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Remarks on invariant descriptive set theory *

by

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Abstract. Let X be a separable, completely metrizable space and E an analytic equivalence relation on X. $A \subset X$ is E-invariant if $y \in A$ whenever $x \in A$ and $E(x, y)$. We prove that the classes of E-invariant coanalytic sets and of E-invariant PCA sets each satisfy the Reduction Principle, and give E-invariant versions of other classical theorems. Our results generalize work of Vaught and others.

Let X be a Polish (separable, completely metrizable) space with $E \subseteq X \times X$ an equivalence relation on X. $B \subseteq X$ is *invariant* (with respect to E) provided $y \in B$ whenever $x \in B$ and $x E y$.

It is known (cf. [1]) that if E is a *countably separated* Σ_1^1 (analytic) equivalence, then X/E is Borel isomorphic to an analytic space (a metrizable continuous image of ω^ω) and, hence, that most theorems of descriptive set theory hold in invariant form.

Invariant version of several classical theorems have been proved under much weaker assumptions that countable separatedness. It has long been known (cf. our remarks after 1.2 below) that the invariant first separation principle, *Disjoint invariant Σ_1^1 sets can be separated by an invariant Borel set*, could be derived quite simply from the classical (non-invariant) theorem assuming only that E be Σ_1^1.

As 1.1 and 1.3 below we prove the invariant reduction principles:

If E is a Σ_1^1 equivalence then both the classes of invariant Π_1^1 (coanalytic) subsets of X and of invariant Σ_2^1 (PCA) *subsets of X have the reduction property.*

These results extend recent work of Y. N. Moschovakis ([18] and [19]) and R. L. Vaught ([23] and [24]). Vaught had proved the invariant reduction principles on the assumption that E be a "Polish action" equiva-

* Theorems 1.1, 1.7, 2.5 and all the new results in § 3 are due to Burgess. 1.3, 1.4, 4.2, the preliminary version of 1.6 and all of § 2 except 2.5 are due to Miller. All other results were proved jointly.

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lence. This case arises when a Polish topological group G acts on X according to a continuous map J: $(g, x) \mapsto gx$, inducing an equivalence $E = E_G = \{(x, gx)\colon x \in X, g \in G\}$. Equivalence classes are called orbits. Notice that, on the weaker assumption that G is an analytic space and J is Borel measurable, E_G is Σ^1_1 so our theorems give new information even when restricted to the action case.

Moreover, our arguments are strikingly simple. In contrast to previous proofs, which involved rather special group-theoretic or model-theoretic methods, our argument for Π^1_1 is an adaption of the well-known proof of the invariant first separation theorem. Our argument for Σ^1_2 is a variant of the classical proof of Σ^1_2 reduction. As a consequence of this simplicity, we are able to derive corresponding results about the analytical (lightface) hierarchy and under suitable set-theoretic assumptions, to extend our theorems to higher levels of the two hierarchies.

Vaught's proof of the reduction theorem for the invariant Π^1_1 sets (but not for Σ^1_2, cf. our remarks in § 4) applies in a context which properly includes the Polish action case. That is, when G is assumed only to be a non-meager topological group with countable basis. His argument relies on an analysis of the transform $B \to B^* = \{x\colon \{g\colon \in B\}$ is comeager in $G\}$. He defines $B^{\Delta} = \sim(\sim B)^*$.

As 2.1 we prove:

If B_1, B_2 reduce A_1, A_2 then B_1^*, B_2^{Δ} reduce A_1^*, A_2^{Δ}.

This says, roughly, that * preserves reduction and it yields Vaught's invariant Π^1_1 reduction theorem as a corollary. We also prove that * preserves several other interesting properties. Thus:

(a) *B^* is comeager if B is;*

(b) *B^* has the Baire property if B does;*

(c) ** preserves levels in various hierarchies of Δ^1_2 sets obtained from generalizations of the operation ($\mathcal{A}$).*

A particularly important class of Polish actions arises as follows. A *logic space* is a countable product of spaces of the form 2^{ω^n}, ω^{ω^n}, or ω^n. An element of such a space is a countable sequence of relations, functions, and constants on the set ω. Let G be the group $\omega!$ of permutations of ω with the relative topology from ω^ω, or any closed subgroup. Then there is a standard action J of G on any logic space X which is of special importance in the model theory of infinitary logic. Typically, if $(x, y, i) \in X = 2^{\omega\times\omega} \times \omega^\omega \times \omega$, then $J(g, (x, y, i)) = (gx, gy, g(i))$, where gx is defined by $gx(m, n) = x(g(m), g(n))$, and gy by $gy(n) = g^{-1}(y(g(n)))$. If G is the full group $\omega!$, then $R, S \in X$ are E_G-equivalent if and only if the structures (ω, R) and (ω, S) are isomorphic. These standard actions of closed subgroups of $\omega!$ on logic spaces will be called *logic actions*. The case $G = \omega!$, $X = {}^{\omega}s2$ is the *canonical logic action*.

Many theorems of model theory may be interpreted as theorems about the canonical logic action. For example, a theorem of Morley ([17]) tells us:

In the canonical logic action, any invariant Σ^1_2 set contains $\leqslant \aleph_1$ or $2^{\aleph_0}$ orbits.

It has been known that Morley's theorem could be derived from a result of Mansfield in the theory of definability over the hereditarily countable (HC) sets. A formula ψ in the language of set theory is said to be Σ_0 provided that only the bounded quantifiers $\exists s \in t$, $\forall s \in t$ occur in ψ. A set $B \subseteq \mathrm{HC}$ is $\Sigma_n(\mathrm{HC})$ if there is a Σ_0 formula ψ and a parameter $w \in \mathrm{HC}$ such that

$$B = \{x \in \mathrm{HC}\colon (\mathrm{HC}, \in) \models \exists y_n \forall y_{n-1} \ldots (\exists/\forall) y_1 \psi(x, y_1, \ldots, y_n, w)\}.$$

Mansfield's theorem asserts:

Any $\Sigma_1(\mathrm{HC})$ set has cardinality $\leqslant \aleph_1$ or $2^{\aleph_0}$.

Using well-known coding devices we remark that conversely, Mansfield's theorem is derivable from Morley's. As a further application of the same devices we prove, for example, 3.1:

For any $n \geqslant 1$, the $\Sigma_n(\mathrm{HC})$ sets have the reduction property if and only if, in the canonical logic action, the invariant Σ^1_{n+1} sets have the reduction property.

We conclude the paper with a survey of known results of invariant descriptive set theory with an eye to the hypotheses needed for their proofs. For example, we construct subgroups of $\omega!$ to illustrate that the weakest hypothesis for Vaught's invariant Π^1_1-reduction theorem ("G is non-meager") properly overlaps with the hypothesis of our 1.1 ("G is analytic").

We are grateful to R. L. Vaught for suggesting several improvements in the exposition and for his constant encouragement and advice. We wish also to thank S. G. Simpson for several observations which we use in § 3 and G. Bergman for suggesting an example in § 4.

1. Σ^1_1 equivalence relations (1). Let X be a Polish space, $E \subseteq X \times X$ a Σ^1_1 equivalence relation on X. For $B \subseteq X$ define

$$B^+ = \{y\colon (\exists x)(x \in B \wedge x E y)\}$$

and

$$B^- = \{y\colon (\forall x)(x E y \to x \in B)\}.$$

(1) In our original version of this section, we considered equivalence induced by the (Borel) action of an analytic topological group. Professor Vaught pointed out that our proofs apply in the more general case considered here.

Then $B^- \subseteq B \subseteq B^+$, B^+ and B^- are invariant, and if B is Σ^1_n (Π^1_n) so is B^+ (resp. B^-).

Recall that $A_1, B_1 \subseteq X$ are said to *reduce* $A, B \subseteq X$ if $A_1 \subseteq A$, $B_1 \subseteq B$, $A_1 \cup B_1 = A \cup B$, and $A_1 \cap B_1 = \emptyset$. A class $\mathfrak{K}$ of sets has the *reduction property* provided that for any $A, B \in \mathfrak{K}$ there exist $A_1, B_1 \in \mathfrak{K}$ which reduce A, B. A classical theorem states that the class of Π^1_1 subsets of X has the reduction property.

THEOREM 1.1. *The class of invariant Π^1_1 subsets of X has the reduction property.*

Proof. Let A, B be invariant Π^1_1 subsets of X. By the classical theorem there are Π^1_1 sets A_0, B_0 (not necessarily invariant) reducing A, B. Since $A - B \subseteq A_0$ is invariant, $A - B \subseteq A_0^-$ and $A_0^- \cup B = A \cup B$. For $n \geqslant 0$ let A_{n+1}, B_{n+1} be Π^1_1 sets reducing A_n^-, B. Then $A' = \bigcap_n A_n = \bigcap_n A_n^-$ and $B' = \bigcup_n B_n = (A \cup B) - A'$ are Π^1_1 and invariant and reduce A, B. ■

COROLLARY 1.2. *Two disjoint invariant Σ^1_1 subsets of X can be separated by an invariant Borel set.*

Proof. Let A_0, A_1 be disjoint invariant Σ^1_1 sets. By the well-known classical argument, 1.1 implies that A_0, A_1 can be separated by an invariant set B which is Δ^1_1. By Suslin's theorem, such a B is in fact Borel. ■

Our proof of 1.1 is an adaptation of a direct proof of 1.2 which was known long ago to Ryll-Nardzewski (cf. [23]) and has been used by Makkai [14] and Garland [4] among others.

Let $\mathfrak{K}$ be one of the projective classes Σ^1_n or Π^1_n and let $\mathfrak{L}$ be the opposite class (so $\mathfrak{L} = \Pi^1_n$ if $\mathfrak{K} = \Sigma^1_n$ and *vice versa*). If $B \subseteq X$, a map φ of B into some ordinal ϱ is called a *norm* on B. It is a $\mathfrak{K}$-norm if there exist relations $\leqslant^\varphi_{\mathfrak{K}}$ and $\leqslant^\varphi_{\mathfrak{L}}$ defining respectively $\mathfrak{K}$ and $\mathfrak{L}$ subsets of $X \times X$ such that

(*) for any $y \in B$ and any $x \in X$

$$[(x \in B \wedge \varphi(x) \leqslant \varphi(y)) \quad \text{iff} \quad x \leqslant^\varphi_{\mathfrak{K}} y \quad \text{iff} \quad x \leqslant^\varphi_{\mathfrak{L}} y\,.$$

$\mathfrak{K}$ has the Prewellordering (PWO) property if every $B \in \mathfrak{K}$ has a $\mathfrak{K}$-norm. The classical proof of the Π^1_1 Reduction Principle (using constituents) establishes, when properly regarded, that the class of Π^1_1 subsets of any Polish space has the PWO property (cf. the exposition in [7]).

THEOREM 1.3. *The class of invariant Σ^1_2 subsets of X has the reduction property.*

Proof. Let A, B be invariant Σ^1_2 subsets of X. Then $C = A \times \{0\} \cup B \times \{1\}$ is a Σ^1_2 subset of $X \times \{0, 1\}$. Say $C = \{(x, i): (\exists y \in X)(y, x, i) \in D\}$

where $D \subseteq X \times X \times \{0, 1\}$ is Π^1_1. Let $\varphi\colon D \to \varrho$ be a Π^1_1 norm on D, and let $\leqslant^\varphi_{\Pi^1_1}$, $\leqslant^\varphi_{\Sigma^1_1}$ be Π^1_1 and Σ^1_1 relations respectively, satisfying (*). Define $\psi\colon C \to \varrho$ by $\psi(x, i) = \inf\{\varphi(y, x', i): x\,E\,x' \wedge (y, x', i) \in D\}$. Let

$$A_1 = \{x: x \in A \wedge \psi(x, 0) \leqslant \psi(x, 1)\}, \qquad B_1 = \{x: x \in B \wedge \psi(x, 1) < \psi(x, 0)\}.$$

Clearly A_1, B_1 reduce A, B. Moreover since A, B are invariant and since $\psi(x, i)$ depends only on i and the E-equivalence class of x, A_1, B_1 are invariant. Now $x \in A_1$ if and only if $x \in A$ and

$$(\exists y, x')\big(\underbrace{x\,E\,x'}_{\Sigma^1_1} \wedge \underbrace{(y, x', 0) \in D}_{\Pi^1_1} \wedge (\forall z, x'')\big(\underbrace{\sim x\,E\,x''}_{\Pi^1_1} \vee \underbrace{(y, x', 0) \leqslant^\varphi_{\Pi^1_1} (z, x'', 1)}_{\Pi^1_1}\big)\big).$$

Also $x \in B_1$ if and only if $x \in B$ and

$$(\exists y, x')\Big(\underbrace{x\,E\,x'}_{\Pi^1_1} \wedge \underbrace{(y, x', 1) \in D}_{\Pi^1_1} \wedge (\forall z, x'')\Big(\underbrace{\sim x\,E\,x''}_{\Pi^1_1} \vee$$
$$\vee \sim \underbrace{\big((z, x'', 0) \leqslant^\varphi_{\Sigma^1_1} (y, x', 1)\big)}_{\Pi^1_1}\Big)\Big).$$

These expressions show A_1, B_1 are Σ^1_2 sets. ■

Our proof of 1.3 is essentially the classical proof of the Σ^1_2 reduction principle with some extra clauses inserted to guarantee the invariance of A_1 and B_1.

Let $B \subseteq X$. A norm $\varphi\colon B \to \varrho$ is *good* if $\varphi(x) = \varphi(y)$ whenever $x, y \in B$ and $x\,E\,y$. A class $\mathfrak{K}$ has the *good* PWO *property* if every invariant $B \in \mathfrak{K}$ has a good $\mathfrak{K}$-norm. The following is implicit in the proof of 1.3.

COROLLARY 1.4 (to the proof of 1.3). (a) *If $\mathfrak{K}$ has the good* PWO *property then the class of invariant $\mathfrak{K}$ sets has the reduction property.*

(b) *The class of Σ^1_2 subsets of X has the good* PWO *property.*

The most comprehensive reference for the Axiom of Projective Determinateness (PD) and its consequence is [7].

COROLLARY 1.5 (to the proofs of 1.1 and 1.3). *Assume* PD. *Then for any $n \geqslant 0$*

(a) *The class of invariant Π^1_{2n+1} subsets of X has the reduction property.*

(b) *The class of Σ^1_{2n+2} subsets of X has the good* PWO *property.*

(c) *The class of invariant Σ^1_{2n+2} subsets of X has the reduction property.*

Proof. Our proofs of 1.1 and 1.3 establish that, if the class of Π^1_{2n+1} sets has the PWO property (and, hence, the reduction property), then (a), (b), and (c) are true. PD implies that this hypothesis is fulfilled. ■

Note that for results about, say, Π^1_3 and Σ^1_4 sets we do not fully need the hypothesis that E is Σ^1_1 (Σ^1_3 would suffice). Similar remarks apply to 1.6 and 4.2 below.

We would expect to get the good PWO property for the class of Π^1_{2n+1} subsets of X $(n \geqslant 1)$ from PD, but we have been able to prove this only in rather special circumstances as discussed in § 4.

For $x \in \omega^\omega$ and $i \in \omega$ let $(x)_i \in \omega^\omega$ be defined by $(x)_i(m) = x(2^i(2m+1))$ and let $[x] = \{(x)_i \colon i \in \omega\}$. A binary relation R on ω^ω is a *strong* Σ^1_k *well-ordering* provided R well-orders ω^ω in type ω_1 and $\{(x, y) \colon x\, R\, y\}$ and $\{(y, z) \colon [z] = \{y' \colon y'\, R\, y\}\}$ are Σ^1_k subsets of $\omega^\omega \times \omega^\omega$.

The existence of a strong Σ^1_2 well-ordering of ω^ω follows from the Axiom of Constructibility $(\mathrm{V} = \mathrm{L})$ by a theorem of Gödel and Addison. Silver has shown that if there is a measurable cardinal $\varkappa$ and a normal ultrafilter D on $\varkappa$ such that $\mathrm{V} = \mathrm{L}^D$, then there is a strong Σ^1_3 well-ordering of ω^ω. For proofs of those theorems, with applications to descriptive set theory, see [3] and [22]. The applications in [3] and [22] are stated for ω^ω but apply to any Polish space X by a standard argument (similar to that used in 1.6(a) below). We now consider some applications of strong well-orderings in invariant descriptive set theory.

Let $X = X_0 \times X_1$ be a product of Polish spaces, and E an equivalence relation on X. If $A \subseteq X$ is E-invariant, we say B E*-invariantly uniformizes* A if $B \subseteq A$, B is E-invariant, domain B = domain A (i.e. $(\forall x \in X_0, y \in X_1)\big((x, y) \in A \to (\exists y' \in X_1)(x, y') \in B\big)$ and

$$(\forall x \in X_0)(\forall y, y' \in X_1)\big((x, y) \in B \wedge (x, y') \in B \to (x, y)\, E\, (x, y')\big).$$

A class $\mathfrak{K}$ of subsets of X satisfies the E*-Invariant Uniformization Principle* (E-IUP) if for every $A \in \mathfrak{K}$ there is a $B \in \mathfrak{K}$ E-invariantly uniformizing A.

If $X = X_0 \times X_1$ as above, and E_0, E_1 are equivalence relations on X_0, X_1 respectively, then $E_0 \times E_1$ denotes the equivalence relation $\{((x_0, y_0), (x_1, y_1)) \colon x_0\, E_0\, x_1 \wedge y_0\, E_1\, y_1\}$ on X.

THEOREM 1.6. *Assume there exists a strong Σ^1_k well-ordering of ω^ω. Then*

(a) *There is a function $s \colon X \to X$ whose graph is a Σ^1_k subset of E such that $x\, E\, y$ if and only if $s(x) = s(y)$.*

Moreover, for any $n \geqslant k$

(b) *The class of invariant Σ^1_n subsets of X has the reduction property.*

(c) *The class of Σ^1_n subsets of X has the good* PWO *property.*

(d) *If E_0, E_1 are Σ^1_1 equivalence relations on Polish spaces X_0, X_1, and $X = X_0 \times X_1$, $E = E_0 \times E_1$, then the class of Σ^1_n subsets of X satisfies the* E-IUP.

A result very close to 1.6(a) was obtained by K. Kuratowski some time before we considered the problem. He showed the existence of any Σ^1_k well-ordering (not necessarily strong) implies the existence of a Σ^1_k selector (set of equivalence class representatives) for any Σ^1_1 equivalence relation E such that every E-equivalence class is countable.

Proof. We first assume X is a closed subspace of ω^ω, so E is a Σ^1_1 subset of $\omega^\omega \times \omega^\omega$. Let R be a strong Σ^1_k well ordering of ω^ω. Set $s(x)$ = the R-least y such that $x\, E\, y$. Then

$$\text{Graph } s = \Big\{(x, y) \colon \underbrace{x\, E\, y}_{\Sigma^1_1} \wedge (\exists z)\big(\underbrace{[z] = \{y' \colon y'\, R\, y\}}_{\Sigma^1_k} \wedge \underbrace{(\forall i \in \omega)\big(\sim x\, E\, (z)_i\big)}_{\Pi^1_1}\big)\Big\}.$$

Since $k \geqslant 2$ this shows that Graph s is Σ^1_k. Clearly this s satisfies the conditions of (a).

To verify (a) in the general case, let C be a closed subset of ω^ω and f a 1-1, continuous map of C onto X. Then $E' = f^{-1}(E)$ is a Σ^1_1 equivalence on C. Let $s' \subseteq E'$ be the function obtained above. Then s defined by $s(x) = fs'f^{-1}(x)$ satisfies the conditions of (a).

Now for $n \geqslant k$ the reduction property, the ordinary PWO, and the ordinary uniformization principle for the class of Σ^1_n subsets of X all follow from the existence of a strong Σ^1_k well-ordering of ω^ω. Combining these with (a) we can derive (b)-(d). We prove (d) to indicate the method.

With the notation as in (d), let $A \subseteq X$ be an E-invariant Σ^1_n set. Let A' be a Σ^1_n set (not necessarily invariant) uniformizing A in the ordinary sense. Apply (a) to X_0, E_0 to obtain the function s_0. Then $B = \{(x, y) \colon (\exists y_1)(y_1\, E_1\, y \wedge (s_0(x), y_1) \in A')\}$ E-invariantly uniformizes A. For clearly, B is Σ^1_n and E-invariant. Moreover, if $(x, y) \in B$, then for some y_1, $(s_0(x), y_1) \in A$ and is E-equivalent to (x, y), so $(x, y) \in A$ and $B \subseteq A$. If $x \in$ domain A, then by E-invariance $s_0(x) \in$ domain A = domain A', so $x \in$ domain B. Finally, if $(x, y) \in B$, $(x, y') \in B$, then there exist y_1, y_1' such that $y_1\, E_1\, y$, $y_1'\, E_1\, y'$, and $(s_0(x), y_1)$, $(s_0(x), y_1') \in A'$. Since A' uniformizes A in the ordinary sense, $y_1 = y_1'$; so $y\, E_1\, y'$ and $(x, y)\, E\, (x, y')$ as required, completing the proof. ■

Note that if we are given only a Σ^1_k well-ordering (not necessarily strong) the graph of s as defined above can still be shown to be Π^1_k and hence Σ^1_{k+1}, so (b)-(d) still hold for $n > k$.

Let J_0, J_1 be Borel action of an analytic group G on Polish spaces X_0, X_1 respectively, and let E_0, E_1 be the induced equivalences. The equivalence $E_0 \times E_1$ on $X = X_0 \times X_1$ is the same as that induced by the action $J_0 \times J_1$ of $G \times G$ on X given by

$$(J_0 \times J_1)\big((g, h), (x, y)\big) = \big(J_0(g, x), J_1(h, y)\big).$$

But in this group action situation there is another natural equivalence on X to be considered, viz. the equivalence E^V induced by the action J^V of G on X given by $J^V(g, (x, y)) = (J_0(g, x), J_1(g, y))$. In fact the original version of invariant uniformization introduced by Vaught in [23] and studied by Myers in [15] and [16] was E^V-invariant uniformization for J_0, J_1 the standard actions of $\omega!$ (as discussed in the introduction) on two logic spaces X_0, X_1. To clarify the relationship between this earlier version of invariant uniformization and our "product" version, we state the following

PROPOSITION 1.7. *In the notation above we have, for any n:*

(a) *The E^V-IUP for the class of Σ_n^1 subsets of X implies the $(E_0\times E_1)$-IUP for the same class.*

(b) *The $(E_0\times E^V)$-IUP for the class of Σ_n^1 subsets of $X_0\times X$ implies the E^V-IUP for the class of Σ_n^1 subsets of X.*

(c) *If for some $k\leqslant n$ there exists a strong Σ_k^1 well-ordering of ω^ω, then the class of Σ_n^1 subsets of X satisfies the E^V-IUP.*

Proof. To see (a), let $A\subseteq X$ be an $(E_0\times E_1)$-invariant Σ_n^1 set. A is *a fortiori* E^V-invariant. If B is a Σ_n^1 set E^V-invariantly uniformizing A, then $C=\{(x,y)\colon (\exists g\in G)(x, J_1(g,x))\in B\}$ is still Σ_n^1 and $(E_0\times E_1)$-invariantly uniformizes A.

To see (b), let $A\subseteq X$ be an E^V-invariant Σ_n^1 set. Then

$$A'=\{(x,(y,z))\colon xE_0y\wedge(y,z)\in A\}\subseteq X_0\times X$$

is an $(E_0\times E^V)$-invariant Σ_n^1 set. If B' is a Σ_n^1 set $(E_0\times E^V)$-invariantly uniformizing A', then $B=\{(x,z)\colon (x,(x,z))\in B'\}$ is a Σ_n^1 set and can be shown to E^V-invariantly uniformize A. To show domain $B=$ domain A, for example, suppose $(x,z)\in A$, so $(x,(x,z))\in A'$. Since domain $B'=$ domain A', there is some $(x,(x',z'))\in B'$. Then xE_0x', so there is $g\in G$ with $J_0(g,x')=x$. By the $(E_0\times E^V)$-invariance of B',

$$(x,(x, J_1(g,z')))\in B', \quad\text{and}\quad (x, J_1(g,z'))\in B\,.$$

Now (c) is immediate from (b) and 1.6(d). It could also be proved directly from 1.6(a). ∎

Myers has recently (see [16]) settled an old question of Vaught by showing that there are logic spaces X_0, X_1 in which, setting $X=X_0\times X_1$, $G=\omega!$, J_0, J_1 the standard actions of G on X_0, X_1, E_0, E_1 the induced equivalences, and E^V as above, it happens that the E^V-IUP fails for the class of Π_1^1 subsets of X. Moreover he shows it is consistent with the usual axioms (ZFC) of set theory that in this example the E^V-IUP fails for every projective class $\mathfrak{K}$. His arguments also show that for certain logic spaces X_0, X_1, with other notation as above, the $(E_0\times E_1)$-IUP fails for the class of Π_1^1 subsets of X, and that it is consistent with ZFC that the $(E_0\times E_1)$-IUP fails for every projective class $\mathfrak{K}$ in this example. Thus no version known of the IUP can be proved in ZFC alone, even if we consider only logic actions. We know of no hypothesis weaker than the existence of a strong Σ_2^1 well-ordering which would imply the IUP in any version for Σ_2^1.

For the remainder of § 1 let us assume that X is a finite product of spaces of the form ω^{ω^n}, 2^{ω^n}, or ω^n. For such spaces the "lightface" or analytical hierarchy Σ_1^1, Π_1^1, Σ_2^1, etc. has been defined and extensively studied (see e.g. [2]). Let us assume E is a Σ_1^1 equivalence relation on X. We shall show that 1.1-1.6 hold in lightface versions.

THEOREM 1.8. *The class of invariant Π_1^1 subsets of X has the reduction property.*

Proof. We use the reduction property for the class of Π_1^1 subsets of X and argue much as in the proof of 1.1. However, a countable intersection or union of Π_1^1 sets is not necessarily Π_1^1 so the argument of 1.1 does not apply directly.

For simplicity we assume $X=\omega^\omega$. Let $\{\varphi_i\colon i\in\omega\}$ be a recursive enumeration of the Π_1^1 formulas of second order arithmetic with one free function variable. Let $W=\{(i,x)\colon \varphi_i(x)\text{ is true}\}$. Then W is a Π_1^1 subset of $\omega\times\omega^\omega$. There is a recursive function w such that for any i, $\{x\colon (i,x)\in W\}^-=\{x\colon (w(i),x)\in W\}$. Let

$$U=\{(i,j,x)\colon (i,x)\in W\},\qquad V=\{(i,j,x)\colon (j,x)\in W\}\,.$$

Let U', V' be Π_1^1 subsets of $\omega\times\omega\times\omega^\omega$ reducing U, V. Let u, v be recursive functions such that

$$\{x\colon (u(i,j),x)\in W\}=\{x\colon (i,j,x)\in U'\}\,,$$
$$\{x\colon (v(i,j),x)\in W\}=\{x\colon (i,j,x)\in V'\}\,.$$

Now let A, B be invariant Π_1^1 subsets of X. Express $A=\{x\colon (i,x)\in W\}$, $B=\{x\colon (j,x)\in W\}$. Define recursive functions a, b by $a(0)=i$, $b(0)=j$ and $a(n+1)=u(w(a(n)),j)$, $b(n+1)=v(w(a(n)),j)$. Then

$$A'=\{x\colon (\forall n\in\omega)(a(n),x)\in W\}\,,\qquad B'=\{x\colon (\exists n\in\omega)(b(n),x)\in W\}$$

are invariant Π_1^1 sets reducing A, B. ∎

In contrast to the above, the proofs of the lightface versions of 1.2-1.6 may be simply obtained by systematically replacing boldface notation with lightface notation. We leave this to the reader.

The reader familiar with admissible sets may wonder whether our theorems also hold in "arbitrary admissible set" form. The lightface theory is essentially the theory of the first admissible set containing ω, i.e. of L_α where $\alpha=$ the least non-recursive ordinal $=$ the least admissible ordinal $>\omega$. Vaught has shown that the methods of [23] reduce questions of *non*-invariant descriptive set theory for an *arbitrary* admissible set to questions of *invariant* descriptive set theory for the *particular* admissible set L_α, and that our methods can be used to derive theorems of the *invariant* theory for an arbitrary admissible set from the *non*-invariant theory for that admissible set. Thus by a rather indirect route we do get the full invariant theory for an arbitrary admissible set.

2. Vaught's transform. Let X be a topological space, G a topological group, $J\colon G\times X\to X$ an action of G on X which is continuous in each variable separately, E the induced equivalence relation. We assume that G is a Baire space (equivalently, cf. [8], G is non-meager) and has a countable weak basis $\mathcal{K}$. Such an $\mathcal{K}$ consists of non-empty open sets and every non-empty open set includes a member of $\mathcal{K}$.

We first recall some of the principle definitions and results of Vaught [24]. It will be convenient to assume $G\in\mathcal{K}$ and to take U, V with subscripts as variables ranging over the members of $\mathcal{K}$. For $B\subseteq X$ and $x\in X$, $B^x=\{g\in G\colon gx\in B\}$. For $g\in G$, $B^g=\{x\in X\colon gx\in B\}$. $B^{*U}=\{x\in X\colon B^x\cap U \text{ is comeager in } U\}$, $B^*=B^{*G}$, $B^{\Delta U}=\sim(\sim B^{*U})=\{x\in X\colon B^x\cap U \text{ is non-meager in } U\}$, $B^\Delta=B^{\Delta G}$.

B is *normal* if for every $x\in X$, B^x has the Baire property. In 2.1 we will characterize the normal sets as those which behave well with respect to $*$. Since the operations of complementation and countable union and intersection, and the operation $(\mathcal{A})$ preserve the property of Baire, and since these operations commute with the transform $B\to B^x$, they all preserve normality. Since the transform $B\to B^x$ takes closed sets to closed sets, all the subsets of X obtained from closed sets by iterating the above operations are normal. Vaught refers to these sets as the $\mathcal{A}$-sets; classically they have been known as $\mathcal{C}$-sets (*ensembles criblés*).

Let sq be the set of all finite sequences of natural numbers. We let $\underline{k}$, $\underline{l}$ with subscripts range over elements of sq. Let ξ, η with subscripts range over elements of ω^ω. For $n\in\omega$ we write $\xi|n$ to denote

$$(\xi(0),\,\xi(1),\,\ldots,\,\xi(n-1))\in sq.$$

Thus, the set obtained by operation $(\mathcal{A})$ from the indexed family $\{A_{\underline{k}}\colon \underline{k}\in sq\}$ is $\bigcup_\xi\bigcap_n A_{\xi|n}$.

Formulas 1.3, 1.4, 1.5, and 1.6 of Vaught [24] tell us:

(0) If B is closed, $B^{*U}=\bigcap_{g\in U}B^g$. If B is open, $B^{\Delta U}=\bigcup_{g\in U}B^g$.

(1) $(\bigcap_n B_n)^{*U}=\bigcap_n B_n^{*U}$, $(\bigcup_n B_n)^{\Delta U}=\bigcup_n B_n^{\Delta U}$.

(2) If B is normal $(\sim B)^{*U}=\sim\bigcup_{V\subseteq U}B^{*V}$, hence $B^{\Delta U}=\bigcup_{V\subseteq U}B^{*V}$. Also, $(\sim B)^{\Delta U}=\sim\bigcap_{V\subseteq U}B^{\Delta V}$ and $B^{*U}=\bigcap_{V\subseteq U}B^{\Delta V}$.

(3) If each $A_{\underline{k}}$ is normal then

$$\Big(\bigcup_\xi\bigcap_n A_{\xi|n}\Big)^{*U}=\bigcap_{U_0\subseteq U}\ \bigcup_{V_0\subseteq U_0}\ \bigcup_{k_0}\ \bigcap_{U_1\subseteq V_0}\ \bigcup_{V_1\subseteq U_1}\ \bigcup_{k_1}\cdots\bigcap_n A^{*V_n}_{(k_0,\ldots,k_n)}.$$

Formally, membership in the right hand side of (3) is defined in terms of the existence of a winning strategy for a certain infinite game. Vaught derives from (0)-(2) the important consequence that if B is Borel, so are B^{*U} and $B^{\Delta U}$.

The following observations are at least implicit in [24]: Assume X is Polish; then if A is Σ^1_1 we can express A as $\bigcup_\xi\bigcap_n A_{\xi|n}$ with the $A_{\xi|n}$ Borel. Then (3) expresses A^{*U} as the result of a "game" operation applied to the $A_{\underline{k}}^{*V}$ for $V\subseteq U$. Since these sets are Borel, and, as is well-known, the "game" operation is really just a variant of $(\mathcal{A})$, A^{*U} is obtained by $(\mathcal{A})$ from Borel sets, hence it is Σ^1_1. Since $A^{\Delta U}=\bigcup_{V\subseteq U}A^{*V}$ and a countable union of Σ^1_1 sets is Σ^1_1, $A^{\Delta U}$ is also Σ^1_1. Finally, if A is Π^1_1, $A^{*U}=\sim(\sim A)^{\Delta U}$ and $A^{\Delta U}=\sim(\sim A)^{*U}$ are Π^1_1. These are most of the facts we shall need from [24].

We begin our own remarks on the Vaught transform with a converse to (2).

Proposition 2.1. *$B\subseteq X$ is normal if and only if for every U, $B^{\Delta U}=\bigcup_{V\subseteq U}B^{*V}$.*

Proof. The "only if" part is the content of (2). For the "if" part, suppose that for every U $B^{\Delta U}=\bigcup_{V\subseteq U}B^{*V}$. Fixing $x\in X$ this implies that either $B^x\cap U$ is meager or there is a $V\subseteq U$ such that $\sim B^x\cap V$ is meager. Since $\mathcal{K}$ is a weak basis it follows that every non-empty open set contains a point where either B^x or $\sim B^x$ is meager. This shows that B^x has the Baire property (cf. [10]). ∎

Note that the proposition is true without the assumption that $\mathcal{K}$ is countable, or that G is a group (cf. [24] § 1 for a more general treatment).

Proposition 2.2. *If A_1, B_1 reduce A, B then A_1^*, B_1^Δ reduce A^*, B^Δ.*

Proof. Clearly $A_1^*\subseteq A^*$, $B_1^\Delta\subseteq B^\Delta$. From the definitions, $A^*-B^\Delta=(A-B)^*\subseteq A_1^*$ and $B^\Delta-A^*=(B-A)^\Delta\subseteq B_1^\Delta$, so $A_1^*\cup B_1^\Delta=A^*\cup B^\Delta$. Finally, since $A_1=(A\cup B)-B_1$, $A_1^*=(A\cup B)^*-B_1^\Delta$ so $A_1^*\cap B_1^\Delta=\emptyset$. ∎

Theorem 2.3 (Vaught). *If X is Polish then the class of invariant Π^1_1 subsets of X has the reduction property.*

Proof. Let A, B be invariant Π^1_1 sets. Let A_1, B_1 be arbitrary (not necessarily invariant) Π^1_1 sets reducing A, B. Since A, B are invariant, $A^*=A$, $B^\Delta=B$, and so by 2.2 A_1^*, B_1^Δ reduce A, B. We have remarked that A_1^*, B_1^Δ are still Π^1_1. ∎

Our 2.3 overlaps with 1.1 and coincides with Vaught's invariant version of the Π^1_1 Reduction Principle. Our proof is shorter than Vaught's, but his proof, properly regarded, establishes a stronger result, viz. that the class of Π^1_1 subsets of X has the good PWO property.

Now we add to the list of properties preserved by * and Δ.

THEOREM 2.4. (a) *If B is meager, so are B^Δ and B^*.*

(b) *If B has the property of Baire so do B^Δ and B^*.*

Proof. Let B be meager. Then there are closed nowhere dense sets C_n such that $B \subseteq \bigcup_n C_n$. Then

$$B^\Delta \subseteq (\bigcup_n C_n)^\Delta = \bigcup_n C_n^\Delta = \bigcup_n \bigcup_U C_n^{*U} = \bigcup_n \bigcup_U \bigcap_{g \in U} C_n^g .$$

Each set C_n^g being a translate of the closed nowhere dense set C_n, is closed nowhere dense. The inclusion $B^\Delta \subseteq \bigcup_n \bigcup_U (\bigcap_{g\in U} C_n^g)$ thus shows that B^Δ is meager. Since $B^* \subseteq B^\Delta$, B^* is also meager (compare [24] 2.4).

Now let B have the property of Baire. Write $B = A \cup N$ where A is a G_δ and N is meager. Then $B^\Delta = A^\Delta \cup N^\Delta$ where A^Δ is Borel and by what we have just shown N^Δ is meager. So B^Δ has the Baire property. Since $B^* = \sim(\sim B)^\Delta$ and the class of sets with the Baire property is closed under complementation, B^* has the Baire property. ■

Remarks (on 1.9 of [24]). Let $E = \bigcup_\xi \bigcap_n A_{\xi|n}$. The classical approximations to E are defined by $A_{\underline{k}}^0 = A_{\underline{k}}$, $A_{\underline{k}}^{\alpha+1} = A_{\underline{k}}^\alpha \cap \bigcup_i A_{\underline{k}\frown i}^\alpha$, $A_{\underline{k}}^\lambda = \bigcap_{\alpha<\lambda} A_{\underline{k}}^\alpha$ for λ a limit ordinal, $E_\alpha = A_{\varnothing}^\alpha$, $T_\alpha = \bigcup_{\underline{k}} (A_{\underline{k}}^\alpha - A_{\underline{k}}^{\alpha+1})$. If X has the countable chain condition, i.e. if every disjoint collection of open subsets is countable, and if the $A_{\underline{k}}$ all have the property of Baire, then for some α T_α is meager. In this case, our 2.4 at once shows that T_α^Δ is meager, and hence $(E_\alpha - T_\alpha)^* = E_\alpha^* - T_\alpha^\Delta$ comeager in E_α^*. In 1.9 of [24] this conclusion is derived from the more restrictive hypothesis that the $A_{\underline{k}}$ are all $\mathcal{A}$-sets, but the proof in [24] applies outside the "action case" we are considering here.

In classical descriptive set theory there occurs a sequence of operations Γ^ν ($\nu < \omega_1$) on countable indexed families of sets in which Γ^0 is countable union, Γ^1 is a variant of operation ($\mathcal{A}$), and, roughly speaking, for $\nu \geqslant 1$ $\Gamma^{\nu+1}$ is to Γ^ν as Γ^1 is to Γ^0. The works of Lapunov ([11] and [12]) contain the whole theory of these operations. Non-readers of Russian will find most of the facts we shall need in [5].

Associated with any operation Γ on families of sets we have a hierarchy $\mathfrak{B}(\Gamma)$ of subsets of X defined by

$$\mathfrak{B}(0, \Gamma) = \{A \subseteq X\colon A \text{ open}\}, \quad \mathfrak{B}'(\alpha, \Gamma) = \{X - A\colon A \in \mathfrak{B}(\alpha, \Gamma)\},$$

$\mathfrak{B}(\alpha, \Gamma) =$ the closure under Γ of $\bigcup_{\beta<\alpha} \mathfrak{B}'(\beta, \Gamma)$, and $\mathfrak{B}(\Gamma) = \bigcup_{\alpha<\omega_1} \mathfrak{B}(\alpha, \Gamma)$. Thus if $\Gamma =$ countable union, $\mathfrak{B}(\Gamma) =$ Borel hierarchy and e.g. $\mathfrak{B}(1, \Gamma) = F_\sigma$. If $\Gamma =$ operation ($\mathcal{A}$) (or its variant Γ', or the "game" operation of (3)) then $\mathfrak{B}(\Gamma) =$ the hierarchy of $\mathcal{A}$-sets, and (for Polish X) $\mathfrak{B}(1, \Gamma)$ $= \Sigma_1^1$. $\mathfrak{B}(1, \Gamma^2)$ property includes $\mathfrak{B}(\Gamma^1)$, just as Σ_1^1 property includes Borel.

THEOREM 2.5. *Let $\nu, \alpha \geqslant 0$. If $B \in \mathfrak{B}(\alpha, \Gamma^\nu)$; $C \in \mathfrak{B}'(\alpha, \Gamma^\nu)$ then $B^\Delta \in \mathfrak{B}(\alpha, \Gamma^\nu)$ and $C^* \in \mathfrak{B}'(\alpha, \Gamma^\nu)$. If ν is a successor ordinal, $B^* \in \mathfrak{B}(\alpha, \Gamma^\nu)$ and $C^\Delta \in \mathfrak{B}'(\alpha, \Gamma^\nu)$ as well.*

Proof. The case $\nu = 0$ is 1.8(b) of [24]. The case $\nu = 1$ is implicit in [24]; we have discussed preservation of Σ_1^1 and Π_1^1 by * and Δ above. We sketch the proof for $\nu = 2$. This sufficiently illustrates the general case, while avoiding some notational complexities.

It will be convenient to work not with Γ^2 as defined in [5], but with a variant which we call the operation ($\mathcal{A}'$). Let SQ be the set of all finite sequences of even length of elements of *sq*. We let $\underline{K}$ with subscripts range over elements of SQ. ($\mathcal{A}'$) is an operation on families indexed by SQ. Applied to $\{A_{\underline{K}}\colon \underline{K} \in \mathrm{SQ}\}$ it yields

$$(4) \qquad \bigcup_{\xi_0} \bigcap_{m_0} \bigcap_{\eta_0} \bigcup_{n_0} \bigcup_{\xi_1} \bigcap_{m_1} \bigcap_{\eta_1} \bigcup_{n_1} \ldots \bigcap_{p} A_{\xi_0|m_0, \eta_0|n_0 \ldots \xi_p|m_p, \eta_p|n_p} .$$

Formally, membership in (4) is defined in terms of the existence of a winning strategy for a certain infinite game. Readers familiar with [5] should recognize that ($\mathcal{A}'$) is a variant of Γ^2 so that $\mathfrak{B}(\alpha, (\mathcal{A}')) = \mathfrak{B}(\alpha, \Gamma^2)$ for every $\alpha < \omega_1$; other readers can simply follow the proof for ($\mathcal{A}'$) and forget about Γ^2.

Since ($\mathcal{A}'$) preserves the property of Baire (see below) and commutes with $B \mapsto B^x$, all the sets in $\mathfrak{B}((\mathcal{A}'))$ are normal. Let A be the result of ($\mathcal{A}'$) applied to $\{A_{\underline{K}}\colon \underline{K} \in \mathrm{SQ}\}$, and assume that the $A_{\underline{K}}$ are all normal. We claim

$$(5) \qquad A^{*U} = \bigcap_{U_{00} \subseteq U} \bigcup_{V_{00} \subseteq U_{00}} \bigcup_{k_{00}} \bigcap_{U_{01} \subseteq V_{00}} \bigcup_{V_{01} \subseteq U_{01}} \bigcup_{k_{01}} \ldots$$
$$\ldots \bigcap_{m_0} \bigcup_{W_{00} \subseteq V_{0m_0}} \bigcap_{X_{00} \subseteq W_{00}} \bigcap_{l_{00}} \bigcup_{W_{01} \subseteq X_{00}} \bigcap_{X_{01} \subseteq W_{01}} \bigcap_{l_{01}} \ldots$$
$$\ldots \bigcup_{n_0} \bigcap_{U_{10} \subseteq X_{0n_0}} \bigcup_{V_{10} \subseteq U_{10}} \bigcup_{k_{10}} \bigcap_{U_{11} \subseteq V_{10}} \bigcup_{V_{11} \subseteq U_{11}} \bigcup_{k_{11}} \ldots$$
$$\ldots \bigcap_{m_1} \bigcup_{W_{10} \subseteq V_{1m_1}} \bigcap_{X_{10} \subseteq W_{10}} \bigcap_{l_{10}} \bigcup_{W_{11} \subseteq X_{10}} \bigcap_{X_{11} \subseteq W_{11}} \bigcap_{l_{11}} \bigcup_{n_1} \ldots$$
$$\ldots \bigcap_{p} A^{*X_{pm_p}}_{(k_{00}, k_{01}, \ldots, k_{0m_0}), (l_{00}, l_{01}, \ldots l_{0n_0}), \ldots, (k_{p0}, k_{p1}, \ldots, k_{pm_p}), (l_{p0}, l_{p1}, \ldots, l_{pn_p})} .$$

(Here W, X with subscripts range over $\mathcal{K}$.) Formally, membership in (5) is defined in terms of the existence of a winning strategy for a certain game of length ω^2. The argument that the game operation (3) is a variant of ($\mathcal{A}$) shows that the operation in (5) is a variant of ($\mathcal{A}'$). Thus, our claim $A^{*U} =$ (5) implies that A^{*U} can be obtained by ($\mathcal{A}'$) from the $A_{\underline{K}}^{*V}$, $V \subseteq U$.

From this fact our desired conclusion, that every $\mathfrak{B}(\alpha, (\mathcal{A}'))$ and $\mathfrak{B}'(\alpha, (\mathcal{A}'))$ is closed under $*$ and Δ, follows easily (cf. our discussion of Σ_1^1 and Π_1^1 above). So it only remains to show $A^{*U} = (5)$.

To this end we introduce approximations to A and to (5). The approximations to A are obtained from the representation (4). Let $A_{\underline{K}}^0 = A_{\underline{K}}$, $A_{\underline{K}}^{\alpha+1} = A_{\underline{K}}^\alpha \cap \bigcup_\xi \bigcap_m \bigcap_\eta \bigcup_n A^\alpha_{\underline{K}^\frown\xi|m^\frown\eta|n}$, $A_{\underline{K}}^\lambda = \bigcap_{\alpha<\lambda} A_{\underline{K}}^\alpha$ for λ a limit ordinal, $A_\alpha = A_\varnothing^\alpha$, $T_\alpha = \bigcup_{\underline{K}} (A_{\underline{K}}^\alpha - A_{\underline{K}}^{\alpha+1})$. Lapunov's theory of Kolmogorov's R-operation, or Moschovakis' theory of generalized inductive definitions will tell us $A = \bigcap_{\alpha<\omega_1} A_\alpha = \bigcup_{\alpha<\omega_1} (A_\alpha - T_\alpha)$, and the reader familiar with approximations to sets obtained by the operation $(\mathcal{A})$ will have no difficulty in supplying his own proof. These approximations can also be used to show $(\mathcal{A}')$ preserves the property of Baire, imitating one of the classical proofs for $(\mathcal{A})$.

For any $x \in X$ we get a representation of A^x in terms of the $A_{\underline{K}}^x$ from (4) simply by superscripting with x. The approximations to A^x obtained from this representation are precisely the A_α^x. Since the $A_{\underline{K}}$ are normal, so that the $A_{\underline{K}}^x$ have the property Baire, standard arguments show that for some $\alpha < \omega_1$ T_α^x is meager. This gives us

(6) $$A^{*U} = \bigcap_{\alpha<\omega_1} A_\alpha^{*U}.$$

The inclusion from left to right is obvious. To go from right to left, let $x \in X$ and let $\alpha < \omega_1$ be such that T_α^x is meager. Since $A_\alpha^x - T_\alpha^x \subseteq A^x \subseteq A_\alpha^x$, $A^x \cap U$ is comeager in U if and only if $A_\alpha^x \cap U$ is, so if $x \in A_\alpha^{*U}$, $x \in A^{*U}$.

We leave A^{*U} aside for the moment and discuss approximations to (5). Let sq' be the set of all finite sequences of the form

$$(U_0, V_0, k_0, \ldots, U_m, V_m, k_m) \quad \text{where} \quad U_0 \supseteq V_0 \supseteq \ldots \supseteq U_m \supseteq V_m,$$

and let SQ′ be the set of all finite sequences of even length of elements of sq'. Let $\underline{s}, \underline{t}$ with subscripts range over sq', and $\underline{S}$ over SQ'. If $\underline{s} = (U_0, V_0, k_0, \ldots, U_m, V_m, k_m)$, set $\underline{k}(\underline{s}) = (k_0, \ldots, k_m) \in sq$, $v(\underline{s}) = V_m$. If $\underline{S} = \underline{s}_m, \underline{t}_0, \ldots, \underline{s}_p, \underline{t}_p$, set $\underline{K}(\underline{S}) = \underline{k}(\underline{s}_0), \underline{k}(\underline{t}_0), \ldots, \underline{k}(\underline{s}_p), \underline{k}(\underline{t}_p)$, and $V(\underline{S}) = v(\underline{s}_p)$. With this notation we can define approximations $B_{\underline{S}}^\alpha$ to (5) for $\underline{S} \in \mathrm{SQ}'$ by $B_{\underline{S}}^0 = A_{\underline{K}(\underline{S})}^{*V(\underline{S})}$, $B_{\underline{S}}^\lambda = \bigcap_{\alpha<\lambda} B_{\underline{S}}^\alpha$ for λ a limit ordinal, and

(7) $$B_{\underline{S}}^{\alpha+1} = \bigcap_{U_0 \subseteq V(\underline{S})} \bigcup_{V_0 \subseteq U_0} \bigcup_{k_0} \bigcap_{U_1 \subseteq V_0} \bigcup_{V_1 \subseteq U_1} \bigcup_{k_1} \ldots$$
$$\ldots \bigcap_m \bigcup_{W_0 \subseteq V_m} \bigcup_{X_0 \subseteq W_0} \bigcap_{l_0} \bigcup_{W_1 \subseteq X_0} \bigcup_{X_1 \subseteq W_1} \bigcap_{l_1} \ldots$$
$$\ldots \bigcap_n B^\alpha_{\underline{S}^\frown(W_0, V_0, k_0, \ldots, U_m, V_m, k_m)^\frown(W_0, X_0, l_0, \ldots, W_n, X_n, l_n)}.$$

Define also $B_\alpha = B_\varnothing^\alpha$. For these approximations too, we have (5) $= \bigcap_{\alpha<\omega_1} B_\alpha$. Comparing with (6) we see it will suffice to show $A_\alpha^{*U} = B_\alpha$ to prove our claim $A^{*U} = (5)$. In fact it is easily shown by induction, using (3) at the successor step, that $B_{\underline{S}}^\alpha = (A^\alpha_{\underline{K}(\underline{S})})^{*V(\underline{S})}$. This concludes our sketch of the proof of 2.5. ∎

We could, if we wished, introduce for $\nu < \omega_1$ a language L^ν, so that $L^0 = L_{\omega_1\omega}$, $L^1 = L_{\omega_1 G}$ (as in [24]), and, generalizing 3.8 of [24], Invariant $\mathfrak{B}(\Gamma^\nu) = L^\nu$.

3. Definability theory over HC. In this section we show that certain facts about logic actions are systematically equivalent to facts in the theory of definability over HC. We must assume that the reader has some acquaintance with this theory. We will deal with Lévy's hierarchy (Σ_n, Π_n) of formulas in the language of set theory which was defined in [11] and recalled in our introduction. We will use, for instance, the fact that a subset of a logic space is Σ_{n+1}^1 if and only if it is $\Sigma_n(\mathrm{HC})$.

We begin with some folklore. A theorem of Mansfield states

(1) Each $\Sigma_1(\mathrm{HC})$ set contains $\leqslant \aleph_1$ or exactly $2^{\aleph_0}$ elements.

A theorem of Morley (slightly reformulated) states

(2) In every logic action, each invariant Σ_2^1 set contains $\leqslant \aleph_1$ or exactly $2^{\aleph_0}$ orbits.

Some logicians at Stanford who knew (1) noticed when they learned of (2) that (2) could be derived from (1). Their argument was roughly as follows.

Let X be a logic space, G a closed subgroup of $\omega!$, J the standard action at G on X, E_G the induced equivalence relation (we write "G-invariant" for "E_G-invariant"). We begin with observation (cf. [20]) that there are relations T_n on ω such that for any $\underline{R}, \underline{S} \in X$, $\underline{R}\, E_G\, \underline{S}$ if and only if $(\omega, \underline{R}, T_n)_{n\in\omega} \simeq (\omega, \underline{S}, T_n)_{n\in\omega}$, *viz*

$$T_n = \{(k_0 \ldots k_{n-1})\colon (\exists g \in G)(\forall i < n)\, g(i) = k_i\}.$$

Now for $\underline{R} \in X$ let $\mathrm{Sc}_G(\underline{R})$ be the Scott sentence of $(\omega, \underline{R}, T_n)_{n\in\omega}$, as defined in [9], Lecture 1. Then $\underline{R}\, E_G\, \underline{S}$ if and only if $\mathrm{Sc}_G(\underline{R}) = \mathrm{Sc}_G(\underline{S})$. Moreover it is immediate from the inductive construction of the Scott sentence that $\{(\underline{R}, \mathrm{Sc}_G(\underline{R}))\colon \underline{R} \in X\}$ is $\Sigma_1(\mathrm{HC})$.

To derive (2) from (1), let $A \subseteq X$ be G-invariant Σ_2^1, hence $\Sigma_1(\mathrm{HC})$. Then $A' = \{\mathrm{Sc}_G(\underline{R})\colon \underline{R} \in A\}$ is $\Sigma_1(\mathrm{HC})$, and the number of G-orbits in A equals the number of elements in A'. This derivation of (2) from (1) constitutes a simplification of the original proof of (2).

It will be instructive to observe that, conversely, (1) follows just from (2) for the canonical logic action. We write $T(x)$ for the transitive closure $\{x\} \cup x \cup \bigcup x \cup \bigcup\bigcup x \cup \ldots$ of $\{x\}$. Thus $\mathrm{HC} = \{x\colon T(x)$ is countable}. If $R \in 2^{\omega\times\omega}$ we say R *codes* x if $(\omega, R) \cong (T(x), \epsilon)$. Let Co be the function $\{(R, x)\colon R \in 2^{\omega\times\omega}, x \in \mathrm{HC}, R \text{ codes } x\}$. The domain of Co, dom Co is the set of $R \in 2^{\omega\times\omega}$ such that (ω, R) is well-founded and a model of the $L_{\omega_1\omega}$ sentence

$$\forall x_0, x_1\big(\forall y\,(y\bar{R}x_0 \leftrightarrow y\bar{R}x_1) \to x_0 = x_1\big) \wedge$$
$$\wedge\ \exists x \forall y\big(y = x \vee y\bar{R}x \vee \exists z(y\bar{R}z \wedge z\bar{R}x) \vee$$
$$\vee\ \exists z_0 z_1(y\bar{R}z_0 \wedge z_0\bar{R}z_1 \wedge z_1\bar{R}x) \ldots\big).$$

It follows that dom Co is an invariant Π^1_1 set, in particular it is $\Delta_1(\mathrm{HC})$. Co itself is $\Sigma_1(\mathrm{HC})$ since $\mathrm{Co}(R) = x$ if and only if $(\exists\pi)\big(\pi\colon (\omega, R) \cong (T(x), \epsilon)\big)$. (So, in fact, like any $\Sigma_1(\mathrm{HC})$ function with a $\Delta_1(\mathrm{HC})$ domain, Co is $\Delta_1(\mathrm{HC})$.) Finally for $R, S \in \mathrm{dom\,Co}$, $\mathrm{Co}(R) = \mathrm{Co}(S)$ if and only if $(\omega, R) \cong (\omega, S)$.

To derive (1) from (2), let $A \subseteq \mathrm{HC}$ be $\Sigma_1(\mathrm{HC})$. Then $A' = \{R \in 2^{\omega\times\omega}\colon \mathrm{Co}(R) \in A\}$ is $\Sigma_1(\mathrm{HC})$, hence Σ^1_2, and $\omega!$-invariant, and the number of elements in A equals the number of $\omega!$-orbits in A'.

Note that in the above argument we have shown that (2) for the canonical logic action implies the full (2). This same remark will apply to 3.1 and 3.2 below.

The same method gives several other results.

THEOREM 3.1. *For any $n \geqslant 1$ the following are equivalent:*

(a) *The class of $\Sigma_n(\mathrm{HC})$ sets has the reduction property.*

(b) *In every logic action, the class of invariant Σ^1_{n+1} sets has the reduction property.*

Proof. To derive (a) from (b) let $A_0, A_1 \subseteq \mathrm{HC}$ be $\Sigma_n(\mathrm{HC})$. For $i = 0, 1$ et $B_i = \{R \in \mathrm{dom\,Co}\colon \mathrm{Co}(R) \in A_i\} \subseteq 2^{\omega\times\omega}$. Then the B_i are $\Sigma_n(\mathrm{HC})$, hence Σ^1_{n+1}, and $\omega!$-invariant. As a consequence of (b) there exist C_0, C_1 reducing B_0, B_1 which are $\omega!$-invariant and Σ^1_{n+1}, hence $\Sigma_n(\mathrm{HC})$. If $D_i = \{\mathrm{Co}(R)\colon R \in C_i\}$ then D_0, D_1 are $\Sigma_n(\mathrm{HC})$ and are easily seen to reduce A_0, A_1 (the invariance of C_0 and C_1 is used to show $D_1 \cap D_2 = \emptyset$).

To derive (b) from (a), let X be a logic space, G a closed subgroup of $\omega!$, and let $B_0, B_1 \subseteq X$ be G-invariant and Σ^1_{n+1}, hence $\Sigma_n(\mathrm{HC})$. Let $A_i = \{\mathrm{Sc}_G(R)\colon R \in B_i\}$, $i = 0, 1$. Then the A_i are $\Sigma_n(\mathrm{HC})$ and assuming (a) there exist $\Sigma_n(\mathrm{HC})$ sets D_0, D_1 reducing A_0, A_1. Let $C_i = \{R\colon \mathrm{Sc}_G(R) \in D_i\}$. Then C_0, C_1 are G-invariant and $\Sigma_n(\mathrm{HC})$ hence Σ^1_{n+1}, and are easily seen to reduce B_0, B_1. ■

Before we knew 3.1, S. Simpson showed us a forcing argument to prove (a) for $n = 1$. Once we have 3.1, however, this is immediate from our 1.3 which proves (b) for $n = 1$ (An earlier, more difficult, proof of (b) for $n = 1$ appears in [23].)

Next we will apply the method to the uniformization principle. Let X_0, X_1 be logic spaces, G a closed subgroup of $\omega!$, J_0, J_1 the standard actions of G on X_0, X_1, and E_0, E_1 the induced equivalence relations. The product equivalence $E_0 \times E_1$ on $X_0 \times X_1$ was defined in Section 1. The equivalence E^V defined in Section 1 is in this case just the equivalence induced by the standard action J^V of G on $X_0 \times X_1$.

THEOREM 3.2. *For any $n \geqslant 1$ the following are equivalent:*

(a) *The class of $\Sigma_n(\mathrm{HC})$ sets satisfies the uniformization principle.*

(b) *For every pair of logic spaces X_0, X_1, and every G, etc., as above, the class of Σ^1_{n+1} subsets of $X_0 \times X_1$ satisfies the $(E_0 \times E_1)$-IUP.*

(c) *For every pair of logic spaces X_0, X_1, and every G, etc., as above, the class of Σ^1_{n+1} subsets of $X_0 \times X_1$ satisfies the E^V-IUP.*

Proof. The equivalence of (a) and (b) is proved much as in 3.1. The equivalence of (b) and (c) follows from 1.7.

A subset A of HC is PR(HC) if there is a function f which is primitive recursive in the sense of [6] and a parameter $w \in \mathrm{HC}$ such that $A = \{x \in \mathrm{HC}\colon f(w, x) = \emptyset\}$. A subset of a logic space is PR(HC) if and only if it is Borel. (Though apparently known to Jensen, this result does not appear in [6].) Vaught's theorem (proved in [23], generalized in [24]) (3) *In a logic action, any invariant Σ^1_2 set is a union of $\aleph_1$ invariant Borel sets* suggests

PROPOSITION 3.4. *Any $\Sigma_1(\mathrm{HC})$ set is a union of $\aleph_1$ PR(HC) sets.*

Proof. The methods of this section do not enable us to derive 3.4 from (3). For if $B \subseteq \mathrm{dom\,Co}$ is Borel, we *cannot* assert that $\{\mathrm{Co}(R)\colon R \in B\}$ is PR(HC). So we proceed as follows.

Let A be $\Sigma_1(\mathrm{HC})$, so $A' = \{R \in \mathrm{dom\,Co}\colon \mathrm{Co}(R) \in A\}$ is $\Sigma_1(\mathrm{HC})$, hence Σ^1_2. By the classical analogue of (3) we can decompose A' as $\bigcup_{\alpha<\omega_1} B'_\alpha$, with the B' Borel. For $\alpha < \omega_1$, $C' = \{R \in \mathrm{dom\,Co}\colon \mathrm{Co}(R) \text{ has rank} < \alpha\}$ is Borel. (Cf. the fact that being a well-founded relation of rank $< \alpha$ is expressible in $L_{\omega_1\omega}$.) Letting $D' = C'_\alpha \cap \bigcup_{\beta<\alpha} B'_\beta$, we obtain a new decomposition $A' = \bigcup_{\alpha<\omega_1} D'_\alpha$ in which the D' are still Borel, hence PR(HC). Now, readers familiar with [6] will see that there is a primitive recursive function f of two variables such that for $R \in C'_\alpha$, $\mathrm{Co}(R) = f(\alpha, R)$. Thus, the sets $D_\alpha = \{\mathrm{Co}(R)\colon R \in D'_\alpha\}$ are PR(HC), and $A = \bigcup_{\alpha<\omega_1} D_\alpha$ affords a decomposition of A as required by 3.4. ■

4. Examples. Let X be a Polish space, H a Polish topological group, K a bicontinuous action of H on X, and F the induced equivalence relation. Then for X, H, K, and F we have

(A) The equivalence relation is Σ^1_1.

(B) The group is a Baire space and has a countable weak basis.

These are precisely the hypotheses of our Section 1 and our Section 2 (and much of [24]) respectively, so all the results of both sections (and many results of [24]) apply in this situation which includes the logic actions.

Now let G be a subgroup of H. Since the closure of G in H is itself a Polish group, we may assume, without loss of generality, that G is dense in H. Let J be the restriction of K to $G\times X$ and let E be the equivalence relation induced by J. We wish to consider which theorems hold for X, G, J, E. The following hypotheses on H will give us (A) and (B) respectively, for X, G, J, E and, hence, will yield the theorems which follow from them.

(A′) G is a Σ_1^1 subgroup of H.

(B′) G is a non-meager subgroup of H.

Note that, since G is dense in H, G is non-meager in itself with the relative topology if and only if it is a non-meager subset of H. So there is no real ambiguity in our statement (B′).

No proper dense subgroup of H satisfies *both* (A′) and (B′), for a theorem of Banach, Kuratowski, and Pettis (cf. [8]) tells us that if a dense subgroup G of H is non-meager and has the Baire property (a consequence of (A′)), then it is closed in H and, hence, simply equals H.

A theorem proved on hypothesis (A) or (B) will have bearing, of course, outside the situation ("subaction of a Polish action") we are considering here, but we feel that many interesting examples are to be sought and many interesting distinctions are to be drawn in this situation.

We begin by presenting examples where (A′) is satisfied but not (B′) and *vice versa*. If $H=\omega!$, then $G=\{G\in\omega!\colon (\exists m)(\forall n)\big(g(2^m(2n+1)) = 2^m(2n+1)\big)\}$ is clearly dense, Σ_1^1 (in fact F_σ) and meager. We can improve this example to get a G which is Σ_1^1 but not Borel as follows.

Let $f\colon sq\to\omega$ be a bijection. For $\xi\in\omega^\omega$ let $T(\xi)=\{f(\xi|n)\colon n\in\omega\}$. Then if $\xi\neq\eta$, $T(\xi)\cap T(\eta)$ is finite. This is the classical construction of $2^{\aleph_0}$ *almost disjoint sets*. For $\xi\in\omega^\omega$ let $e_\xi\colon\omega\to T(\xi)$ enumerate $T(\xi)$ in increasing order. Let $g_\xi\in\omega!$ be defined by $g_\xi\big(e_\xi(2n+1)\big)=e_\xi(2n)$, $g_\xi\big(e(2n)\big)=e_\xi(2n+1)$ and $g(m)=m$ for $m\notin T(\xi)$. Let

$$G_0=\{g\in\omega!\colon (\exists m)(\forall n>m)\,g(n)=n\}\,.$$

For $W\subseteq\omega^\omega$ let G_W be the subgroup of $\omega!$ generated by $G_0\cup\{g_\xi\colon \xi\in W\}$. G_W is always dense since G_0 is.

It is easily seen that G_W is a Σ_1^1 subgroup of $\omega!$ whenever W is a Σ_1^1 subset of ω^ω. We claim that if W is not Borel then neither is G_W. It will suffice to show that W can be recovered from G_W by the equation $W=\{\xi\colon g_\xi\in G_W\}$. Clearly $W\subseteq\{\xi\colon g_\xi\in G_W\}$ so suppose that $g_\xi\in G_W$ but $\xi\notin W$. Since $g_\xi\in G_W$, $g_\xi=g_0g_1\ldots g_k$ where each g_n is either an element of G_0 or g_η for some $\eta\in W$. By choosing m so large that each element of G_0 among $g_0,\ldots,g_k$ fixes $e_\xi(m)$ and that $e_\xi(m)$ is greater than every element of $T(\xi)\cap T(\eta)$ for each η with g_η among $g_0,\ldots,g_k$ we obtain $e_\xi(m)\neq g_\xi\big(e_\xi(m)\big)=g_0g_1\ldots g_k\big(e_\xi(m)\big)=e_\xi(m)$, a contradiction which proves the claim. This also shows that $G_W\subsetneq\omega!$.

Examples satisfying (B′) but not (A′) are well known for $H=(\boldsymbol{R},+)$, the additive group of reals (cf. [8]). For H the non-Abelian group $\omega!$ the problem is a bit more difficult. Let $\mathfrak{U}$ be a non-principle ultrafilter on ω. For $g\in\omega!$ let $S(g)=\{n\colon g(n)=n\}$ and define $G=\{g\in\omega!\colon S(g)\in\mathfrak{U}\}$. G is obviously a proper subgroup and since $\mathfrak{U}$ is non-principle, $G_0\subseteq G$ and G is dense. G. Bergman suggested that G might prove to be non-meager. We will verify that this is true and, hence, that G is not Σ_1^1.

We show that the assumption $G\subseteq\bigcup C_n$ with each C_n closed nowhere dense leads to a contradiction. Let σ, τ with subscripts range over the set of finite permutations and set $U_\sigma=\{g\in\omega!\colon g \text{ extends } \sigma\}$. Then $\{U_\sigma\colon \sigma \text{ a finite permutation}\}$ is a countable weak basis for $\omega!$. We define inductively permutations σ_n and τ_n, $n\in\omega$ such that for each n, domain $\sigma_n=$ domain τ_n, $\sigma_n\subseteq\sigma_{n+1}$, $\tau_n\subseteq\tau_{n+1}$ and $U_{\sigma_{2n+1}}\cap C_n=U_{\tau_{2n+1}}\cap C_n=\emptyset$.

Suppose σ_i, τ_i have been defined for $i<2n$. Since C_n is assumed nowhere dense there exists $\sigma_{2n}\supseteq\sigma_{2n-1}$ such that $U_{\sigma_{2n}}\cap C_n=\emptyset$. Define τ_{2n} to have the same domain as σ_{2n}, to extend τ_{2n-1} and to be the identity on domain σ_{2n} — domain σ_{2n-1}. Let $\tau_{2n+1}\supseteq\tau_{2n}$ be such that $U_{\tau_{2n+1}}\cap C_n=\emptyset$ and define σ_{2n+1} to have the same domain as τ_{2n+1}, to extend σ_{2n}, and to be the identity on domain τ_{2n+1} — domain τ_{2n}. Clearly $g=\bigcup_{n\in\omega}\sigma_n$ and $h=\bigcup_{n\in\omega}\tau_n$ are elements of $\omega!-\bigcup_{n\in\omega}C_n$. By construction $S(g)\cup S(h)=\omega$, so since $\mathfrak{U}$ is an ultrafilter, one of $S(g)$, $S(h)$ belongs to $\mathfrak{U}$, say the former. But then $g\in G-\bigcup_n C_n$, a contradiction.

The ultrafilter $\mathfrak{U}$ on ω can be constructed from a well-ordering of 2^ω. If we have a strong Σ_k^1 well-ordering, then the construction can be performed to provide that G above is a Σ_k^1 subgroup of $\omega!$.

These examples already show us that the hypotheses of several theorems cannot be weakened. Consider the theorem that orbits are Borel. This is proved in [21] for actions by Polish groups, and in [24] assuming that both (A) and (B) hold. Any Polish group H acts on itself by translation. If G is a subgroup of H, the orbit of id $\in H$ in the induced action of G on H is simply G itself. Thus, neither (A′) nor (B′) alone suffices to guarantee that this orbit is Borel, by our examples in which G satisfies (A′) or (B′) alone but is not Borel. We do not know whether orbits must be Borel in an arbitrary subaction of a Polish action where G is a *Borel* subgroup of H.

Again, consider the question of when an invariant set B can be written as a union of $\aleph_1$ invariant Borel sets. In [24] it is shown that this is possible provided B is Σ_1^1 and (B) holds. Our second example above shows that (A′) is not sufficient for this result. The result can be extended to Σ_2^1 on the assumption that (A) holds in addition to (B). Our third example shows that (B′) alone is not sufficient for the extension, since if $V = L$ the non-meager subgroup of $\omega!$ can be taken to be Σ^1_2.

Any Σ^1_2 set A can be represented in the form $A = \bigcup_{\xi\in\omega_1}\bigcap_{n\in\omega} A_{\xi|n}$ where each $A_{\xi|n}$ is Borel. One might hope that assuming (B), A^* could be represented as (3) of § 2, with k_i ranging over ω_1 rather than ω. It would then follow, however, that A, if invariant, would be a union of $\aleph_1$ invariant Borel sets. We have just remarked that (B) is not sufficient. In fact we *are* able to prove this representation theorem (and hence, also the decomposition theorem) if we assume (B) + (Any union of $\aleph_1$ meager sets is meager.).

Now consider the invariant Π^1_1 reduction theorem. We have proved it from (A) alone as 1.1 and from (B) alone as 2.3. However, it is interesting to note that in the case we have considered in this section ("subaction of Polish action") if (B′) holds then the passage from H-invariance to G-invariance introduces no new invariant Π^1_1 sets in view of the following.

PROPOSITION 4.1. *Let G be a dense non-meager subgroup of H. Then any G-invariant Σ^1_1 or Π^1_1 subset B of X is H-invariant.*

Proof. By 1.7 of [24], B is a union of $\aleph_1$ G-invariant Borel sets, so it suffices to prove the proposition assuming that B is Borel. Let $G' = \{g\in H\colon (\forall x)(x\in B \leftrightarrow gx\in B)\}$. This is the largest subgroup with respect to which B is invariant. Since $G \subseteq G'$, G' is dense in H and it is non-meager. Since B is Borel, G' is Π^1_1 and, hence, has the Baire property. By the Banach-Kuratowski-Pettis Theorem, $G' = H$. (A more direct argument based on (0) and (3) of § 2 is also possible.) ■

Assuming PD, which implies that every projective set has the Baire property, 4.1 extends to the case where B is any projective set. However, if $V = L$ our third example shows that 4.1 can fail for Σ^1_2 sets.

The invariant Σ^1_2 reduction theorem was proved from (A) as our 1.3. We do not know whether it follows from (B) or even (B′) alone.

The situation with respect to the good PWO property is rather curious. As remarked just after 2.3 it follows from (B) for Π^1_2. According to our 1.4 it follows from (A) for Σ^1_2. Does it hold for Π^1_1 assuming (A) or for Σ^1_2 assuming (B)? Assuming PD we get good PWO for Σ^1_{2n} ($n \geq 1$) from (A) (our 1.5(b)). We conjecture that it follows also for Π^1_{2n+1} but we have been able to prove only the following partial result.

THEOREM 4.2. *Assume PD. Let X be a Polish space, G a Σ^1_1 subgroup of $\omega!$ acting on X according to a Borel measurable map. Then for $n\geq 1$ the class of Π^1_{2n+1} subsets of X has the good PWO property.*

Proof. Roughly speaking, the proof of 4.2 stands in the same relation to the proof of the non-invariant version that the proof of 1.4 did to its non-invariant analog.

Let C be an invariant Π^1_{2n+1} subset of X, say $C = \{x\colon (\forall \alpha\in\omega^\omega)(x, \alpha)\in D\}$ where $D\subseteq X\times\omega^\omega$ is Σ^1_{2n}. Let $\varphi\colon D\to\varrho$ be a Σ^1_{2n} norm. Fix a bijection $i\colon \omega\times\omega\to\omega$. We say that $\gamma\in\omega^\omega$ *codes* $g\in\omega!$ (and write $g = g_\gamma$) provided $(\forall n, m)\big(g(n) = m \text{ iff } (\exists p)(\gamma(p) = i_{\langle n,m\rangle})\big)$. Let

$$G\text{-codes} = \{\gamma\colon (\exists g\in G)\ \gamma \text{ codes } g\}.$$

Let $D'\subseteq X\times\omega^\omega\times\omega^\omega$ be the Σ^1_{2n} set $\{(x, \gamma, \alpha)\colon \gamma\notin G\text{-codes} \vee (g_\gamma x, \alpha)\in D\}$. Note that, since C is invariant, $C = \{x\colon (\forall\gamma, \alpha)(x, \gamma, \alpha)\in D'\}$. The reader may check that, defining

$$\varphi'(x, \gamma, \alpha) = \begin{cases} 0 & \text{if } \gamma\notin G\text{-codes},\\ \varphi(g_\gamma x, \alpha) & \text{if } \gamma\in G\text{-codes},\end{cases}$$

φ' is a Σ^1_{2n} norm on D'.

Now we are in a position to play the usual game for defining a prewellordering, and hence a norm, on C. Let $x, y\in X$. Consider the infinite game $G(x, y)$ indicated by the following diagram.

I:	$(\alpha_I(0), \gamma_I(0))$		$(\alpha_I(1), \gamma_I(1))$		$\ldots\ \alpha_I,\ \gamma_I$
II:		$(\alpha_{II}(0), \gamma_{II}(0))$		$(\alpha_{II}(1), \gamma_{II}(1))$	$\ldots\ \alpha_{II},\ \gamma_{II}$

I's plays define $(\alpha_I, \gamma_I)\in\omega^\omega\times\omega^\omega$, II's plays define $(\alpha_{II}, \gamma_{II})$. II wins if and only if

$$(y, \gamma_{II}, \alpha_{II})\notin D'$$

or

$$[(y, \gamma_{II}, \alpha_{II})\in D' \wedge (x, \gamma_I, \alpha_I)\in D' \wedge \varphi'(x, \gamma_I, \alpha_I)\leq\varphi'(y, \gamma_{II}, \alpha_{II})].$$

We define $x\leq_\psi y$ provided II has a winning strategy in $G(x, y)$.

Standard arguments (cf. [7] 2c-1) show that $\leq_\psi$ induces a Π^1_{2n+1} norm ψ on C such that $\psi(x)\leq\psi(y)$ if and only if $x\leq_\psi y$. To verify that ψ is a good norm it suffices to show that for every $g\in G$ and every $x\in C$, $x\leq_\psi gx$. We exhibit a winning strategy for II in $G(x, gx)$. Suppose at move k, I plays $\alpha_I(k) = a$, $\gamma_I(k) = b = i_{\langle n,m\rangle}$. Then II should play $\alpha_{II}(k) = a$, $\gamma_{II}(k) = i_{\langle gn, m\rangle}$. With this strategy it is apparent that $\gamma_{II}\in G$-codes if and only if $\gamma_I\in G$-codes, and that if γ_I codes h then γ_{II} codes hg^{-1}. Thus, $\varphi'(x, \gamma_I, \alpha_I) = \varphi'(gx, \gamma_{II}, \alpha_{II})$ and the proof is complete. ■

Morley's theorem, cited as (2) in § 3, occupies an exceptional position in that it is known *only* for logic actions. In [24] Vaught asked whether the following generalization is true.

(1) If X is a Polish space, G a Polish group, J a bicontinuous action of G on X, then the number of orbits in any invariant Σ^1_2 of X is $\leq\aleph_1$ or exactly $2^{\aleph_0}$.

We conjecture the following stronger statement.

(2) If X is a Polish space, E a Σ^1_1 equivalence relation on X, then the number of E-equivalence classes in any invariant Σ^1_2 set is $\leq\aleph_1$ or exactly $2^{\aleph_0}$.

Friedman's 74th problem is to settle the status of the following special case of (2).

(3) If $X = \omega^\omega$, E a Σ^1_1 equivalence on X, then the number of E-equivalence classes in X is $\leqslant \aleph_1$ or $2^{\aleph_0}$.

In fact, (3) is equivalent to (2). For let X be a Polish space, which by the trick used in 1.6 we may suppose to be ω^ω. Let $E \subseteq X^2$ be a Σ^1_1 relation, let $A \subseteq X$ be a Σ^1_2 set, and suppose $E \cap A^2$ is an equivalence relation with $\leqslant \aleph_1$ equivalence classes. Write $A = \bigcup_{\alpha<\omega_1} B_\alpha$ with each B_α Borel. Define Σ^1_1 equivalence relations E_α on all of X by setting

$$E_\alpha = \{(x, y)\colon (x \notin B_\alpha \wedge y \notin B_\alpha) \wedge xEy\}.$$

The number of E-equivalence classes in A is the sum of the number of E_α-equivalence classes in X for $\alpha < \omega_1$. If (3) holds this sum is $\aleph_1$ or $2^{\aleph_0}$.

The strongest conjecture along these lines is that of Martin:

(4) (3) holds with Π^1_2 replacing Σ^1_1.

We do know that the conclusion of Morley's theorem does not follow in ZFC from (B), or even (B′), alone. For assume that $2^{\aleph_0} \geqslant \aleph_3$ and that Martin's Axiom (MA) holds. Let $H = (\boldsymbol{R}, +)$, W a Hamel basis (i.e. a basis for $\boldsymbol{R}$ as a vector space over $\boldsymbol{Q}$), $W_0 \subseteq W$ a subset of power $\aleph_2$. Let G be the subspace of R generated over $\boldsymbol{Q}$ by $W - W_0$, regarded as a subgroup of H. Then $\aleph_2$ translates of G cover H, and since by MA any union of $< 2^{\aleph_0}$ meager sets is meager, G must be non-meager. When G acts on H by translation, the whole space, H, contains exactly $\aleph_2$ orbits.

Some questions raised above have recently been answered. The first author has shown, using a theorem of Silver, that an analytic equivalence relation has $\leqslant \aleph_1$ or $2^{\aleph_0}$ equivalence classes, so that (2) of § 4 holds. Details of these and other developments will appear in his doctoral dissertation, "Infinitary Languages and Descriptive Set Theory", University of California at Berkeley, 1974.

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