{\sim}{\Sigma}^0_\alpha$ (or $\underset{\sim}{\Pi}^0_\alpha$) subset of X^L if and only if $[R]$ is a $\underset{\sim}{\Sigma}^0_\alpha$ ($\underset{\sim}{\Pi}^0_\alpha$) subset of $[\![\underset{\sim}{\bigwedge} Th(R)]\!]$. In view of this fact, and of (27), we lose no information by studying the complexity of orbits relative to $[\![\underset{\sim}{\bigwedge} T]\!]$ where T is a complete L-theory which is not ω-categorical.

<u>For the remainder of §4 we assume T is a fixed, complete not ω-categorical theory of $L_{\omega\omega}$ and $X = X^T = [\![\underset{\sim}{\bigwedge} T]\!]$ with the relative topology from $X^{L_{\omega\omega}}$.</u>

X is exactly the space $\underset{\sim}{S}$ studied in [8].

Following Benda [8] we say R is full (weakly saturated) if every elementary type over T is realized in R. An elementary type Δ is powerful if every model of T which realizes Δ is full.

Theorem 4.2. No orbit is Σ_2^o.

Proof.

Suppose $[R] \in \underset{\sim}{\Sigma}_2^o(X)$; then by (26), $[R] = [\![\bigvee_n \exists \underline{v}_o \ldots \underline{v}_{n-1} \bigwedge \phi_{nm}]\!]$ for some collection $\{\phi_{nm}\colon n,m \in \omega\}$ such that each ϕ_{nm} is an n-formula of $L_{\omega\omega}$. Since $[R]$ is minimal invariant, there is some n_o such that

$$[R] = [\![\underset{\sim}{\exists} \underline{v}_o \ldots \underline{v}_{n-1} \bigwedge_m \phi_{n_o m}]\!]$$

i.e.

$$[R] = \{S\colon \ \Delta \text{ is realized in } S\}$$

where Δ is the n_o-type $\{\phi_{n_o m}\colon \ m \in \omega\}$.

If R is not full, let Σ be a complete type over T which is omitted by R, and let S realize both Δ and Σ. Then $S \in [R]$ and $S \not\cong R$, a contradiction.

If R is full, then Δ is powerful and, since T is not ω-categorical, there are both saturated and non-saturated models which realize Δ, again contradicting the fact that $[R]$ is an orbit. $\square$

Lemma 4.3. If R is full and G is an invariant $\underset{\sim}{\Pi}_2^o$ set which contains R, then $G = X$.

Proof.

It suffices to prove the lemma for $G = [\![\forall v_0 \ldots v_{n-1} \bigvee_m \phi_m]\!]$, each $\phi_m \in L_{\omega\omega}$, since every invariant Π^0_2 set is an intersection of sets of this form. Let $\Delta = \{\neg\phi_m : m \in \omega\}$. Then $G =$ $\{S: S$ omits $\Delta\}$. Since R is full, Δ is powerful and every model of T omits Δ. □

Theorem 4.4. No full model has a Δ^0_3 orbit.

Proof.

Suppose R is full and $[R] \in \Delta^0_3$. Then $[R] \in \mathcal{D}_{(\omega_1)}(\mathrm{inv}(\Pi^0_2(X)))$ and since $[R]$ is minimal invariant, $[R] = G_1 - G_2$ for some invariant Π^0_2 sets G_1, G_2. By 4.3 $G_1 = X$ and $[R] = -G_2$ contradicting 4.2. □

Corollary 4.5.

(i) If R is saturated, then $[R] \in \Pi^0_3(X) - \Sigma^0_3(X)$.

(ii) If Δ is a powerful n-type and $(\omega, S, i_0, \ldots, i_{n-1})$ is a prime model of a complete extension of $\Delta\binom{v_0 \ldots v_{n-1}}{\underline{0} \ldots \underline{n-1}} \subseteq L_{\omega\omega}(\rho \cup \{\underline{0}, \ldots, \underline{n-1}\})$, then $[S] \in$ $\Sigma^0_3(X) - \Pi^0_3(X)$.

Proof.

It is easy (see [8]) to see that R, S belong to Σ^0_3, Π^0_3 respectively. The conclusion then follows by 4.4. □

We have a partial converse to 4.5(i).

Theorem 4.6. Assume R is full and $[R] \in \underset{\sim}{\Pi}^0_3$. Then R is saturated.

Proof.

Suppose R is not saturated.

Since R is full, T has a countable saturated model S. Then [R] and [S] are disjoint minimal invariant $\underset{\sim}{\Pi}^0_3$ sets. It follows from the invariant $\underset{\sim}{\Pi}^0_3$ separation theorem that there are invariant $\underset{\sim}{\Pi}^0_2$ sets G_1, G_2 such that $[R] \subseteq G_1 - G_2 \subseteq -[S]$. Since R is full, it follows from 4.3 that $G_1 = X$. Then $[S] \subseteq G_2$, and since S is full, $G_2 = X$ and $[R] = \emptyset$, a contradiction. □

The invariant $\underset{\sim}{\Pi}^0_\alpha$ separation principle appears to be a useful tool for attacking general classification problems in descriptive set theory. For example, consider the following proof of one of the first results in the subject (cf. Addison [4] or Lusin [29]).

(28) (Baire 1906) The set $A = \{R \in 2^{\omega\times\omega}$: (R defines a function $f: \omega \to \omega$) & $(\forall n)(f^{-1}\{n\}$ is finite)$\}$ belongs to $\underset{\sim}{\Pi}^0_3(2^{\omega^2}) - \underset{\sim}{\Sigma}^0_3(2^{\omega^2})$.

Proof.

A is obviously invariant $\underset{\sim}{\Pi}^0_3$. If A were $\underset{\sim}{\Sigma}^0_3$ then A would be an alternated union of invariant $\underset{\sim}{\Pi}^0_2$ sets. Such sets cannot separate structures which satisfy the same $\underset{\sim}{\Pi}'^0_2$ sentences (i.e. which realize the same types of $\underset{\sim}{\forall}^0_1$ formulas). It is easy to show that A can.

Consider, for example, the functions f_o, f_1 defined as follows,

(i) If $j = p^n$ where p is the ith odd prime and $1 \leq n \leq i$, then $f_1(j) = p$; otherwise $f_1(j) = j$.

(ii) If j is odd, then $f_2(j) = f_1(j)$; if j is even, then $f_2(j) = 0$.

Let R_i be the characteristic function of f_i, $i = 1,2$. Then $R_1 \in A$, $R_2 \notin A$ and it is a straightforward exercise to show that (ω,R_1), (ω,R_2) realize the same types of $\underset{\sim}{\forall}_1^o$ formulas. $\square$

§5. On Theorems of Lusin and Makkai

In this section we will use the transform method to derive a recent "Global Definability Theorem" of M. Makkai, (see [30]), from the following classical theorem of Lusin (cf. [26] §39 VII Cor. 5).

(29) If f is a continuous function defined on a Lusin space X such that the preimage of every point in $f(X)$ is countable, then there is a collection $B = \{B_i : i \in \omega\}$ of Borel sets such that $X = \bigcup B$ and each $f{\restriction}_{B_i}$ is one-one.

Theorem 5.1. (Makkai) Let ρ be a countable similarity type. Let $\underline{P}$ be an n-ary relation symbol not in ρ and let σ be a sentence of $L_{\omega_1\omega}(\rho \cup \{\underline{P}\})$. For $\mathfrak{A} \in V_\rho$ let $M_\sigma(\mathfrak{A}) = \{P \subseteq |\mathfrak{A}|^n : (\mathfrak{A}, P) \models \sigma\}$. Then the following are equivalent:

(i) For every countable $\mathfrak{A} \in V_\rho$, $M_\sigma(\mathfrak{A})$ is countable.

(ii) There exists a set $\Phi = \{\phi_i(\underline{v}_o \ldots \underline{v}_{n+k_i}) : i \in \omega\} \subseteq L_{\omega_1\omega}(\rho)$ such that

$$\sigma \models \bigvee_{i\in\omega} \exists \underline{v}_n \ldots \underline{v}_{n+k_i} \forall \underline{v}_o \ldots \underline{v}_{n-1} (\underline{P}(\underline{v}_o, \ldots, \underline{v}_{n-1}) \leftrightarrow \phi_i).$$

Proof.

(ii) $\Rightarrow$ (i) is obvious. Now assume (i).

Since the set of isomorphism types of finite $\rho \cup \{\underline{P}\}$ structures is countable and every finite isomorphism type is definable, we may assume that all models of σ are infinite.

Let $X = [\![\sigma]\!] \subseteq X_{\rho \cup \{\underline{P}\}}$, and let $\pi: [\![\sigma]\!] \to X_\rho$ be the canonical projection $(R,P) \mapsto (R)$. By assumption, for each $R \in X_\rho$, $\pi^{-1}(\{R\}) = M_\sigma(R)$

is countable. By (29) there exists $B = \{B_i : i \in \omega\}$ such that $[\![\sigma]\!] = \bigcup B$, each $B_i \in \mathcal{B}(X_\rho \cup \{\underline{P}\})$, and each $\pi \restriction B_i$ is one-one.

Since $[\![\sigma]\!]$ is invariant, $[\![\sigma]\!] = [\![\sigma]\!]^\Delta = (\bigcup B)^\Delta = \bigcup_{i\in\omega} B_i^{\ \Delta} = \bigcup_{i\in\omega} \bigcup_{n\in\omega} \bigcup_{s\in {}^n\omega} B_i^{\ *[s]}$. By Vaught's basic result (2), there is a set $\Psi = \{\psi_{im}(\underline{v}_o \ldots \underline{v}_{m-1}) : i,m \in \omega\}$ such that for each $m,i \in \omega$, $(R,P) \in X_{\rho \cup \{\underline{P}\}}$, $s \in {}^m\omega$,

$$(\omega,R,P,s_o,\ldots s_{m-1}) \models \psi_{mi} \iff s \in {}^m\omega \quad \& \quad (R,P) \in B_i^{\ *[s]}.$$

It follows that $\sigma \models \bigvee_{i\in\omega} \bigvee_{m\in\omega} \exists \underline{v}_n \ldots \underline{v}_{n+m-1} \psi_{nm} \binom{\underline{v}_o \ldots \underline{v}_{m-1}}{\underline{v}_n \ldots \underline{v}_{n+m-1}}$.
We claim that for every $i,m \in \omega$, $s \in {}^m\omega$, $R \in X_\rho$, $P_1, P_2 \in 2^{\omega^n}$

$$(30) \qquad [(\omega,R,P_1,s) \models \psi_{im} \quad \& \quad (\omega,R,P_2,s) \models \psi_{im}] \Rightarrow P_1 = P_2.$$

This suffices since (ii) then follows by the infinitary analogue of the Beth definability theorem.

The following computation verifies (30):

$$(R,P_1),(R,P_2) \in B_i^{\ *[s]} \Rightarrow B_i^{\ (R,P_1)} \cap B_i^{\ (R,P_2)} \cap [s] \text{ is comeager in } [s]$$

$$\Rightarrow (\exists g \in \omega!)[(gR,gP_1),(gR,gP_2) \in B_i]$$

$$\Rightarrow (\exists g \in \omega!)[gP_1 = gP_2]$$

$$\Rightarrow P_1 = P_2. \qquad \square$$

Note that the finitary Chang-Makkai theorem (*cf*. Chang-Keisler [12] 5.3.6) follows from 5.1 via Keisler's approximations. Thus, 5.1 is the intermediate step in a derivation of the Chang-Makkai theorem from (29).

Note also that, since each $M_\sigma((\omega,R))$ is a $\underset{\sim}{\Sigma}^1_1$ subset of 2^{ω^n}, condition (i) of 5.1 is equivalent to

(i′) For every $R \in X_\rho$, $M_\sigma((\omega,R))$ does not contain a perfect subset.

Cf. [26] §36.V.

§6. On $L_{\omega_1\omega}$ Definability and Invariant Sets

In this section we collect some further applications of the *-transform in logic. The first result refines and extends a theorem of Lopez-Escobar on the explicit definability of invariant Borel functions between logic spaces. The remainder of the section is concerned with some recent results on definability due to V. Harnik [17] and [18]. These results were orginally obtained by a "forcing in model theory" construction which was derived from Vaught's method. We show that the same facts can be obtained directly from the method of [46].

Let ρ be arbitrary and let $1 \le \alpha < \omega_1$. A function $F\colon X_\rho \to 2^{\omega^n}$ is invariant if its graph is an invariant subset of $X_\rho \times 2^{\omega^n}$. An equivalent condition is that $gF(R) = F(gR)$ for every $R \in X_\rho$.

Given $K \subseteq X_\rho^{(n)}$, we define $F_K\colon X_\rho \to 2^{\omega^n}$ by setting

$$F_K(R)(i_1,\dots,i_n) = 1 \iff (\omega,R,i_1,\dots,i_n) \in K.$$

$F\colon X_\rho \to 2^{\omega^n}$ is said to be <u>elementary</u>, (respectively α-<u>elementary</u>), provided $F = F_K$ for some $K \in L_{\omega_1\omega}(X_\rho^{(n)})$, $(\underset{\sim}{\Delta}{}'^{\,0}_{\alpha}(X_\rho^{(n)})$. It is apparent that

(31) Every α-elementary function is invariant and α-Borel.

The 1-elementary functions were introduced by Craig in [13] where the converse of (31) was proved for $\alpha = 1$, ρ finite, relational. In [28] Lopez-Escobar applied his infinitary version of Beth definability to prove, for countable ρ, that every invariant Borel function is elementary.

Theorem 6.1. Let ρ be arbitrary and let $1 \leq \alpha < \omega_1$, $n \in \omega$, $F\colon X_\rho \to 2^{\omega^n}$. Then F is invariant and α-Borel if and only if F is α-elementary.

Proof.

The "if" part is (31). For the "only if" part of the theorem assume F is α-Borel and invariant.

Let $K = \{(R,i_1,\ldots,i_n)\colon F(R)(i_1,\ldots,i_n) = 1\}$. The invariance of F implies that K is an invariant subset of $X_\rho \times \omega^n$. Since $K = \bigcup\{F^{-1}([(s,1)]) \times \{s\}\colon s \in {}^n\omega\}$, $= -\bigcup\{F^{-1}([(s,0)]) \times \{s\}\colon s \in {}^n\omega\}$ where $[(s,i)] = \{x\colon x\colon {}^n\omega \to 2 \;\&\; x(s) = i\}$, K is $\underset{\sim}{\Delta}^0_\alpha$.

It follows from (2) that there is a $\underset{\sim}{\Sigma}'^0_\alpha$ formula ϕ and a $\underset{\sim}{\Pi}'^0_\alpha$ formula ψ such that $K = [\![\phi^{(n)}]\!] = [\![\psi^{(n)}]\!]$. 6.1 follows immediately since $F = F_K$. □

We turn to a discussion of Harnik's definability results. A key lemma is an observation regarding the behavior of equivalence relations under $*$:

(32) Assume the basic hypothesis of chapter II concering (G,X,X',J). Let $B \subseteq X'$ and suppose E is an equivalence on X such that

$$(\forall x,y)(xEy \Rightarrow (\exists \pi)(\pi \text{ is an autohomeomorphism of } G \;\&\; \pi B^x = B^y).$$

Then B^{*J} is E-invariant.

(32) follows immediately from the definitions of B^* and the fact that meagerness is a topological property. It has the following

corollary:

(33) Suppose that $\mathcal{J}_1 = (G,X_1,J_1)$, $\mathcal{J}_2 = (G,X_2,J_2)$ are actions, and $B \subseteq X_1 \times X_2$ is $E_{\mathcal{J}_1 \times \mathcal{J}_2}$-invariant. Assume further that X_1 is a Baire topological space and that each J_1^g: $x \mapsto J_1(g,x)$ is continuous. Let I: $X_1 \times X_2 \to X_1 \times X_2$ be the identity function. Then B^{*I} is $E_{\mathcal{J}_2}$-invariant.

Consider the following case of (33): Let ρ be a countable similarity type, θ a sentence of $L_{\omega_1\omega}(\rho)$ which has an infinite model, and L a countable fragment which contains θ. Let $X_1 = [\![\theta]\!]$ with the relative topology as a subspace of X^L. Let $X_2 = X_{\rho_1}$, ρ_1 arbitrary, $\mathcal{J}_1,\mathcal{J}_2$ the canonical actions of $\omega!$ on X_1,X_2. If ψ is a sentence of $L_{\omega_1\omega}(\rho+\rho_1)$ then $[\![\psi]\!]$ satisfies the assumptions made on B in (33), hence $[\psi]^{*I}$ is an invariant Borel subset of X_{ρ_1} and

$$-\pi_2(-[\![\theta \wedge \psi]\!]) \subseteq [\![\psi]\!]^{*I} \subseteq \pi_2([\![\theta \wedge \psi]\!]).$$

From (1) we conclude:

(34) (Harnik) Under the assumptions of the proceeding paragraph, for every sentence $\psi \in L_{\omega_1\omega}(\rho+\rho_1)$ there exists $\psi' \in L_{\omega_1\omega}(\rho_1)$ such that for every $(A,S) \in V_{\rho_1}$

$(\forall R)((A,R,S) \models \theta \wedge \psi) \to \mathfrak{A} \models \psi'$ and

$\mathfrak{A} \models \psi' \Rightarrow (\exists R)((A,R,S) \models \phi \wedge \psi')$

Remarks.

VIII. Let Γ be $L_{\omega_1 G}$ or any of the stronger languages L_α, $L_{\omega_1 f}$ studied in [10]. Using the techniques of [46] and [10], both (34) and (36), below, may be extended to analogous results where $\psi \in \Gamma$. Also, as Harnik observed, the passage from K to the ($L_{\omega_1\omega}$-definable) orbit of a countably infinite member of K shows that θ can be replaced by any class $K \subseteq V_\rho$ which contains a countable model, (if K contains finite model everything becomes trivial). As Vaught first observed, this passage allows one to derive (34) even more directly form the results in [46] — Let $R_o \in X_\rho \cap K$. Given ψ, let $[\![\psi]\!]^{R_o} = \{S: (R_o,S) \in [\![\psi]\!]\} \in \mathcal{B}(X_{\rho_1})$ and define $[\![\psi]\!]' = ([\![\psi]\!]^{R_o})^{*I_{\rho_1}}$. It follows from (1), and the invariance of θ,ψ under isomorphism, that $[\![\psi]\!]' = [\![\psi']\!]$ for some $\psi' \in L_{\omega_1\omega}(\rho_1)$ having the required property. This argument gives a slightly stronger result than (34) in that ρ need not be assumed countable. On the other hand, the argument used to prove (34) can be carried out over any prim(ω)-closed set.

IX. In [18] Harnik showed that a weak version of (34) is valid if θ is allowed to be any sentence of $L_{\omega_1\omega}(\rho+\rho_1)$ such that $(\forall S \in X_{\rho_1})(\exists R \in X_\rho)((R,S) \in [\![\theta]\!])$ — in this case one can find a suitable ψ' in the infinitary game language $L_{\omega_1 G}(\rho_1)$ (though not necessarily in $L_{\omega_1\omega}(\rho_1)$). This version can be proved like (34) by considering a modified *-transform. In the modified transform, one considers spaces G, X, X' and a map J as before, but, instead of assuming G is a Baire space, one associates to each $x \in X$ a subspace, $G_x \subseteq G$, which is a relative Baire space. One then defines, for $U \in \Sigma_1^0(G)$, $B^{*U} = \{x\colon\ U \cap G_x \neq \emptyset\ \ \&\ \ B^x \cup U \cap G_x$ is comeager in $U \cap G_x\}$.

The inductive clauses (II (2)-(4)) go over with slight modification allowing one to show:

(35) If every Borel set B is normal, ($B^x \cap G_x$ is almost open in G_x for all x), and H is a countable weak basis for G, then B^* belongs to the σ-algebra generated by $\{C^{*U}\colon\ C \in \Pi_1^0(X'),\ U \in H\}$ whenever B is Borel in X'.

In the applications to the results in [18], one easily shows that the normality condition holds, that each C^{*U} is Σ_1^1, and that the inductive proof of (35) yields a definability result analogous to (2). For example, in the case at hand, we would let L be a countable fragment of $L_{\omega_1\omega}(\rho+\rho_1)$ which contained θ, and for each $S \in X_{\rho_1}$ define G_S = the cross section at S of the space $[\![\theta]\!] \subseteq X^L$, G = the disjoint union of the spaces G_S, $S \in X_{\rho_1}$; $J(R) = (R,S)$ if $R \in G_S$, otherwise $J(R)$ arbitrary.

Now suppose $\rho = \{\rho_i\colon\ i \in \omega\}$ is a disjoint collection of purely relational similarity types. A ρ-sentence is a sentence ϕ of $L_{\omega_1\omega}(\bigcup\rho)$, such that, for each atomic subformula $\underline{R}(\underline{v}_{i_1},\ldots,\underline{v}_{i_n})$ of ϕ, if

$\underline{R} \in \rho_m$, then each $i_1,\ldots,i_n$ is a power of the mth prime. Let $\rho' = \bigcup \rho$, ρ'' an arbitrary relational type. $\mathfrak{A}, \mathfrak{B} \in V_{\rho'+\rho''}$ are said to be ρ-isomorphic ($\mathfrak{A} \underset{\rho}{\cong} \mathfrak{B}$) if $(\forall i \in \omega)(\mathfrak{A}\restriction\rho_i \cong \mathfrak{B}\restriction\rho_i)$.

(36) (Harnik) With the definitions of the preceding paragraph and $1 \leq \alpha < \omega_1$, $\psi \in \underset{\sim}{\Pi}'^{o}_{\alpha}(\rho'+\rho'')$, there is a ρ-sentence $\psi' \in \Pi'^{o}_{\alpha}(\rho')$ such that

$(\forall \mathfrak{A})(\mathfrak{A} \underset{\rho}{\cong} \mathfrak{B}\ \&\ \mathfrak{A} \models \psi) \Rightarrow \mathfrak{B} \models \psi'$ and

$$\mathfrak{B} \models \psi' \Rightarrow (\exists \mathfrak{A})(\mathfrak{A} \underset{\rho}{\cong} \mathfrak{B}\ \&\ \mathfrak{A} \models \psi).$$

Proof.

Let $X_1 = X_2 = \overline{X}_{\rho'+\rho''}$, $G = \omega!^{\omega} \times X_{\rho''}$, J: $((\langle g_i: i \in \omega\rangle, S), (R, S_1)) \mapsto (\langle g_i R_i: i \in \omega\rangle, S)$.

As a basis for G we may take the sets of the form $[s_o,\ldots,s_{n-1}] \times [\psi]$, where $n \in \omega$, $s_o,\ldots,s_{n-1} \in {}^{n}\omega$, ψ is a basic $\overline{\rho}''$-name and $[s_o,\ldots,s_{n-1}] = \{\langle g_i: i \in \omega\rangle \in \prod_{i\in\omega} \omega!: (\forall i < n)(g_i \in [s_i])\}$. For $B \subseteq \overline{X}_{\rho}$, $n \in \omega$, ψ a basic ρ''-name, let $B^{*n\psi} = \{(R, s_o,\ldots,s_{n-1}): R \in B^{*[s_o,\ldots,s_{n-1}]}\} \subseteq \overline{X}^{(n^2)}_{\rho'}$. An argument analogous to the proof of 0.1(a) establishes the fact that B^{*n} is $\mathrm{Mod}^{(n^2)}(\overline{\phi}) \cap \overline{X}^{(n^2)}_{\rho}$ for some ρ-formula $\phi \in \underset{\sim}{\Pi}'^{o}_{\alpha}(\rho')$ whenever $B \in \underset{\sim}{\Pi}^{o}_{\alpha}(\overline{X}_{\rho})$. (36) then follows by the Löwenheim-Skolem theorem. □

§7. A Selector for Elementary Equivalence

Let L be a countable fragment of $L_{\omega_1\omega}(\rho)$ which is closed under quantification, and let S(L) be the set of sentences of L. Assume for convenience that $C_\rho = \emptyset$. A ρ-sentence θ is propositional over L, ($\theta \in P(L)$), if every subformula of θ which begins with a quantifier belongs to L.

Let $\equiv_L$ be the relation of L-elementary equivalence between ρ-structures, and let $E_L \subseteq \overline{X}^2_\rho$ be $\{((S_1,\sim_1),(S_2,\sim_2)): S_1/\sim_1 \equiv_L S_2/\sim_2\}$. Given an n-formula $\phi \in L$, let $\langle\phi\rangle = \{(S,\sim) \in \overline{X}_\rho: (S,0,\ldots,n-1)/\sim \models \phi\}$, and let $\overline{X}^L$ be the topological space formed over the set $\overline{X}_\rho$ by taking $\{\langle\phi\rangle: \phi \in L\}$ as a basis. Observe that if $\phi \in L$ is an n-formula, then

(37) $\langle\phi\rangle^{+I_\rho} \subseteq \langle\phi\rangle^{+E_L}$, and $\langle\phi\rangle^{+I_\rho} = \langle\exists \underline{v}_0 \cdots \underline{v}_{n-1}\phi\rangle$,

so $\langle\phi\rangle^{+E_L} = \langle\exists \underline{v}_0,\ldots,\underline{v}_{n-1}\phi\rangle$ also. Thus, $\langle\phi\rangle^{+E_L}$ is clopen in $\overline{X}^L$ for every $\phi \in L$; and the set $\{\langle\theta\rangle/E_L: \theta \in S(L)\}$ forms a basis for $\overline{X}^L/E_L$. For $S = (R,\sim) \in \overline{X}^L$, let $Th^L(\tilde{S})$ be the L-theory of $\tilde{S} = R/\sim$. Now $[S]_{E_L} = \langle\bigwedge Th^L(\tilde{S})\rangle$ and for $\theta \in S(L)$, $[S]_{E_L} \in \langle\theta\rangle/E_L$ if and only if $\theta \in Th^L(\tilde{S})$. It follows that $\overline{X}^L/E_L$ is just the usual Stone space

$S(L)$ associated with L. Since we have verified the hypothesis of II.3.1, we may conclude

Theorem 7.1. Assume L is a countable fragment of $L_{\omega_1\omega}$ which is closed under quantification. Then there exists a continuous selector for E_L, $s: \overline{X}^L \to \overline{X}^L$.

From 7.1 we conclude that all of the remarks of paragraphs (11) and (12) of II §3 apply with $E = E_L$, $X = \overline{X}^L$. For example,

(38) $S(L)$ is Polish.

(39) $E_L\text{-inv}(\underset{\sim}{\Sigma}^0_\alpha(\overline{X}^L))$ has the reduction property for each $\alpha < \omega_1$.

(40) $E_L\text{-inv}(\mathcal{O}(\underset{\sim}{\Delta}^0_1(\overline{X}^L))$ = $\mathcal{O}(\overline{L\text{-}\underset{\sim}{\Delta}'^0_1(\overline{X}^L)})$ for every Boolean operation $\mathcal{O}$.

In (40) we have implicitly used the fact (immediate from (37) and the continuity of s) that

(41) $s^{-1}(B) = \langle\phi\rangle$ for some $L\text{-}\underset{\sim}{\Sigma}'^0_1$ sentence ϕ whenever B is open in $\overline{X}^L$.

Let Γ be Σ^0_α, Π^0_α or any of the Kolmogorov classes $\mathcal{B}(\Gamma_\alpha)$ or Borel game classes $\mathcal{B}(G\Gamma_\delta)$ studied in [10], and let Γ' be the corresponding collection of formulas ($L\text{-}\underset{\sim}{\Sigma}'^0_\alpha$, $L\text{-}\underset{\sim}{\Pi}'^0_\alpha$, L_α, $L_{\omega_1 f}$). A straightforward induction based on (41) shows

(42) If $B \in c\Gamma(\overrightarrow{X}^L)$ then $s^{-1}(B) = (\phi)$ for some $\phi \in P(L) \cap \Gamma$.

Corollary 7.2. Let Γ' be as in (42). For every $\phi \in \Gamma'$ there exists $\phi^* \in P(L) \cap \Gamma'$ such that for every $\mathfrak{A} \in V_\rho$,

$$(\forall \mathfrak{B})(\mathfrak{B} \equiv_L \mathfrak{A} \rightarrow \mathfrak{B} \models \phi) \Rightarrow \mathfrak{A} \models \phi^* \quad \text{and}$$

$$\mathfrak{A} \models \phi^* \quad \Rightarrow \quad (\exists \mathfrak{B})(\mathfrak{B} \equiv_L \mathfrak{A} \ \& \ \mathfrak{B} \models \phi).$$

Proof.

Given ϕ let $B = \{(R,\simeq) \in \overrightarrow{X}^L : R/\simeq \models \phi\}$. Let $\phi^* \in P(L) \cap \Gamma'$ be such that $(\phi^*) = s^{-1}(B)$. Then, since $s^{-1}(B)$ is an E_L-invariantization of B and the Löwenheim-Skolem theorem holds for Γ, ϕ^* has the required property. □

Remarks.

X. 7.2 extends 4.1 of Harnik [17]. The list of languages in 7.2 is not exhaustive. For example, the method of 7.2 applies to each level of the hierarchies on L_α, $L_{\omega_1 f}$ implicit in their constructions by iterations of operations. It appears to be rather difficult to make an exhaustive list.

XI. For $L = L_{\omega\omega}$, (38) is well-known. It was first proved for the larger fragments studied here by M. Morley in [33] using an infinitary Henkin construction. In fact, the Henkin method is essentially similar to the argument establishing the Kuratowski-Ryll-Nardzewski selector theorem. The proof in [27] of the selector theorem (couched in terms of our special case) proceeds by considering a countable dense subset $R = (r_1, r_2, \ldots)$ of $\overrightarrow{X}^L$, defining a convergent sequence of functions

$f_i\colon \overline{X}^L/E_L \to R$ and then setting $s = \lim\langle f_i\colon i \in \omega\rangle$. The argument is changed in no essential way if instead of considering R, we look at a basis C for $\overline{X}^L$ consisting of clopen sets C_{ij}, $i,j \in \omega$ such that under some complete metric on $\overline{X}^L$ we have

$$(\forall i,j,k)(r_i \in C_{ij} \quad \& \quad \text{diameter}(C_{ij}) < 2^{-j} \quad \& \quad (j < k \Rightarrow C_{ik} \subseteq C_{ij})).$$

The construction then proceeds by specifying for each $T \in \overline{X}^L/E_L$ and each n, a set $C(n,T) \in C$ in such a way that for each S, the sequence $\langle C(n,T)\colon n \in \omega\rangle$ is decreasing, and $(\forall n,S)(\text{diam}(C(n,T)) < 2^{-n})$, and $C(n,T) \cap \pi^{-1}(\{T\}) \neq 0$; then defining $s(T)$ = the unique member of $\bigcap_n C(n,T)$ (cf. Bourbaki [9] IX §6.8 where a nearly identical argument is given along these lines).

Let $\langle \underline{P}_i\colon i \in \omega\rangle$ be an enumeration of the atomic $\overline{\rho}$-names. Then the canonical metric on $X_{\overline{\rho}}$ is such that the clopen sets of diameter 2^{-n} have the form $[\underset{\sim}{\bigwedge}\Phi]$ where Φ is a finite set of subbasic names and $(\forall i < n)(\underline{P}_i \in \Phi$ or $\neg\underline{P}_i \in \Phi)$. Let $\{B_i\colon i \in \omega\}$ be the Skolem conditions such that $\overline{X}^L = [\underset{\sim}{\bigwedge} B_i]$. The canonical metric on $X = [\underset{\sim}{\bigwedge} B_i]$ is such that clopen sets of diameter less than 2^{-n} have the form $[\underset{\sim}{\bigwedge}\Phi] \cap X$ where Φ is a collection of basic names such that $[\underset{\sim}{\bigwedge}\Phi]$ is a clopen set in $X_{\overline{\rho}}$ with diameter less than 2^{-n} and $[\underset{\sim}{\bigwedge}\Phi]$ is disjoint from $[\bigvee_{i<n} \neg B_i]$. Since each B_i has one of the forms $P_{\underset{\sim}{\bigvee}\Psi} \to \underset{\sim}{\bigvee}\{P_\phi\colon \phi \in \Psi\}$, $\neg P_\phi \leftrightarrow P_{\neg\phi}$, or $P_\phi \to P_{\underset{\sim}{\bigvee}\Psi}$, the construction of the last paragraph may be recongnized as the familiar Henkin construction. Clearly, 7.1 is also closely related to the known fact: "Every

recursive complete theory has a recursively presented model" (cf. [19]).

Note also that the collection $\{\phi\binom{v_0, \ldots, v_{n-1}}{\underline{0}, \ldots, \underline{n-1}} : \phi$ is an n-formula of $L\}$ generates a fragment $L^{(n)}$ such that $S(L^{(n)})$ is exactly the space $S^n(L)$ of n-types for L. Thus, the fact that $S(L)$ is Polish for each fragment L, implies that $S^{(n)}(L)$ is Polish for each L and every $n \in \omega$.

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