# Coordination via Selling Information 

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#### Abstract

We consider games of incomplete information in which the players' payoffs depend both on a privately observed type and an unknown but common "state of nature". External to the game, a data provider knows the state of nature and sells information to the players, thus solving a joint information and mechanism design problem: deciding which information to sell while eliciting the player' types and collecting payments. We restrict ourselves to a general class of symmetric games with quadratic payoffs that includes games of both strategic substitutes (e.g. Cournot competition) and strategic complements (e.g. Bertrand competition, Keynesian beauty contest). By to the Revelation Principle, the sellers' problem reduces to designing a mechanism that truthfully elicits the player' types and sends action recommendations that constitute a Bayes Correlated Equilibrium of the game. We fully characterize the class of all such Gaussian mechanisms - where the joint distribution of actions and private signals is a multivariate normal distribution-as well as the welfare- and revenue- optimal mechanisms within this class. For games of strategic complements, the optimal mechanisms maximally correlate the players' actions, and conversely maximally anticorrelate them for games of strategic substitutes. In both cases, for sufficiently large uncertainty over the players' types, the recommendations are deterministic (and linear) conditional on the state and the type reports, but they are not fully revealing.


## 1 Introduction

How economic agents coordinate their actions in strategic environments is a critical question with numerous examples both within and across organizations:

1. Competing firms wish to tailor their prices or quantities to the level of market demand, but they must also respond to each others' strategies and take their own costs into account.
2. Division managers within the same company want to choose investment projects that balance their own division's profits and the interests of the company as a whole, while taking cross-division synergies into account.

In all these settings, players have common preferences over an unknown state, idiosyncratic preferences over actions, and also a motive for coordination or anti-coordination (depending on the strategic complements vs. substitutes nature of their interaction). As such, all these setting entail similar challenges. In the case of strategic substitutes (e.g., Cournot competition), players would like to positively coordinate with a state but not with each other. Differences

[^0]in idiosyncratic preferences (e.g., cost) can help, while the presence of a common state does not. In the case of strategic complements (e.g., Bertrand competition), players would like to coordinate both with a state and with each other, but differences in cost can prevent full action alignment.

In this paper, we study how information can help achieve more efficient outcomes in setting such as those described above. In particular, we study how to optimally design (and sell) informative signals to the players so as to induce socially preferable actions. To do so, we explore the joint study of information and mechanism design problems for symmetric $n$-player games with quadratic utility functions and Gaussian information. We characterize the set of implementable state-type-action distributions when agents are privately informed about an idiosyncratic preference parameter. We then derive the welfare-maximizing ( $2^{\text {nd }}-$ best) information design.

In our model, a designer observes the unknown state and optimally sells informative to the players. Because the players also have independent private types known only to them, the role of the seller is thus threefold: (i) to elicit the private types of the players; (ii) to design signals about the unknown state that induce a "desirable" equilibrium of the game; and (iii) to collect payments. We leverage our model to answer questions such as the following: (a) What is the structure of the optimal mechanisms? (b) How is it impacted by the properties of the downstream game? (c) How do the type-elicitation stage and the action-recommendation stage interact?

Our main contribution is to provide a complete characterization of implementable joint distributions of state, types, and actions. Our characterization shows that each player's choice of report interacts with their choice of action in the ensuing game. Specifically, there are mechanisms for which double deviations are profitable for the players, even when the mechanism is separately truthful and obedient. Furthermore, we characterize the optimal double deviations, and we provide necessary and sufficient conditions on the joint distribution of state, types, and actions that make a mechanism (globally) incentive compatible.

As such, our problem does not reduce to simply "selling a Bayes Correlated Equilibrium (BCE) truthfully." This problem would roughly correspond to picking a BCE of the game with commonly known types and associating it to the players' reports. In such a game, after misreporting their type, a player will typically want to deviate from the action recommendation as well. In other words, the type elicitation and the action recommendation problems cannot be studied separately.

A byproduct of our characterization that is of independent interest is an analysis of the welfare-maximizing mechanism with transfers. ${ }^{1}$ In the optimal mechanism, the designer induces different degrees of correlation in the players' actions depending on the strategic complements vs. substitutes nature of the downstream game: maximum positive correlation for strategic complements and maximum anticorrelation for strategic substitutes. When the prior variance of the players' types is large enough (and for all parameter values if the game has strategic complements), the recommendations are a linear combination of the fundamentals. This means that, each player only learns a linear combination of the state and of other player's types. This allows for obedient coordination without revealing full information to the players.

The welfare-optimal action distribution differs from the complete information game in a systematic way. In particular, both in games of strategic complements and strategic substitutes, the welfare-optimal design places more weight on the players' types relative to the common state. This is intuitive, because relying on the private types enables greater co-

[^1]ordination with strategic complements, and greater anti-coordination in a game of strategic substitutes (e.g., in Cournot, it exacerbates the differences in the players' equilibrium actions).

Related Literature. Our paper is most closely related to the literature on information design in games, directly building on Bergemann and Morris (2013) and Mathevet et al. (2020). See also Bergemann and Morris (2019) and Kamenica (2019) for surveys. It also contributes to the literature on mechanism design with externalities (see Jehiel and Moldovanu (2006) for an overview).

The presence of private information and the focus on information design are the key differences between our setting and the literature on the social value of information (Angeletos and Pavan, 2007) and on the tradeoff between adaptation and coordination in multi-division organizations (Alonso et al., 2008; Rantakari, 2008).

Viewed as a model of selling information, our paper adds competing buyers to the mechanism design approach of Babaioff et al. (2012); Bergemann et al. (2018); Liu et al. (2021), privately known types to the approach of Bimpikis et al. (2019), and a coordination motive to the setting of Agarwal et al. (2020). Relative to Rodríguez Olivera (2021); Bonatti et al. (2022), the present model extends beyond dominant-strategy games with binary states and actions, but it restricts attention to linear-quadratic-Gaussian settings. Finally, the value of (complete and voluntary) information sharing in imperfectly competitive markets is exhaustively studied in Raith (1996). In our setting, beyond introducing private types, information sharing is partial and mediated by a (benevolent) designer.

Notation. For a vector space $V$ and a subset of vectors $S \subseteq V$, $\operatorname{span}(S)$ denotes the linear span of $S$. For $n, m \geq 1, \mathcal{M}_{n, m}(\mathbb{R})$ denote the vector space of $n \times m$ matrices with real entries. For convenience, we write $\mathcal{M}_{n}(\mathbb{R})$ when $m=n$ and implicitly identify $\mathcal{M}_{n, 1}(\mathbb{R})$ with $\mathbb{R}^{n}$. The identity and all-ones matrix of $\mathcal{M}_{n}(\mathbb{R})$ are denoted respectively by $I_{n}$ and $J_{n}$. Finally $1_{n}$ denotes the all-ones vector in $\mathbb{R}^{n}$ and we write $[n]:=\{1, \ldots, n\}$. Finally $\mathcal{S}_{n}^{+}(\mathbb{R})$ denotes the cone of positive semidefinite matrices.

Unless stated otherwise, all random variables in this paper are assumed to be defined on the same sample space $(\Omega, \mathcal{F}, \mathbb{P})$. For random variables $X \in \mathbb{R}$ and $Y \in \mathbb{R}, \operatorname{Cov}(X, Y):=$ $\mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])]$ denotes the covariance between $X$ and $Y$ and $\operatorname{Var}(X):=\operatorname{Cov}(X, X)$ is the variance of $X$. We also alternatively write $\sigma_{X}^{2}$ for $\operatorname{Var}(X)$ and $\sigma_{X Y}$ for $\operatorname{Cov}(X, Y)$. By extension, for random vectors $X \in \mathbb{R}^{n}$ and $Y \in \mathbb{R}^{m}, \operatorname{Cov}(X, Y):=\mathbb{E}\left[(X-\mathbb{E}[X])(Y-E[Y])^{\top}\right]$ denotes the cross-covariance matrix of $X$ and $Y$, that is, the matrix in $\mathcal{M}_{n, m}(\mathbb{R})$ whose entry $(i, j)$ is given by $\operatorname{Cov}\left(X_{i}, Y_{j}\right)$. Finally, $\operatorname{Var}(X)=\operatorname{Cov}(X, X)$ denotes the covariance matrix of $X \in \mathbb{R}^{n}$.

## 2 Model

### 2.1 Basic game

Actions and Payoffs We consider $n$ players who compete in a game of incomplete information. In this game, each player $i \in[n]$ has a private-value type $\theta_{i}$ and faces an unknown (common) payoff-relevant state $\omega$. We write $u_{i}\left(a ; \omega, \theta_{i}\right)$ for the payoff of player $i$ given action profile $a \in \mathbb{R}^{n}$, state $\omega \in \mathbb{R}$ and type $\theta_{i} \in \mathbb{R}$.

We restrict ourselves to symmetric games with quadratic payoffs. In line with the literature on information design in quadratic games (e.g., Bergemann and Morris (2013)), we assume
that each player $i$ 's best response to $a_{-i}$ given $\omega$ and $\theta_{i}$ is given by the linear function

$$
\begin{equation*}
a_{i}=r \sum_{j \neq i} a_{j}+s \omega+t \theta_{i} . \tag{1}
\end{equation*}
$$

A critical parameter for our analysis is the coefficient $r$, whose sign determines whether actions are strategic complements or substitutes. For our welfare calculations, we assume that the best response function (1) for each player $i$ is generated by the following utility function:

$$
\begin{equation*}
u_{i}\left(a ; \omega, \theta_{i}\right)=-\frac{1}{2} a_{i}^{2}+r a_{i} \sum_{j \neq i} a_{j}+\left(s \omega+t \theta_{i}\right) a_{i} . \tag{2}
\end{equation*}
$$

We now provide three classic examples of this framework.
Example 2.1 (Cournot Competition). Firms produce goods that are (partial) substitutes. Let $q_{i}$ denote the quantity of good $i$ produced by firm $i$. Assuming a linear demand curve with symmetric substitution patterns, the inverse demand curve of good $i$ can be written as $P_{i}(q)=\omega+r \sum_{j \neq i} q_{j}-q_{i} / 2$, with $r<0$. Finally, with a marginal production cost of $\theta_{i}$, the profit of firm $i$ can be written as

$$
u_{i}(q)=q_{i} P_{i}(q)-\theta_{i} q_{i}
$$

yielding the best response $q_{i}=\omega+r \sum_{j \neq i} q_{j}-\theta_{i}$, which is of the form (1) with $s=1$ and $t=-1$.
Example 2.2 (Bertrand competition). Consider a version of the previous example with differentiated products Bertrand competition. Let $p_{i}$ the price charged by firm $i$. The demand curve of good $i$ can be written as $Q_{i}(p)=\omega+r \sum_{j \neq i} p_{j}-p_{i} / 2$, with $r>0$. Finally, with a marginal production cost of $\theta_{i}$, the profit of firm $i$ can be written as

$$
u_{i}(p)=\left(p_{i}-\theta_{i}\right) Q_{i}(p),
$$

yielding the best response $p_{i}=\omega+r \sum_{j \neq i} p_{j}+\theta_{i} / 2$, which is of the form (1) with $s=1$ and $t=1 / 2$.

Example 2.3 (Beauty Contest). Each player wishes to adapt their action to both the common state and their idiosyncratic type, and to coordinate with the other players' average action. Thus, each player minimizes the following quadratic loss function:

$$
u_{i}(a)=-\left(a_{i}-\omega\right)^{2}-\left(a_{i}-\Sigma_{j \neq i} a_{j} /(n-1)\right)^{2}-\left(a_{i}-\theta_{i}\right)^{2} .
$$

Any quadratic terms not containing $a_{i}$ are irrelevant for player $i$ 's actions. Indeed, the bestresponse function of player $i$ is given by

$$
a_{i}=(1 / 3)\left(\omega+\theta_{i}+\Sigma_{j \neq i} a_{j} /(n-1)\right),
$$

which is of the form (1) with $s=t=1 / 3$ and $r=1 /(3(n-1))$.
Complete Information Benchmark. Under complete information about the state and all types, our game admits a unique Nash equilibrium. Collecting the best responses (1) for $i \in[n]$ yields the linear system

$$
J_{n}(1,-r) a=s \omega 1_{n}+t \theta
$$

where $\theta=\left(\theta_{1}, \ldots, \theta_{n}\right) \in \mathbb{R}^{n}$ is the vector of types and $J_{n}(1,-r):=(1+r) I_{n}-r J_{n}$ is the $n \times n$ matrix with 1 on the diagonal and $-r$ off the diagonal. This matrix is invertible whenever $r \notin\left\{-1, \frac{1}{n-1}\right\}$ (see Proposition C.7), in which case the solution to the linear system is the unique Nash equilibrium of the complete information game

$$
\begin{equation*}
a_{i}=\frac{s \omega+t \theta_{i}}{1-(n-1) r}+\frac{r \cdot t \sum_{j \neq i}\left(\theta_{j}-\theta_{i}\right)}{(1+r)(1-(n-1) r)} . \tag{3}
\end{equation*}
$$

An important property of our game is that the linear combination

$$
s \omega+\frac{r t \Sigma_{j \neq i} \theta_{j}}{1+r}
$$

is a sufficient statistic for each player to choose the complete-information Nash equilibrium action. In the welfare-optimal mechanism (Propositions 4.2 and 4.4), we show that the designer reveals a different linear combination of the state and the other players' types.

### 2.2 Information Structure and Mechanism Design

We assume that the vector $(\theta, \omega)$ is drawn from an independent Gaussian prior distribution. Each player $i$ observes her type $\theta_{i}$, while the designer observes the state of nature $\omega$.

By the revelation principle for dynamic games (Myerson, 1991), it is without loss to assume that the designer recommends action $a_{i}$ to player $i$ given the observed state $\omega$ and the vector of type reports $\theta$ as long as the resulting mechanism is incentive compatible. Thus, a mechanism consists of

- a function $\tau: \mathbb{R}^{n} \times \mathbb{R} \rightarrow \Delta\left(\mathbb{R}^{n}\right)$ that maps the type reports $\theta \in \mathbb{R}^{n}$ and state of nature $\omega \in \mathbb{R}$ to a distribution $\tau(\omega, \theta)$ over actions profiles.
- for each player $i \in[n]$, a function $p_{i}: \mathbb{R} \rightarrow \mathbb{R}$ mapping their reported type to the payment they are being charged in exchange for the recommendation.

Note that the function $\tau$ induces the conditional distribution $a \mid \theta, \omega$. So designing $\tau$ is equivalent to designing the joint distribution of $(a, \theta, \omega)$, subject to the constraint that the induced marginal distribution of $(\theta, \omega)$ is the one given by the prior.

Gaussian mechanisms. Throughout this paper, we restrict ourselves to the mechanisms where the conditional distribution is normal, or equivalently, the joint distribution $(a, \theta, \omega)$ is multivariate normal. This distribution is characterized by the mean vector $\mu=\mathbb{E}[a, \theta, \omega] \in$ $\mathbb{R}^{2 n+1}$ and covariance matrix $\mathrm{K}=\operatorname{Var}(a, \theta, \omega) \in \mathcal{M}_{2 n+1}(\mathbb{R})$. Note that $\mu$ and K have the following block structure

$$
\mu=\left[\begin{array}{l}
\mu_{a}  \tag{4}\\
\mu_{\theta} \\
\mu_{\omega}
\end{array}\right] \quad \text { and } \quad \mathrm{K}=\left[\begin{array}{ccc}
\mathrm{K}_{a a} & \mathrm{~K}_{a \theta} & \mathrm{~K}_{a \omega} \\
\mathrm{~K}_{a \theta} & \mathrm{~K}_{\theta \theta} & 0 \\
\mathrm{~K}_{a \omega}^{\top} & 0 & \sigma_{\omega}^{2}
\end{array}\right] .
$$

The means $\mu_{\theta}:=\mathbb{E}[\theta]=\left(\mu_{\theta_{i}}\right)_{i \in[n]}$ and $\mu_{\omega}:=\mathbb{E}[\omega] \in \mathbb{R}$ are given by the prior distribution and the vector $\mu_{a}:=\mathbb{E}[a]=\left(\mu_{a_{i}}\right)_{i \in[n]}$ is chosen by the information seller. Similarly, $\mathrm{K}_{\theta \theta}=\operatorname{Var}(\theta)=\operatorname{diag}\left(\sigma_{\theta_{1}}^{2}, \ldots, \sigma_{\theta_{n}}^{2}\right)$ and $\sigma_{\omega}^{2}$ are given by the prior distribution, whereas $\mathrm{K}_{a a}:=\operatorname{Var}(a) \in \mathcal{M}_{n}(\mathbb{R}), \mathrm{K}_{a \theta}:=\operatorname{Cov}(a, \theta) \in \mathcal{M}_{n}(\mathbb{R})$ and $\mathrm{K}_{a \omega}:=\operatorname{Cov}(a, \omega) \in \mathbb{R}^{n}$ are chosen by the designer.

A standard property of multivariate normals is that their conditional expectations are linear, so we can equivalently write the action recommendations in a Gaussian mechanism as a linear combination of the fundamentals $(\theta, \omega)$ with zero-mean (but possibly correlated) noise added:

$$
\begin{equation*}
a_{i}=\alpha_{i}+\beta_{i}\left(\omega-\mu_{\omega}\right)+\sum_{j \in[n]} \gamma_{i j}\left(\theta_{j}-\mu_{\theta_{j}}\right)+\varepsilon_{i}, \tag{5}
\end{equation*}
$$

for all $i \in[n]$ and where $\varepsilon=\left(\varepsilon_{i}\right)_{i \in[n]}$ is a zero-mean multivariate normal $\mathcal{N}\left(0, K_{\varepsilon}\right)$ independent of $(\theta, \omega)$. Writing $\alpha=\left(\alpha_{i}\right)_{i \in[n]}, \beta=\left(\beta_{i}\right)_{i \in[n]}$ and $\Gamma=\left(\gamma_{i j}\right)_{(i, j) \in[n]^{2}}$, we have

$$
\mu_{a}=\alpha, \quad \mathrm{K}_{a \omega}=\sigma_{\omega}^{2} \beta, \quad \mathrm{~K}_{a \theta}=\Gamma \mathrm{K}_{\theta \theta}, \quad \mathrm{K}_{a a}=\sigma_{\omega}^{2} \beta \beta^{\top}+\Gamma \Gamma^{\top} \mathrm{K}_{\theta \theta}+\mathrm{K}_{\varepsilon} .
$$

## 3 Characterizations

### 3.1 Symmetry and positive semi-definiteness

Because the quadratic game introduced in Section 2.1 is symmetric, we will be able to show in Section 4.1 that we can restrict ourselves to symmetric mechanisms without loss of generality.

Let $\mathfrak{S}_{n}$ be the symmetric group on $[n]$ and for each permutation $\pi \in \mathfrak{S}_{n}$, denote by $P_{\pi} \in \mathcal{M}_{n}(\mathbb{R})$ the permutation matrix whose entry $(i, j) \in[n]^{2}$ is $\left(P_{\pi}\right)_{i, j}=\mathbf{1}\{i=\pi(j)\}$. In particular, for $x \in \mathbb{R}^{n}, P_{\pi} x$ is the permuted vector whose $i$ th coordinate is $\left(P_{\pi} x\right)_{i}=x_{\pi^{-1}(i)}$ for $i \in[n]$.

Definition 3.1. A mechanism is symmetric if $(a, \theta, \omega)$ and $\left(P_{\pi} a, P_{\pi} \theta, \omega\right)$ are identically distributed for each permutation $\pi \in \mathfrak{S}_{n}$.

The covariance matrix of a symmetric mechanism has a simple structure presented next.
Lemma 3.2. Let $\mu \in \mathbb{R}^{2 n+1}$ and $\mathrm{K} \in \mathcal{M}_{2 n+1}(\mathbb{R})$ be respectively the mean vector and covariance matrix of a Gaussian mechanism, with the block structure indicated in (4). Then the mechanism is symmetric iff $\mu_{a}, \mu_{\theta}, \mathrm{K}_{a, \omega} \in \mathbb{R} 1_{n}$ and $\mathrm{K}_{a a}, \mathrm{~K}_{a \theta} \in \mathbb{R} I_{n}+\mathbb{R} J_{n}$ and $\mathrm{K}_{\theta \theta} \in \mathbb{R} I_{n}$.

Proof. For a permutation $\pi \in \mathfrak{S}_{n}$, let us denote by $\mu^{\pi}$ and $\Sigma^{\pi}$ the mean vector and covariance matrix of $\left(P_{\pi} a, P_{\pi} \theta, \omega\right)$

$$
\mu^{\pi}=\left[\begin{array}{c}
P_{\pi} \mu_{a}  \tag{6}\\
P_{\pi} \mu_{\theta} \\
\mu_{\omega}
\end{array}\right] \quad \text { and } \quad \mathrm{K}^{\pi}=\left[\begin{array}{ccc}
P_{\pi} \mathrm{K}_{a a} P_{\pi}^{\top} & P_{\pi} \mathrm{K}_{a \theta} P_{\pi}^{\top} & P_{\pi} \mathrm{K}_{a \omega} \\
P_{\pi} \mathrm{K}_{a \theta} P_{\pi}^{\top} & P_{\pi} \mathrm{K}_{\theta \theta} P_{\pi}^{\top} & 0 \\
\mathrm{~K}_{a \omega}^{\top} P_{\pi}^{\top} & 0 & \sigma_{\omega}^{2}
\end{array}\right] .
$$

Because the mechanism is Gaussian, it is fully determined by its means and covariance matrix, hence the mechanism is symmetric iff $\mathrm{K}^{\pi}=\mathrm{K}$ and $\mu^{\pi}=\mu$ for all $\pi \in \mathfrak{S}_{n}$. From Proposition C.7, we immediately obtain that $\mu_{a}, \mu_{\theta}, \mathrm{K}_{a \omega} \in \operatorname{span}\left(1_{n}\right)$ and $\mathrm{K}_{a a}, \mathrm{~K}_{a \theta}, \mathrm{~K}_{\theta \theta} \in \operatorname{span}\left(I_{n}, J_{n}\right)$. Because we assumed that the prior on $\theta$ is independent, $\mathrm{K}_{\theta \theta}$ is diagonal, implying that $\mathrm{K}_{\theta \theta} \in \operatorname{span}\left(I_{n}\right)$.

In other words, symmetry reduces the number of parameters required to specify a mechanism from $2 n(n+1)$ to only 6 : any of the coordinates of $\mu_{a}$ and $\mathrm{K}_{a \omega}$, and the on- and off-diagonal entries of $\mathrm{K}_{a a}$ and $\mathrm{K}_{a \theta}$. Crucially, this number is independent of $n$. In what follows, we choose an arbitrary $i \in[n]$ and $j \neq i$ and write these parameters as $\mu_{a_{i}}, \sigma_{a_{i} \omega}, \sigma_{a_{i}}^{2}$, $\sigma_{a_{i} a_{j}}, \sigma_{a_{i} \theta_{i}}$ and $\sigma_{a_{i} \theta_{j}}$. Since the covariance matrix K is required to be positive semi-definite, this implies constraints on the covariance parameters as stated in the next lemma proved in Appendix A.

Lemma 3.3. The covariance matrix K of a symmetric mechanism is positive semidefinite iff

1. $\sigma_{a_{i} \theta_{i}}=\sigma_{a_{i} \theta_{j}}=0$ whenever $\sigma_{\theta_{i}}^{2}=0$, and $\sigma_{a_{i} \omega}=0$ whenever $\sigma_{\omega}^{2}=0$.
2. The following inequality constraints hold with the convention $0 / 0=0$.

$$
\left\{\begin{array}{l}
\frac{1}{\sigma_{\theta_{i}}^{2}}\left(\sigma_{a_{i} \theta_{i}}-\sigma_{a_{i} \theta_{j}}\right)^{2} \leq \sigma_{a_{i}}^{2}-\sigma_{a_{i} a_{j}} \\
\frac{1}{\sigma_{\theta_{i}}^{2}}\left(\sigma_{a_{i} \theta_{i}}+(n-1) \sigma_{a_{i} \theta_{j}}\right)^{2}+\frac{n}{\sigma_{\omega}^{2}} \sigma_{a_{i} \omega}^{2} \leq \sigma_{a_{i}}^{2}+(n-1) \sigma_{a_{i} a_{j}}
\end{array}\right.
$$

Furthermore, there is equality in the first inequality iff $\operatorname{Cov}\left(a_{i}, a_{j} \mid \theta, \omega\right)=\operatorname{Var}\left(a_{i} \mid \theta, \omega\right)$ and in the second inequality iff $\operatorname{Cov}\left(a_{i}, a_{j} \mid \theta, \omega\right)=-\operatorname{Var}\left(a_{i} \mid \theta, \omega\right) /(n-1)$.

Remark. The case where equality holds in both inequalities is thus equivalent to

$$
-\frac{\operatorname{Var}\left(a_{i} \mid \theta, \omega\right)}{n-1}=\operatorname{Cov}\left(a_{i}, a_{j} \mid \theta, \omega\right)=\operatorname{Var}\left(a_{i} \mid \theta, \omega\right)
$$

or equivalently, $\operatorname{Var}\left(a_{i} \mid \theta, \omega\right)=0$. In this case, $a_{i}=\mathbb{E}\left[a_{i} \mid \theta, \omega\right]$ a.s., (that is, $a_{i}$ is a determin-istic-and affine - function of $\theta, \omega$ ).

### 3.2 Obedience

Recall that a mechanism is obedient if following the recommended action is a best response for each player, conditioned on their type and their recommendation. In other words, the condition

$$
a_{i} \in \underset{a_{i}^{\prime} \in \mathbb{R}}{\arg \max } \mathbb{E}\left[u_{i}\left(a_{i}^{\prime}, a_{-i} ; \theta_{i}, \omega\right) \mid a_{i}, \theta_{i}\right] .
$$

must hold almost surely for each player $i \in[n]$.
Lemma 3.4. Assume that $\left(u_{i}\right)_{i \in[n]}$ defines a symmetric game. If $(a, \theta, \omega)$ is an obedient mechanism, then for all permutation $\pi \in \mathfrak{S}_{n}$, the permuted mechanism $\left(P_{\pi} a, P_{\pi} \theta, \omega\right)$ is also obedient.

Proof. Consider a permutation $\pi \in \mathfrak{S}_{n}$, and an obedient mechanism $(a, \theta, \omega)$ : for all $i \in[n]$, and $a^{\prime} \in \mathbb{R}$

$$
\mathbb{E}\left[u_{i}\left(a_{i}, a_{-i} ; \theta_{i}, \omega\right) \mid a_{i}, \theta_{i}\right] \geq \mathbb{E}\left[u_{i}\left(a^{\prime}, a_{-i} ; \theta_{i}, \omega\right) \mid a_{i}, \theta_{i}\right]
$$

By symmetry of the game, the previous inequality is equivalent to

$$
\mathbb{E}\left[u_{\pi(i)}\left(a_{j}, a_{-j} ; \theta_{j}, \omega\right) \mid a_{j}, \theta_{j}\right] \geq \mathbb{E}\left[u_{\pi(i)}\left(a_{j}^{\prime}, a_{-j} ; \theta_{j}, \omega\right) \mid a_{j}, \theta_{j}\right]
$$

where $j=\pi^{-1}(i)$. This inequality is exactly the obedience constraint of player $\pi(i)$ for the permuted mechanism $\left(P_{\pi} a, P_{\pi} \theta, \omega\right)$. Since this is true for all $i$, we conclude that the permuted mechanism is also obedient.

The following proposition gives a characterization of obedience, showing in particular that the recommendations' means are determined by obedience.

Proposition 3.5. Assume that $r \notin\left\{-1, \frac{1}{n-1}\right\}$, then the mechanism $(\mu, \mathrm{K})$ is obedient iff

1. The mean action $\mu_{a_{i}}$ of each player $i \in[n]$ is determined by the prior's mean:

$$
\begin{equation*}
\mu_{a_{i}}=\frac{s \mu_{\omega}+t \mu_{\theta_{i}}}{1-(n-1) r}+\frac{r \cdot t \sum_{j \neq i}\left(\mu_{\theta_{j}}-\mu_{\theta_{i}}\right)}{(1+r)(1-(n-1) r)} . \tag{7}
\end{equation*}
$$

2. The covariance matrix K satisfies the following linear constraints for each $i \in[n]$

$$
\left\{\begin{array}{l}
\sigma_{a_{i}}^{2}=r \sum_{j \neq i} \sigma_{a_{i} a_{j}}+s \sigma_{a_{i} \omega}+t \sigma_{a_{i} \theta_{i}}  \tag{8}\\
\sigma_{a_{i} \theta_{i}}=r \sum_{j \neq i} \sigma_{a_{j} \theta_{i}}+t \sigma_{\theta_{i}}^{2}
\end{array} .\right.
$$

Proof. Observe that the conditional expected utility when deviating from $a_{i}$ to $a_{i}^{\prime}$,

$$
\mathbb{E}\left[u_{i}\left(a_{i}^{\prime}, a_{-i} ; \omega, \theta_{i}\right) \mid a_{i}, \theta_{i}\right]=-\frac{1}{2}\left(a_{i}^{\prime}\right)^{2}+r a_{i}^{\prime} \sum_{j \neq i} \mathbb{E}\left[a_{j} \mid a_{i}, \theta_{i}\right]+\left(s \mathbb{E}\left[\omega \mid a_{i}, \theta_{i}\right]+t \theta_{i}\right) a_{i}^{\prime},
$$

is concave in $a_{i}^{\prime}$. Obedience of player $i \in[n]$ is thus equivalent to the first-order condition

$$
\begin{equation*}
a_{i}=r \sum_{j \neq i} \mathbb{E}\left[a_{j} \mid a_{i}, \theta_{i}\right]+s \mathbb{E}\left[\omega \mid a_{i}, \theta_{i}\right]+t \theta_{i} . \tag{9}
\end{equation*}
$$

Because the mechanism is Gaussian, the random variable on the right-hand side of (9) is normal and $\left(a_{i}, \theta_{i}\right)$-measurable. It is thus fully determined by its mean and its covariances with $a_{i}$ and $\theta_{i}$. Consequently (9) is equivalent to

$$
\left\{\begin{array}{l}
\mu_{a_{i}}=r \sum_{j \neq i} \mu_{a_{j}}+s \mu_{\omega}+t \mu_{\theta_{i}}  \tag{10}\\
\sigma_{a_{i}}^{2}=r \sum_{j \neq i} \sigma_{a_{i} a_{j}}+s \sigma_{a_{i} \omega}+t \sigma_{a_{i} \theta_{i}} \\
\sigma_{a_{i} \theta_{i}}=r \sum_{j \neq i} \sigma_{a_{j} \theta_{i}}+t \sigma_{\theta_{i}}^{2}
\end{array},\right.
$$

where the first equation expresses the equality of means in (9), and the second (resp. third) equation expresses the equality of the covariance with $a_{i}$ (resp. $\theta_{i}$ ) in (9). To compute the covariances, we used that $\operatorname{Cov}(\mathbb{E}[X \mid Y], Z)=\operatorname{Cov}(X, Z)$ for random variables $(X, Y, Z)$ such that $Z$ is $Y$-measurable.

Writing the first equation in (10) for each player $i \in[n]$ gives the following linear system in $\mu_{a}$

$$
J_{n}(1,-r) \mu_{a}=s \mu_{\omega} 1_{n}+t \mu_{\theta} .
$$

Solving this system using Proposition C. 7 yields the first half of the obedience characterization. The second and third equations in (10) give the second half of the characterization.

Remark 3.6. For symmetric mechanisms, expressions (7) and (8) are the same for each player $i \in[n]$ and simplify to

$$
\mu_{a_{i}}=\frac{s \mu_{\omega}+t \mu_{\theta_{i}}}{1-(n-1) r} \quad \text { and } \quad\left\{\begin{array}{l}
\sigma_{a_{i}}^{2}=(n-1) r \sigma_{a_{i} a_{j}}+s \sigma_{a_{i} \omega}+t \sigma_{a_{i} \theta_{i}} \\
\sigma_{a_{i} \theta_{i}}=(n-1) r \sigma_{a_{i} \theta_{j}}+t \sigma_{\theta_{i}}^{2}
\end{array} .\right.
$$

Example 3.7. In (3), we computed the Nash equilibrium of the complete information game:

$$
a_{i}=\frac{s \omega+t \theta_{i}}{1-(n-1) r}+\frac{r \cdot t \sum_{j \neq i}\left(\theta_{j}-\theta_{i}\right)}{(1+r)(1-(n-1) r)} .
$$

The mechanism that recommends the actions from the Nash equilibrium for each realization of $(\theta, \omega)$ is trivially obedient: an ex-post best-response is a best-response at the interim stage. We can indeed verify from the expression obtained in (3) that it satisfies Proposition 3.5 as expected.
Example 3.8. The Bayes-Nash equilibrium of the game where each player only observes their own type $\theta_{i}$ corresponds to the specific case of an obedient mechanism for which $a_{i}$ is $\theta_{i^{-}}$ measurable. This implies in particular, $\sigma_{a_{i} \omega}=\sigma_{a_{i} a_{j}}=\sigma_{a_{i} \theta_{j}}=0$. In this case, the second covariance obedience constraint implies $\sigma_{a_{i} \theta_{i}}=t \sigma_{\theta_{i}}^{2}$ and since $\sigma_{a_{i}}^{2}=t \sigma_{a_{i} \theta_{i}}$ by the first covariance constraint, this shows that $a_{i}$ is deterministic given $\theta_{i}$ with mean $\mu_{a_{i}}$ given by (7) and

$$
a_{i}=\mu_{a_{i}}+t\left(\theta_{i}-\mu_{\theta_{i}}\right) .
$$

### 3.3 Truthfulness

As a first step towards understanding incentive compatibility, we focus in this section on characterizing truthful mechanisms among obedient mechanisms. To this end, we introduce

$$
\tilde{v}_{i}\left(\theta_{i}^{\prime} ; \theta_{i}\right)=\mathbb{E}\left[u_{i}\left(a ; \theta_{i}, \omega\right) \mid \theta_{i}, \theta_{i}^{\prime}\right]
$$

the expected interim utility of player $i$ when their true cost is $\theta_{i}$, they report cost $\theta_{i}^{\prime}$ and follow the recommendation $a_{i}$ at the second stage of the game. Truthfulness then requires that for all $\theta_{i}^{\prime} \in \mathbb{R}$,

$$
\tilde{v}_{i}\left(\theta_{i} ; \theta_{i}\right)-p_{i}\left(\theta_{i}\right) \geq \tilde{v}_{i}\left(\theta_{i}^{\prime} ; \theta_{i}\right)-p_{i}\left(\theta_{i}^{\prime}\right) .
$$

Proposition 3.9. An obedient mechanism $(\mu, \mathrm{K}, p)$ is truthful iff for each player $i \in[n]$ :

1. The derivative of the payment function is given by

$$
p_{i}^{\prime}\left(\theta_{i}\right)=\left(\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}-t\right) \mathbb{E}\left[a_{i} \mid \theta_{i}\right]=\left(\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}-t\right)\left(\mu_{a_{i}}+\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}\left(\theta_{i}-\mu_{\theta_{i}}\right)\right)
$$

2. The following covariance constraint holds: $t \sigma_{a_{i} \theta_{i}} \geq 0$.

When $\sigma_{\theta_{i}}^{2}=0$, the previous two conditions reduce to $p_{i}^{\prime}\left(\theta_{i}\right)=-t \mu_{a_{i}}$.
Proof. Let us first compute $\tilde{v}_{i}\left(\theta_{i}^{\prime} ; \theta_{i}\right)$ the expected utility of player $i$ when their true cost is $\theta_{i}$, their reported cost is $\theta_{i}^{\prime}$, and assuming they follow the recommendation at the second stage:

$$
\begin{aligned}
\tilde{v}_{i}\left(\theta_{i}^{\prime} ; \theta_{i}\right) & =\mathbb{E}\left[u_{i}\left(a ; \theta_{i}, \omega\right) \mid \theta_{i}, \theta_{i}^{\prime}\right]=\mathbb{E}\left[\mathbb{E}\left[u_{i}\left(a ; \theta_{i}, \omega\right) \mid a_{i}, \theta_{i}, \theta_{i}^{\prime}\right] \mid \theta_{i}, \theta_{i}^{\prime}\right] \\
& =\mathbb{E}\left[-a_{i}^{2} / 2+t a_{i} \theta_{i} \mid \theta_{i}, \theta_{i}^{\prime}\right]+\mathbb{E}\left[a_{i}\left(s \mathbb{E}\left[\omega \mid a_{i}, \theta_{i}, \theta_{i}^{\prime}\right]+r \sum_{j \neq i} \mathbb{E}\left[a_{j} \mid a_{i}, \theta_{i}, \theta_{i}^{\prime}\right]\right) \mid \theta_{i}, \theta_{i}^{\prime}\right] \\
& =\mathbb{E}\left[-a_{i}^{2} / 2+t \theta_{i} a_{i} \mid \theta_{i}, \theta_{i}^{\prime}\right]+\mathbb{E}\left[a_{i}\left(s \mathbb{E}\left[\omega \mid a_{i}, \theta_{i}^{\prime}\right]+r \sum_{j \neq i} \mathbb{E}\left[a_{j} \mid a_{i}, \theta_{i}^{\prime}\right]\right) \mid \theta_{i}, \theta_{i}^{\prime}\right]
\end{aligned}
$$

where the last equality uses the conditional independence $\left(\omega, a_{j} \perp \theta_{i}\right) \mid a_{i}, \theta_{i}^{\prime}$ for $j \neq i$.
Then, using that $s \mathbb{E}\left[\omega \mid a_{i}, \theta_{i}^{\prime}\right]+r \sum_{j \neq i} \mathbb{E}\left[a_{j} \mid a_{i}, \theta_{i}^{\prime}\right]=a_{i}-t \theta_{i}^{\prime}$ by obedience (9), we get

$$
\begin{aligned}
\tilde{v}_{i}\left(\theta_{i}^{\prime} ; \theta_{i}\right) & =\mathbb{E}\left[a_{i}^{2} / 2+t a_{i}\left(\theta_{i}-\theta_{i}^{\prime}\right) \mid \theta_{i}, \theta_{i}^{\prime}\right] \\
& =\frac{1}{2} \mathbb{E}\left[a_{i}^{2} \mid \theta_{i}^{\prime}\right]+t\left(\theta_{i}-\theta_{i}^{\prime}\right) \mathbb{E}\left[a_{i} \mid \theta_{i}^{\prime}\right] \\
& =\frac{1}{2} \mathbb{E}^{2}\left[a_{i} \mid \theta_{i}^{\prime}\right]+t\left(\theta_{i}-\theta_{i}^{\prime}\right) \mathbb{E}\left[a_{i} \mid \theta_{i}^{\prime}\right]+\frac{1}{2} \operatorname{Var}\left(a_{i} \mid \theta_{i}^{\prime}\right),
\end{aligned}
$$

where the second equality uses the conditional independence $a_{i} \perp \theta_{i} \mid \theta_{i}^{\prime}$.
Truthfulness is equivalent to $\theta_{i}$ maximizing $\theta_{i}^{\prime} \mapsto \tilde{v}_{i}\left(\theta_{i}^{\prime} ; \theta_{i}\right)-p_{i}\left(\theta_{i}^{\prime}\right)$ for all $\theta_{i}$, that is,

$$
p_{i}^{\prime}\left(\theta_{i}\right)=\left.\frac{\partial \tilde{v}_{i}\left(\theta_{i}^{\prime} ; \theta_{i}\right)}{\partial \theta_{i}^{\prime}}\right|_{\theta_{i}^{\prime}=\theta_{i}} \quad \text { and }\left.\quad \frac{\partial^{2} \tilde{v}_{i}\left(\theta_{i}^{\prime} ; \theta_{i}\right)}{\partial \theta_{i}^{2}}\right|_{\theta_{i}^{\prime}=\theta_{i}}-p_{i}^{\prime \prime}\left(\theta_{i}\right) \leq 0
$$

Because the mechanism is Gaussian, $\operatorname{Var}\left(a_{i} \mid \theta_{i}^{\prime}\right)$ does not depend on $\theta_{i}^{\prime}$. Furthermore, when $\sigma_{\theta_{i}}^{2}>0$, we have for all $\theta_{i} \in \mathbb{R}$,

$$
\mathbb{E}\left[a_{i} \mid \theta_{i}^{\prime}\right]=\mu_{a_{i}}+\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}\left(\theta_{i}^{\prime}-\mu_{\theta_{i}}\right)
$$

In this case, the first-order condition for optimality gives,

$$
p_{i}^{\prime}\left(\theta_{i}\right)=\left(\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}-t\right) \mathbb{E}\left[a_{i} \mid \theta_{i}\right]=\left(\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}-t\right)\left(\mu_{a_{i}}+\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}\left(\theta_{i}-\mu_{\theta_{i}}\right)\right)
$$

as desired. And the second-order condition becomes

$$
\left(\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}\right)^{2}-2 t \frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}-\left(\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}-t\right) \frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}=-t \frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}} \leq 0 .
$$

When $\sigma_{\theta_{i}}^{2}=0$, then $\mathbb{E}\left[a_{i} \mid \theta_{i}\right]=\mu_{a_{i}}$ for all $\theta_{i} \in \mathbb{R}$ in which case the first-order condition becomes $p_{i}^{\prime}\left(a_{i}\right)=-t \mu_{a_{i}}$ and the second-order condition is always satisfied.

### 3.4 Incentive compatibility

We now consider incentive compatibility mechanisms, in which no profitable deviations of the players exist. This implies in particular truthful reporting of players' types and obedience as studied in the previous section, but also prevents double deviations in which a player both misreport their type in the first stage and then deviate from the action recommendation in the second stage. In fact, the following proposition reveals that for obedient mechanisms, whenever player $i$ reports type $\theta_{i}^{\prime}$ instead of their true type $\theta_{i}$, then it is always profitable to deviate from the action recommendation $a_{i}$ in the second stage and play instead

$$
\begin{equation*}
a_{i}^{\prime}=a_{i}+t\left(\theta_{i}-\theta_{i}^{\prime}\right) . \tag{11}
\end{equation*}
$$

For example, in the Cournot case where $t<0$, a firm that overstates their marginal cost $\left(\theta_{i}^{\prime}>\theta_{i}\right)$ will then deviate upward and produce more than recommended.

Consequently, an incentive compatible mechanism must be robust to such double deviations and the following proposition shows that this is a strictly stronger requirement than simply requiring obedience and truthfulness separately.

Proposition 3.10. The mechanism ( $\mu, \mathrm{K}, p$ ) is incentive compatible if and only if it is obedient and for each player $i \in[n]$ :

1. The derivative of the payment function is given by

$$
p_{i}^{\prime}\left(\theta_{i}\right)=\left(\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}-t\right) \mathbb{E}\left[a_{i} \mid \theta_{i}\right]=\left(\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}-t\right)\left(\mu_{a_{i}}+\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}\left(\theta_{i}-\mu_{\theta_{i}}\right)\right) .
$$

2. The covariance satisfies $t \sigma_{a_{i} \theta_{i}} \geq t^{2} \sigma_{\theta_{i}}^{2}$, or equivalently by obedience, $r t \sum_{j \neq i} \sigma_{a_{j} \theta_{i}} \geq 0$.

When $\sigma_{\theta_{i}}^{2}=0$, the previous two conditions reduce to $p_{i}^{\prime}\left(\theta_{i}\right)=-t \mu_{a_{i}}$.
Remark. As Proposition 3.10 reveals, incentive compatibility requires obedience as well as the constraint $t \sigma_{a_{i} \theta_{i}} \geq t^{2} \sigma_{\theta_{i}}^{2}$, which is more restrictive than the truthfulness constraint $t \sigma_{a_{i} \theta_{i}} \geq 0$ from Proposition 3.9. In particular, for a mechanism such that $t^{2} \sigma_{\theta_{i}}^{2}>t \sigma_{a_{i} \theta_{i}} \geq 0$, misreporting one's cost is not profitable when following the recommendation at the second stage, but it might be profitable to misreport one's cost and deviate at the second stage according to (11). This result is in sharp contrast with our earlier work (Bonatti et al., 2022), where it was established that for multiplicatively decomposable utilities of the form $u_{i}\left(a ; \theta_{i}, \omega\right)=\theta_{i} \cdot \pi(a ; \omega)$, incentive compatibility is equivalent to requiring truthfulness and obedience separately.

### 3.5 Participation

Specifying a player's outside option amounts to choosing the mechanism that will be used with the remaining players in case of non-participation. Throughout this section, we assume that the designer uses an obedient mechanism in the outside option. The next lemma shows that a player's reservation utility does not depend on the choice of the "outside" mechanism. Intuitively, this is because the reservation utility of a non-participating player-who is uninformed about the state by definition-only depends on the outside option mechanism through the means of the recommendations sent to the participating players. In an obedient mechanisms, these means are fully determined as was shown in Proposition 3.5.

Lemma 3.11. Assume that the designer uses an obedient mechanism with the remaining players in case player $i$ chooses not to participate. Then player $i$ 's reservation utility is independent of the chosen mechanism and is given by

$$
u_{i}^{o}\left(\theta_{i}\right)=\frac{1}{2}\left(\mu_{a_{i}}+t\left(\theta_{i}-\mu_{\theta_{i}}\right)\right)^{2}
$$

where $\mu_{a_{i}}$ is given by (7).
Proof. If player $i$ does not participate, then her action $a_{i}$ must be $\theta_{i}$-measurable. Furthermore, if $a_{i}$ is a best-response to the outside mechanism, then the conditional distribution $a \mid \theta, \omega$ is the one of an obedient mechanism with the restriction that $\sigma_{a_{i} \omega}=\sigma_{a_{i} a_{j}}=\sigma_{a_{i} \theta_{j}}=0$ due to $a_{i}$ being $\theta_{i}$-measurable. In other words, from player $i$ 's perspective, this is exactly the same situation as the Bayes-Nash equilibrium studied in Example 3.8, for which we obtained

$$
a_{i}=\mu_{a_{i}}+t\left(\theta_{i}-\mu_{\theta_{i}}\right) .
$$

We computed in the previous section that the interim expected utility in an obedient mechanism is $\frac{1}{2} \mathbb{E}\left[a_{i}^{2} \mid \theta_{i}\right]$, which directly implies the desired expression for player $i$ 's reservation utility.

Remark 3.12. Note that the previous proof relies on interpreting the strategy of a nonparticipating player $i \in[n]$ as following an obedient mechanism, with the constraint that their action only depend on their type $\theta_{i}$. Such a mechanism is of course very asymmetric, which is why it was crucial to establish a characterization of all-possibly asymmetric-obedient mechanisms in Proposition 3.5.

Moreover, it goes to show that in our linear best-response setting it is not possible to "threaten" a player with adverse actions by their competitors if a player does not participate (and pay a transfer to the designer). Thus, the intuition that (in Cournot) other firms will be called to flood the market does not hold in our case.

In the next proposition, we describe the general form of payments able to implement a given incentive compatible mechanism. Because incentive compatibility only constrains the derivative of the payments, these are determined only up to an additive constant. As we show, the payments properly incentivize participation as long as this constant is upper-bounded by a quantity depending only on the covariance matrix K of the mechanism.

Proposition 3.13. Let $(\mu, \mathrm{K}, p)$ be an incentive compatible mechanism, and assume that an obedient mechanism is used with the remaining players in case of non-participation. Then, the payment of player $i \in[n]$ is given by

$$
p_{i}\left(\theta_{i}\right)=p_{i}\left(\mu_{\theta_{i}}\right)+\frac{1}{2}\left(1-\frac{t \sigma_{\theta_{i}}^{2}}{\sigma_{a_{i} \theta_{i}}}\right)\left(\mathbb{E}^{2}\left[a_{i} \mid \theta_{i}\right]-\mu_{a_{i}}^{2}\right),
$$

and $(\mu, \mathrm{K}, p)$ is individually rational iff

$$
\begin{equation*}
p_{i}\left(\mu_{\theta_{i}}\right) \leq \frac{1}{2} \operatorname{Var}\left[a_{i} \mid \theta_{i}\right]:=\frac{1}{2}\left(\sigma_{a_{i}}^{2}-\frac{\sigma_{a_{i} \theta_{i}}^{2}}{\sigma_{\theta_{i}}^{2}}\right) \tag{12}
\end{equation*}
$$

Proof. We already computed the interim expected utility of an incentive compatible mechanism,

$$
\tilde{u}_{i}\left(\theta_{i}\right)=\frac{1}{2}\left(\mathbb{E}^{2}\left[a_{i} \mid \theta_{i}\right]+\operatorname{Var}\left[a_{i} \mid \theta_{i}\right]\right)
$$

Furthermore, incentive compatibility determines the derivative of the payment as described in Proposition 3.10. Consequently, the payment function $p_{i}$ is known up to an additive constant. It will be convenient for us to parametrize this indetermination by the price at $\mu_{a_{i}}$ and we write

$$
p_{i}\left(\theta_{i}\right)=p_{i}\left(\mu_{\theta_{i}}\right)+\int_{\mu_{\theta_{i}}}^{\theta_{i}} p_{i}^{\prime}(s) \mathrm{d} s=p_{i}\left(\mu_{\theta_{i}}\right)+\frac{1}{2}\left(1-\frac{t \sigma_{\theta_{i}}^{2}}{\sigma_{a_{i} \theta_{i}}}\right)\left(\mathbb{E}^{2}\left[a_{i} \mid \theta_{i}\right]-\mu_{a_{i}}^{2}\right)
$$

Individual rationality is then equivalent to $\tilde{u}_{i}\left(\theta_{i}\right)-p_{i}\left(\theta_{i}\right)-u_{i}^{o}\left(\theta_{i}\right) \geq 0$ for all $\theta_{i} \in \mathbb{R}$. The function on the left-hand side of this inequality is quadratic in $\theta_{i}$ with leading coefficient

$$
\frac{\sigma_{a_{i} \theta_{i}}^{2}}{2 \sigma_{\theta_{i}}^{4}}-\frac{1}{2}\left(1-\frac{t \sigma_{\theta_{i}}^{2}}{\sigma_{a_{i} \theta_{i}}}\right) \frac{\sigma_{a_{i} \theta_{i}}^{2}}{\sigma_{\theta_{i}}^{4}}-\frac{t^{2}}{2}=\frac{t \sigma_{a_{i} \theta_{i}}}{2 \sigma_{\theta_{i}}^{2}}-\frac{t^{2}}{2}
$$

which is non-negative by incentive compatibility (Proposition 3.10). Hence the function is convex and can easily be seen to be minimized at $\theta_{i}=\mu_{\theta_{i}}$. Individual rationality is thus equivalent to requiring that the value at $\mu_{\theta_{i}}$ be non-negative, that is

$$
\frac{1}{2}\left(\mu_{a_{i}}^{2}+\operatorname{Var}\left[a_{i} \mid \theta_{i}\right]\right)-p_{i}\left(\mu_{\theta_{i}}\right)-\frac{1}{2} \mu_{a_{i}}^{2} \geq 0
$$

which simplifies to (12) as desired.

### 3.6 Positivity of transfers

Since we would like payments to be non-negative, we can set the integration constant $p_{i}\left(\mu_{\theta_{i}}\right)$ at the largest value allowed by individual rationality (12). This implies the following expression for the payments

$$
\begin{equation*}
p_{i}\left(\theta_{i}\right)=\frac{1}{2}\left(\sigma_{a_{i}}^{2}-\frac{\sigma_{a_{i} \theta_{i}}^{2}}{\sigma_{\theta_{i}}^{2}}\right)+\frac{1}{2}\left(1-\frac{t \sigma_{\theta_{i}}^{2}}{\sigma_{a_{i} \theta_{i}}}\right)\left(\mathbb{E}^{2}\left[a_{i} \mid \theta_{i}\right]-\mu_{a_{i}}^{2}\right) \tag{13}
\end{equation*}
$$

Proposition 3.14. Let $(\mu, \mathrm{K})$ be an incentive compatible mechanism, then for the payment given in (13), we have $\mathbb{E}\left[p_{i}\left(\theta_{i}\right)\right] \geq 0$.

Proof. From (13) we compute, $\mathbb{E}\left[p_{i}\left(\theta_{i}\right)\right]=\frac{1}{2}\left(\sigma_{a_{i}}^{2}-t \sigma_{a_{i} \theta_{i}}\right)$. But then,

$$
\sigma_{a_{i}}^{2}-t \sigma_{a_{i} \theta_{i}} \geq \frac{\sigma_{a_{i} \theta_{i}}^{2}}{\sigma_{\theta_{i}}^{2}}-t \sigma_{a_{i} \theta_{i}}=\frac{t \sigma_{a_{i} \theta_{i}}}{t^{2} \sigma_{\theta_{i}}^{2}}\left(t \sigma_{a_{i} \theta_{i}}-t^{2} \sigma_{\theta_{i}}^{2}\right) \geq 0
$$

where the first inequality is Cauchy-Schwarz inequality, and the last inequality uses $t \sigma_{a_{i} c \theta_{i}} \geq$ $t^{2} \sigma_{\theta_{i}}^{2}$ by incentive compatibility.

## 4 Welfare maximization

The expected utility of player $i$ in any obedient mechanism can be computed as follows using the law of total expectation,

$$
\begin{align*}
\mathbb{E}\left[u_{i}(a ; \theta, \omega)\right] & =\mathbb{E}\left[-\frac{1}{2} a_{i}^{2}+a_{i}\left(s \mathbb{E}\left[\omega \mid a_{i}, \theta_{i}\right]+t \theta_{i}+r \sum_{j \neq i} \mathbb{E}\left[a_{j} \mid a_{i}, \theta_{i}\right]\right)\right]  \tag{14}\\
& =\frac{1}{2} \mathbb{E}\left[a_{i}^{2}\right]=\frac{\mu_{a_{i}}^{2}+\sigma_{a_{i}}^{2}}{2}
\end{align*}
$$

where the penultimate equality uses the first-order characterization of obedience (9).

### 4.1 Reduction to symmetric mechanisms

The derivation of the expected utility of player $i$ in (14) reveals in particular that the expected welfare of an obedient mechanism is linear in the design parameters $\left(\sigma_{a_{i}}^{2}\right)_{i \in[n]}$-the means $\mu_{a_{i}}$ are determined by obedience (7). This allows for a symmetrization argument presented in the next proposition, showing that restricting oneself to symmetric mechanisms is without loss of generality for the purpose of studying the range of welfare achievable by Gaussian obedient mechanisms.

Proposition 4.1. Assume that the prior distribution on players' types $\theta$ is symmetric, then for any Gaussian and obedient mechanism, there exists a symmetric, Gaussian, and obedient mechanism achieving identical welfare.

The proof of this proposition is presented in Appendix B and follows from Lemma C.6. Essentially, we symmetrize a given mechanism by averaging its mean vector and covariance matrix over all possible permutations of the players. Note that the resulting symmetric mechanism is different from the usual lottery that first draws a permutation of the players uniformly at random and then applies the original mechanism to this permutation. While a lottery would certainly preserve obedience (a convex combination of two obedient mechanisms is still obedient), the resulting mechanism would no longer be Gaussian, because a nontrivial mixture of normal distributions is not normal. In contrast, our symmetrization argument is expressed directly on the moments of the mechanism. The fact that obedience is still preserved is less obvious but follows from the convexity of the obedience constraints on the covariance matrix $K$.

### 4.2 Optimal mechanism

We first determine the welfare-maximizing mechanism subject to the obedience constraint and then show that it satisfies incentive compatibility. By Proposition 4.1, we can focus on symmetric mechanisms without loss of generality. For such mechanisms, maximizing welfare is equivalent to maximizing the expected utility of a single player $i \in[n]$, which is in turn equivalent to maximizing the variance $\sigma_{a_{i}}^{2}$ due to the form of player $i$ 's utility Eq. (14) and the fact that the mean $\mu_{a_{i}}$ is determined by obedience Eq. (7). We thus obtain the following
optimization problem

$$
\begin{aligned}
\max & \sigma_{a_{i}}^{2} \\
\text { s.t. } & \frac{1}{\sigma_{\theta_{i}}^{2}}\left(\sigma_{a_{i} \theta_{i}}-\sigma_{a_{i} \theta_{j}}\right)^{2} \leq \sigma_{a_{i}}^{2}-\sigma_{a_{i} a_{j}} \\
& \frac{1}{\sigma_{\theta_{i}}^{2}}\left(\sigma_{a_{i} \theta_{i}}+(n-1) \sigma_{a_{i} \theta_{j}}\right)^{2}+\frac{n}{\sigma_{\omega}^{2}} \sigma_{a_{i} \omega}^{2} \leq \sigma_{a_{i}}^{2}+(n-1) \sigma_{a_{i} a_{j}} \\
& \sigma_{a_{i}}^{2}=(n-1) r \sigma_{a_{i} a_{j}}+s \sigma_{a_{i} \omega}+t \sigma_{a_{i} \theta_{i}} \\
& \sigma_{a_{i} \theta_{i}}=(n-1) r \sigma_{a_{i} \theta_{j}}+t \sigma_{\theta_{i}}^{2}
\end{aligned}
$$

in the five variables $\left(\sigma_{a_{i}}^{2}, \sigma_{a_{i} \theta_{i}}, \sigma_{a_{i}, \theta_{j}}, \sigma_{a_{i} \omega}, \sigma_{a_{i} a_{j}}\right)$. The two inequality constraints express the positive semidefiniteness of the covariance matrix K (Lemma 3.3) and the two equality constraints are the obedience constraints in the specific case of symmetric mechanisms (Remark 3.6) ${ }^{2}$.

The following proposition describes the welfare-optimal mechanism in the case of $r<0$ (i.e., strategic substitutes).

Proposition 4.2. For $r \in(-1,0)$, there exists a unique symmetric mechanism maximizing social welfare subject to obedience. In this mechanism, the recommended actions are maximally negatively correlated conditioned on $(\theta, \omega)$, hence

$$
a_{i}=\frac{s \mu_{\omega}+t \mu_{\theta_{i}}}{1-(n-1) r}+\frac{\sigma_{a_{i} \omega}}{\sigma_{\omega}^{2}}\left(\omega-\mu_{\omega}\right)+\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}\left(\theta_{i}-\mu_{\theta_{i}}\right)+\sum_{j \neq i} \frac{\sigma_{a_{i} \theta_{j}}}{\sigma_{\theta_{j}}^{2}}\left(\theta_{j}-\mu_{\theta_{j}}\right)+\delta \sum_{j \neq i} \frac{\varepsilon_{i}-\varepsilon_{j}}{\sqrt{n(n-1)}}
$$

for some $\delta \in \mathbb{R}$ and with $\varepsilon \sim \mathcal{N}\left(0, I_{n}\right)$ a standard normal vector. Writing $f:=\frac{1}{n-1}$, the action $a_{i}$ is a deterministic function of $(\theta, \omega)$, that is $\delta=0$, iff

$$
\begin{equation*}
-\frac{(1+r)^{3}(1+(n+1) r)}{n^{2} r^{2}(2 r+3)(f(2 r+3)-r)} \leq \frac{t^{2} \sigma_{\theta_{i}}^{2}}{s^{2} \sigma_{\omega}^{2}} . \tag{15}
\end{equation*}
$$

1. If (15) is satisfied, then

$$
\begin{gather*}
\sigma_{a_{i} \omega}=\frac{s \sigma_{\omega}^{2}}{2 n} \frac{\lambda n f+1}{\lambda(f-r)-r}, \quad \sigma_{a_{i} \theta_{i}}=t \sigma_{\theta_{i}}^{2} \frac{\lambda n f\left[r^{2}+2(f-r)(1+r)\right]-(2+r) r}{2(1+r)[\lambda n f(f-r)-(1+r) r]},  \tag{16}\\
\sigma_{a_{i} \theta_{j}}=f r t \sigma_{\theta_{i}}^{2} \frac{\lambda n f+(2 r+3)}{2(1+r)[\lambda n f(f-r)-(1+r) r]}, \quad \delta=0 . \tag{17}
\end{gather*}
$$

where $\lambda$ is the unique positive root of

$$
\begin{equation*}
\frac{(1+r)^{3} f s^{2} \sigma_{\omega}^{2}}{n^{2}(\lambda(f-r)-r)^{2}}+\frac{r^{2}\left(r^{2}+2 r(1-f)-3 f\right)^{2} t^{2} \sigma_{\theta_{i}}^{2}}{[\lambda n f(f-r)-(1+r) r]^{2}}=f s^{2} \sigma_{\omega}^{2}(1+r)+r^{2} t^{2} \sigma_{\theta_{i}}^{2} \tag{18}
\end{equation*}
$$

2. Otherwise,

$$
\begin{gathered}
\sigma_{a_{i} \omega}=-\frac{s \sigma_{\omega}^{2}}{2 n r}, \quad \sigma_{a_{i} \theta_{i}}=t \sigma_{\theta_{i}}^{2} \frac{(2+r)}{2(1+r)^{2}}, \quad \sigma_{a_{i} \theta_{j}}=-f t \sigma_{\theta_{i}}^{2} \frac{(2 r+3)}{2(1+r)^{2}}, \\
\delta^{2}=-\frac{t^{2} \sigma_{\theta_{i}}^{2}(2 r+3)(f(2 r+3)-r) n^{2} r^{2}+s^{2} \omega^{2}(1+(n+1) r)(1+r)^{3}}{4(1+r)^{4} n^{2} r^{2}} .
\end{gathered}
$$

[^2]Proof. First, we use the last two (obedience) equality constraints to eliminate the variables $\sigma_{a_{i} \theta_{j}}$ and $\sigma_{a_{i}^{2}}$ from the objective function and the remaining constraints, thus reducing the problem to an optimization in the three variables $\sigma_{a_{i} a_{j}}, \sigma_{a_{i} \theta_{i}}$ and $\sigma_{a_{i} \omega}$. We use $f:=\frac{1}{n-1}$ throughout for legibility.

$$
\begin{aligned}
\max & (n-1) r \sigma_{a_{i} a_{j}}+s \sigma_{a_{i} \omega}+t \sigma_{a_{i} \theta_{i}} \\
\text { s.t. } & \frac{1}{r^{2} \sigma_{\theta_{i}}^{2}}\left[(f-r) \sigma_{a_{i} \theta_{i}}-f t \sigma_{\theta_{i}}^{2}\right]^{2} \leq s \sigma_{a_{i} \omega}+t \sigma_{a_{i} \theta_{i}}-(n-1)(f-r) \sigma_{a_{i} a_{j}} \\
& \frac{1}{r^{2} \sigma_{\theta_{i}}^{2}}\left[(1+r) \sigma_{a_{i} \theta_{i}}-t \sigma_{\theta_{i}}^{2}\right]^{2}+\frac{n}{\sigma_{\omega}^{2}} \sigma_{a_{i} \omega}^{2} \leq s \sigma_{a_{i} \omega}+t \sigma_{a_{i} \theta_{i}}+(n-1)(1+r) \sigma_{a_{i} a_{j}}
\end{aligned}
$$

We introduce non-negative Lagrange multipliers $\lambda$ and $\nu$ for the first and second inequality constraints respectively and look for a solution to the KKT conditions. The stationarity condition yields the following system of equations,

$$
\left\{\begin{array}{l}
-r+(f-r) \lambda-(r+1) \nu=0 \\
-s-s \lambda+\nu\left(\frac{2 n}{\sigma_{\omega}^{2}} \sigma_{a_{i} \omega}-s\right)=0 \\
2 \lambda \frac{f-r}{r^{2} \sigma_{\theta_{i}}^{2}}\left[(f-r) \sigma_{a_{i} \theta_{i}}-f t \sigma_{\theta_{i}}^{2}\right]+2 \nu \frac{1+r}{r^{2} \sigma_{\theta_{i}}^{2}}\left[(1+r) \sigma_{a_{i} \theta_{i}}-t \sigma_{\theta_{i}}^{2}\right]=t(\lambda+\nu+1)
\end{array}\right.
$$

From this, we express $\nu, \sigma_{a_{i} \omega}$ and $\sigma_{a_{i} \theta_{i}}$ as a function of $\lambda$.

$$
\begin{gather*}
\nu=\frac{\lambda(f-r)-r}{1+r}, \quad \sigma_{a_{i} \omega}=\frac{s \sigma_{\omega}^{2}}{2 n} \frac{\lambda n f+1}{\lambda(f-r)-r}, \\
\sigma_{a_{i} \theta_{i}}=t \sigma_{\theta_{i}}^{2} \frac{\lambda n f\left[r^{2}+2(f-r)(1+r)\right]-(2+r) r}{2(1+r)[\lambda n f(f-r)-(1+r) r]} . \tag{19}
\end{gather*}
$$

Note that because $r$ is negative, the denominator in the expressions for $\sigma_{a_{i} \omega}$ and $\sigma_{a_{i} \theta_{i}}$ cannot vanish for $\lambda \geq 0$ and the expressions in (19) are well-defined for a dual feasible $\lambda$. Furthermore, since $1+r>0$ by assumption, we get that $\nu \geq-\frac{r}{1+r}>0$ and the second inequality constraint is binding by complementary slackness. We can thus eliminate $\sigma_{a_{i} a_{j}}$ from the first inequality constraint

$$
\frac{f-r}{r^{2} \sigma_{\theta_{i}}^{2}}\left[\sigma_{a_{i} \theta_{i}}-\frac{f t \sigma_{\theta_{i}}^{2}}{f-r}\right]^{2}+\frac{1+r}{r^{2} \sigma_{\theta_{i}}^{2}}\left[\sigma_{a_{i} \theta_{i}}-\frac{t \sigma_{\theta_{i}}^{2}}{1+r}\right]^{2}+\frac{n \sigma_{a_{i} \omega}^{2}}{\sigma_{\omega}^{2}(1+r)} \leq \frac{n f\left(s \sigma_{a_{i} \omega}+t \sigma_{a_{i} \theta_{i}}\right)}{(1+r)(f-r)},
$$

which we rewrite in canonical form after multiplying by the positive constant $\frac{1+r}{n f}$ as

$$
\begin{align*}
\frac{1+r}{r^{2} \sigma_{\theta_{i}}^{2}}\left[\sigma_{a_{i} \theta_{i}}-t \sigma_{\theta_{i}}^{2} \frac{\left(r^{2}+2(f-r)(1+r)\right)}{2(f-r)(1+r)}\right]^{2}+\frac{1}{f \sigma_{\omega}^{2}} & {\left[\sigma_{a_{i} \omega}-\frac{f s \sigma_{\omega}^{2}}{2(f-r)}\right]^{2} } \\
& \leq \frac{r^{2} t^{2} \sigma_{\theta_{i}}^{2}}{4(1+r)(f-r)^{2}}+\frac{f s^{2} \sigma_{\omega}^{2}}{4(f-r)^{2}} \tag{20}
\end{align*}
$$

Using the expression for $\sigma_{a_{i} \omega}$ and $\sigma_{a_{i} \theta_{i}}$ from (19) and multiplying by the positive constant $4(1+r)(f-r)^{2}$, the previous inequality is equivalent to

$$
\begin{equation*}
\frac{(1+r)^{3} f s^{2} \sigma_{\omega}^{2}}{n^{2}[\lambda(f-r)-r]^{2}}+\frac{r^{2}\left(r^{2}+2 r(1-f)-3 f\right)^{2} t^{2} \sigma_{\theta_{i}}^{2}}{[\lambda n f(f-r)-(1+r) r]^{2}} \leq f s^{2} \sigma_{\omega}^{2}(1+r)+r^{2} t^{2} \sigma_{\theta_{i}}^{2} \tag{21}
\end{equation*}
$$

Because $r$ is negative by assumption, we have a fortiori that $f-r>0$, which easily implies that the left-hand side in (21) is a decreasing function of $\lambda$ when $\lambda \geq 0$. Furthermore, this function converges to 0 for $\lambda \rightarrow \infty$. We consider two cases.

1. Either (21) is violated for $\lambda=0$, but since the left-hand side is decreasing and converges to 0 , there exists a unique $\lambda>0$ for which equality holds. Hence, complementary slackness holds for this $\lambda$ when $\sigma_{a_{i} \omega}$ and $\sigma_{a_{i} \theta_{i}}$ are computed as in (19).
2. Or (21) is satisfied for $\lambda=0$, in which case complementary slackness trivially holds.

Furthermore, in both cases $\nu$ is determined by (19), and $\sigma_{a_{i} a_{j}}$ is determined by the fact that the second inequality constraint is binding. Hence, we identified in both cases a unique primal feasible triple ( $\sigma_{a_{i} \omega}, \sigma_{a_{i} \theta_{i}}, \sigma_{a_{i} a_{j}}$ ) and dual feasible pair $(\lambda, \nu)$ satisfying both stationarity and complementary slackness. Since the problem is convex, strong duality holds and this primal feasible triple is the unique maximizer. For convenience, inequality (21) at $\lambda=0$ simplifies to

$$
\frac{t^{2} \sigma_{\theta_{i}}^{2}}{s^{2} \sigma_{\omega}^{2}} \leq-\frac{(1+r)^{3}(1+(n+1) r)}{n^{2} r^{2}(2 r+3)(3 f-(1-2 f) r)}
$$

as given in the proposition statement.
To finish describing the mechanism, we use the fact that the second inequality constraint binds iff the recommended actions are maximally negatively correlated conditioned on $(\theta, \omega)$ by Lemma 3.3. This lets us write $a_{i}$ as in the proposition statement. We determine $\sigma_{a_{i} \theta_{j}}$ from $\sigma_{a_{i} \theta_{i}}$ using the second obedience constraint,

$$
\sigma_{a_{i} \theta_{j}}=f t \sigma_{\theta_{i}}^{2} \frac{\lambda n f r+r(2 r+3)}{2(1+r)[\lambda n f(f-r)-(1+r) r]},
$$

and $\delta^{2}$ from the other variables using the first obedience constraint.
Geometric interpretation. Some intuition about the structure of the welfare-optimal mechanism described in Proposition 4.2 can be gained with the following geometric interpretation of the proof.

1. The two linear obedience equality constraints let us eliminate the variables $\sigma_{a_{i} \theta_{j}}$ and $\sigma_{a_{i}}^{2}$ from the objective function and the rest of the constraints, resulting in an optimization problem in the three variables ( $\sigma_{a_{i}, \theta_{i}}, \sigma_{a_{i} \omega}, \sigma_{a_{i} a_{j}}$ ).
2. In this three-dimensional parametrization of the optimization problem. The objective function is given by

$$
(n-1) r \sigma_{a_{i} a_{j}}+s \sigma_{a_{i} \omega}+t \sigma_{a_{i} \theta_{i}} .
$$

Crucially since $r<0$, the objective function is decreasing in $\sigma_{a_{i} a_{j}}$, so we want to choose it as small as possible. The only lower bound on $\sigma_{a_{i} a_{j}}$ is the second inequality constraint, which is thus necessarily binding at the optimum. As discussed in Lemma 3.3, this implies in particular that the action recommendations are maximally anticorrelated conditioned on $(\theta, \omega)$.
3. The fact that the second inequality constraint binds also lets us eliminate the variable $\sigma_{a_{i} a_{j}}$, resulting in a two-dimensional optimization problem in the variables $\left(\sigma_{a_{i} \theta_{i}}, \sigma_{a_{i} \omega}\right)$. The first inequality constraint expressed in terms of these two variables describes an ellipse $\mathcal{E}$, containing the set of all obedient mechanisms for which the action recommendations are maximally anticorrelated (20). The boundary of $\mathcal{E}$ parametrizes all obedient mechanisms in which the action recommendations are deterministic given $(\theta, \omega)$.


Figure 1: Representation of mechanisms in the ( $\sigma_{a_{i} \omega}, \sigma_{a_{i} \theta_{i}}$ ) plane with $r<0$ and $t<0$. When action recommendations are maximally correlated conditioned on $(\theta, \omega)$, an obedient mechanism is uniquely determined by a pair $\left(\sigma_{a_{i} \omega}, \sigma_{a_{i} \theta_{i}}\right)$ lying inside an elliptic region $\mathcal{E}$ whose boundary $\mathcal{E}$ corresponds to mechanisms that are deterministic conditioned on $(\theta, \omega)$. Stationarity constrains the welfare-optimal mechanism to be a point $P(\lambda)$ lying on a portion of hyperbola (in green) parametrized by the non-negative multiplier $\lambda$. Complementary slackness implies that the welfare-optimal mechanism is either at $P(0)$ if $P(0) \in \mathcal{E}$ (interior solution) or at the intersection of the hyperbola with $\mathcal{E}$ otherwise (boundary solution). The point labeled N is the mechanism that recommends the Nash equilibrium of the complete information game for each realisation of $(\theta, \omega)$.
4. Stationarity further implies that at the optimum the point $\left(\sigma_{a_{i} \theta_{i}}, \sigma_{a_{i} \theta_{\omega}}\right)$ lies on a portion of hyperbola parametrized by the non-negative multiplier $\lambda$ associated with the ellipse constraint $\mathcal{E}$ :

$$
\mathcal{H}:=\left\{\left(\sigma_{a_{i} \theta_{i}}(\lambda), \sigma_{a_{i} \omega}(\lambda)\right) \mid \lambda \geq 0\right\},
$$

where $\sigma_{a_{i} \theta_{i}}(\lambda), \sigma_{a_{i} \omega}(\lambda)$ are the functions described in (16). Note that when $\lambda \rightarrow \infty$, the corresponding point $P(\lambda):=\left(\sigma_{a_{i} \theta_{i}}(\lambda), \sigma_{a_{i} \omega}(\lambda)\right)$ on $\mathcal{H}$ converges to the center of the ellipse $\mathcal{E}$.

Consequently, we recover the two cases described in Proposition 4.2 using complementary slackness:

- either at $\lambda=0$, the point $P(0):=\left(\sigma_{a_{i} \theta_{i}}(0), \sigma_{a_{i} \omega}(0)\right)$ lies within $\mathcal{E}$, in which case complementary slackness holds and we have an interior solution.
- or $P(0)$ is outside of $\mathcal{E}$, but then by continuity and since $P(\lambda)$ converges to the center of $\mathcal{E}$ as $\lambda \rightarrow \infty$, there exists a unique $\lambda>0$ for which $P(\lambda)$ lies on the boundary of $\mathcal{E}$. For such a $\lambda$, complementary slackness holds and we have a boundary solution.

Deterministic vs randomized mechanisms. As discussed in the previous paragraph, depending on the position of $P(0)$ relative to $\mathcal{E}$, we either have a boundary solution for which the action recommendations are a deterministic (and linear) function of $(\theta, \omega)$, or an interior solution in which the mechanism adds independent maximally anticorrelated noise to a deterministic function of $(\theta, \omega)$. Whether or not the solution is on the boundary of the feasible set is determined by the inequality

$$
\begin{equation*}
f(r):=-\frac{(1+r)^{3}(1+(n+1) r)}{n^{2} r^{2}(2 r+3)(f(2 r+3)-r)} \leq \frac{t^{2} \sigma_{\theta_{i}}^{2}}{s^{2} \sigma_{\omega}^{2}} . \tag{22}
\end{equation*}
$$



Figure 2: Plot of $f(r)$ defined in (22) for $n=2$ (and $f=1 /(n-1)=1$ ). The function is positive on $\left(-1,-\frac{1}{n+1}\right)$ and unimodal with a maximum at $r^{\star} \approx-0.45$. When the threshold value $t^{2} \sigma_{\theta_{i}}^{2} /\left(s^{2} \sigma_{\omega}^{2}\right.$ is less than $f\left(r^{\star}\right) \approx 0.013$, then there are two critical values $r_{l}$ and $r_{h}$ at which $f$ reaches the threshold. In this case, condition (22) is violated on ( $r_{l}, r_{h}$ ) and the mechanism is randomized on this interval.
which only depends on parameters of the problem-namely the prior's variance $\left(\sigma_{\theta_{i}}^{2}, \sigma_{\omega}^{2}\right)$ and the parameters $(r, s, t)$. More intuition about this condition can be gained by defining $f(r)$ to be the function of $r$ on the left-hand side of (22) and studying its variations. It is easy to see that the only factor in $f(r)$ whose sign changes on the interval $(-1,0)$ is $1+(n+1) r$. Consequently, $f$ is non-negative on $[-1,-1 /(n+1)]$ and negative on $(-1 /(n+1), 0)$. Furthermore, it can also be shown that $f$ is unimodal on ( $-1,0$ ), reaching a unique positive maximum at some $r^{\star}$ in $(-1,-1 /(n+1))$. We thus have two cases shown in Fig. 2,

- if $t^{2} \sigma_{\theta_{i}}^{2} /\left(s^{2} \sigma_{\omega}^{2}\right) \geq f\left(r^{\star}\right)$, then (22) is always satisfied and the mechanism is deterministic conditioned on $(\theta, \omega)$ for all values of $r \in(-1,0)$.
- otherwise there exist two critical values $r_{l}$ and $r_{h}$ with $-1<r_{l}<r^{\star}<r_{h}<-1 /(n+1)$, solutions to $f(r)=t^{2} \sigma_{\theta_{i}}^{2} /\left(s^{2} \sigma_{\omega}^{2}\right)$. When $r \in\left(r_{l}, r_{h}\right)$, then (22) is violated and the mechanism is randomized, otherwise the mechanism is deterministic conditioned on $(\theta, \omega)$.

This can be intuitively understood as follows: the mechanism designer wishes to maximally anticorrelate the players' actions, which is why $\sigma_{a_{i} \theta_{i}}$ and $\sigma_{a_{i} \theta_{j}}$ are always of opposite signs. However, when $\sigma_{\theta_{i}}^{2}$ is small, the second obedience constraint prevents these two parameters from being far enough from each other to achieve maximum anticorrelation for all values of $r$ and the mechanism therefore adds additional negatively correlated independent noise.

Finally, the description of the welfare-maximizing mechanism in Proposition 4.2 allows us to compare the covariances of the action recommendation $a_{i}$ with the fundamentals $(\theta, \omega)$ in the optimal mechanism and in the Nash equilibrium of the complete information game. This is stated in the following proposition, where quantities in the Nash equilibrium (resp. welfare-maximizing mechanism) are denoted with an N (resp. W subscript).

Proposition 4.3. For $r \in(-1,0)$ the welfare-optimal mechanism of Proposition 4.2 satisfies

$$
\left|\sigma_{a_{i} \omega}^{\mathrm{W}}\right|<\left|\sigma_{a_{i} \omega}^{\mathrm{N}}\right|, \quad\left|\sigma_{a_{i} \theta_{i}}^{\mathrm{W}}\right|>\left|\sigma_{a_{i} \theta_{i}}^{\mathrm{N}}\right| \quad \text { and } \quad\left|\sigma_{a_{i} \theta_{j}}^{\mathrm{W}}\right|>\left|\sigma_{a_{i} \theta_{i}}^{\mathrm{N}}\right| .
$$

Proof. Using the notation $f:=1 /(n-1)$ we can write the Nash equilibrium as

$$
\sigma_{a_{i} \omega}^{\mathrm{N}}=\frac{f}{f-r} s \sigma_{\omega}^{2} \quad \text { and } \quad \sigma_{a_{i} \theta_{i}}^{\mathrm{N}}=\frac{f(1+r)-r}{(1+r)(f-r)} t \sigma_{\theta_{i}}^{2}
$$

We denote by $\sigma_{a_{i} \omega}(\lambda)$ and $\sigma_{a_{i} \theta_{i}}(\lambda)$ the functions of $\lambda$ in (16). Theses functions interpolate continuously and monotonously between their value at 0 and their value $\infty$ which is the center of the ellipse described in (20). We have

$$
\begin{gathered}
\sigma_{a_{i} \omega}(0)=-\frac{1}{2 n r} s \sigma_{\omega}^{2}, \quad \sigma_{a_{i} \omega}(\infty)=\frac{f}{2(f-r)} s \sigma_{\omega}^{2} \\
\sigma_{a_{i} \theta_{i}}(0)=\frac{2+r}{2(1+r)^{2}} t \sigma_{\theta_{i}}^{2} \quad \text { and } \quad \sigma_{a_{i} \theta_{i}}(\infty)=\frac{r^{2}+2(f-r)(1+r)}{2(f-r)(1+r)} t \sigma_{\theta_{i}}^{2}
\end{gathered}
$$

From these expressions it is easy to verify the following inequalities when $r \in(-1,0)$ :

$$
\begin{equation*}
\left|\sigma_{a_{i} \theta_{i}}(\infty)\right|<\left|\sigma_{a_{i} \theta_{i}}^{\mathrm{N}}\right|<\left|\sigma_{a_{i} \theta_{i}}(0)\right| \quad \text { and } \quad\left|\sigma_{a_{i} \omega}(\infty)\right|<\left|\sigma_{a_{i} \omega}(0)\right| \quad \text { and } \quad\left|\sigma_{a_{i} \omega}(\infty)\right|<\left|\sigma_{a_{i} \omega}^{\mathrm{N}}\right| \tag{23}
\end{equation*}
$$

We distinguish two cases depending on the relative position of $\left|\sigma_{a_{i} \omega}(0)\right|$ and $\left|\sigma_{a_{i} \omega}^{\mathrm{N}}\right|$.
1st case: $\left|\sigma_{a_{i} \omega}(0)\right| \leq\left|\sigma_{a_{i} \omega}^{\mathrm{N}}\right|$. In this case, either we have an interior solution. But then $\left|\sigma_{a_{i} \omega}^{\mathrm{W}}\right|=\left|\sigma_{a_{i} \omega}(0)\right|<\left|\sigma_{a_{i} \omega}^{\mathrm{N}}\right|$ by assumption of this case, and $\left|\sigma_{a_{i} \theta_{i}}^{\mathrm{W}}\right|=\left|\sigma_{a_{i} \theta_{i}}(0)\right|>\left|\sigma_{a_{i} \theta_{i}}^{\mathrm{N}}\right|$ by $(23)$. Or we have a boundary solution parametrized by $\lambda$ with $\left|\sigma_{a_{i} \omega}(\lambda)\right|<\left|\sigma_{a_{i} \omega}(0)\right|<\left|\sigma_{a_{i} \omega}^{\mathrm{N}}\right|$ by (23). But for points on the ellipse, as $\sigma_{a_{i} \omega}$ gets closer to $\sigma_{a_{i} \omega}(\infty)$ then $\sigma_{a_{i} \theta_{i}}$ gets further from $\sigma_{a_{i} \theta_{i}}(\infty)$. Consequently, $\left|\sigma_{a_{i} \theta_{i}}(\lambda)\right|>\left|\sigma_{a_{i} \theta_{i}}^{\mathrm{N}}\right|$ as desired.

2nd case: $\left|\sigma_{a_{i} \omega}(0)\right|>\left|\sigma_{a_{i} \omega}^{\mathrm{N}}\right|$. This condition is equivalent to $r>-1 /(n+1)$. In this case we necessarily have a boundary solution because the assumption of this case as well (23) imply that $\left(\sigma_{a_{i} \omega}(\lambda), \sigma_{a_{i} \theta_{i}}(\lambda)\right)$ lies outside of the ellipse. We compute $\lambda^{\mathrm{N}}$ solution to $\sigma_{a_{i} \omega}\left(\lambda^{\mathrm{N}}\right)=\sigma_{a_{i} \omega}^{\mathrm{N}}$ :

$$
\lambda^{\mathrm{N}}=\frac{1+(n+1) r}{n(f-r)}
$$

But for this $\lambda^{\mathrm{N}},(15)$ is violated, implying that $\left(\sigma_{a_{i} \omega}\left(\lambda^{\mathrm{N}}\right), \sigma_{a_{i} \theta_{i}}\left(\lambda^{\mathrm{N}}\right)\right)$ lies outside the ellipse. Consequently, the welfare-optimal mechanism on obtained for some $\lambda^{\mathrm{W}}>\lambda^{\mathrm{N}}$ and $\left|\sigma_{a_{i} \omega}^{\mathrm{W}}\right|<$ $\left|\sigma_{a_{i} \theta_{i}}^{\mathrm{W}}\right|$. This implies by the same argument used at the end of the first case that $\left|\sigma_{a_{i} \theta_{i}}^{\mathrm{W}}\right|>\left|\sigma_{a_{i} \theta_{i}}^{\mathrm{N}}\right|$.

Finally, the fact that the absolute values of $\sigma_{a_{i} \theta_{j}}$ are ordered in the same way as for $\sigma_{a_{i} \theta_{i}}$ is an immediate consequence of the second obedience constraint.

The case of strategic complements $(r>0)$ is covered by the next proposition.
Proposition 4.4. For $r \in\left(0, \frac{1}{n-1}\right)$, there exists a unique symmetric mechanism maximizing welfare subject to obedience. In this mechanism, the recommended actions are deterministic given $(\theta, \omega)$, hence

$$
a_{i}=\frac{s \mu_{\omega}+t \mu_{\theta_{i}}}{1-(n-1) r}+\frac{\sigma_{a_{i} \omega}}{\sigma_{\omega}^{2}}\left(\omega-\mu_{\omega}\right)+\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}\left(\theta_{i}-\mu_{\theta_{i}}\right)+\sum_{j \neq i} \frac{\sigma_{a_{i} \theta_{j}}}{\sigma_{\theta_{j}}^{2}}\left(\theta_{j}-\mu_{\theta_{j}}\right)
$$

The expressions for $\left(\sigma_{a_{i} \omega}, \sigma_{a_{i} \theta_{i}}, \sigma_{a_{i} \theta_{j}}\right)$ are identical to (16) with $\lambda$ the solution to (18) in $\left(\frac{r}{f-r},+\infty\right)$.

The proof of Proposition 4.4 follows closely the one of Proposition 4.2 and is provided in Appendix B. The key difference is that the objective function

$$
(n-1) r \sigma_{a_{i} a_{j}}+s \sigma_{a_{i} \omega}+t \sigma_{a_{i} \theta_{i}}
$$

is now increasing in $\sigma_{a_{i} a_{j}}$ so we want to choose it as large as possible, and the first inequality constraint is now binding. This implies that in the optimal mechanism the action recommendations are maximally positively correlated conditioned on $(\theta, \omega)$. Again, the two remaining
free parameters $\left(\sigma_{a_{i} \theta_{i}}, \sigma_{a_{i} \omega}\right)$ are constrained to lie on a portion of a hyperbola due to stationarity. Unlike the case of strategic substitutes, the endpoint of this portion of hyperbola always lies outside the feasible ellipse $\mathcal{E}$. The optimal mechanism is thus always obtained at the intersection of the hyperbola with the boundary of $\mathcal{E}$, and the action recommendation is thus always deterministic conditioned on $(\theta, \omega)$. Intuitively, it is easier to correlate the players' actions positively than negatively by correlating them all to the state $\omega$, and there is no need for additional independent randomization.

Corollary 4.5. In the welfare optimal mechanism described in Proposition 4.2 and Proposition 4.4, the signs of $\sigma_{a_{i} \omega}, \sigma_{a_{i} \theta_{i}}$ and $\sigma_{a_{i} \theta_{j}}$ are the same as $s, t$ and rt, respectively. In particular, the welfare optimal mechanism is always incentive compatible.

Proof. The signs of the covariance coefficients can easily be verified using that $\lambda \geq 0$ and $r \in(-1,0)$ for the expressions in Proposition 4.2, and that $\lambda>\frac{r}{f-r}$ and $r \in(0, f)$ in Proposition 4.4. The fact that $\sigma_{a_{i} \theta_{j}}$ has the same sign at $r t$ implies $r t \sum_{j \neq i} \sigma_{a_{j} \theta_{i}} \geq 0$, which is one of the equivalent characterizations of incentive compatibility in Proposition 3.10.

Remark 4.6. When $r=0$, the mechanisms in Proposition 4.2 and Proposition 4.4 converge to the same mechanism, with $\sigma_{a_{i} \omega}=s \sigma_{\omega}^{2}, \sigma_{a_{i} \theta_{j}}=0, \sigma_{a_{i} \theta_{i}}=t \sigma_{\theta_{i}}^{2}$ and $\delta=0$. That is, the action recommendation $a_{i}$ is independent of $\theta_{j}$ and deterministic conditioned on $\left(\theta_{i}, \omega\right)$ :

$$
a_{i}=\frac{s \mu_{\omega}+t \mu_{\theta_{i}}}{1-(n-1) r}+s\left(\omega-\mu_{\omega}\right)+t\left(\theta_{i}-\mu_{\theta_{i}}\right) .
$$

In such a mechanism, the players can learn the realization $\omega$ of the state after subtracting their own type from the action recommendation. In other words, the mechanism could be equivalently implemented by revealing the state $\omega$ to the each player $i \in[n]$ and letting them play the Bayes-Nash equilibrium conditioned on $\left(\theta_{i}, \omega\right)$.

As in in Proposition 4.3, we obtain the following comparison-proved in Appendix B-of the covariance parameters in the optimal mechanism and in the complete information Nash equilibrium.

Proposition 4.7. For $r \in\left(0, \frac{1}{n-1}\right)$, the welfare-optimal mechanism of Proposition 4.4 satisfies

$$
\left|\sigma_{a_{i} \omega}^{\mathrm{W}}\right|<\left|\sigma_{a_{i} \omega}^{\mathrm{N}}\right|, \quad\left|\sigma_{a_{i} \theta_{i}}^{\mathrm{W}}\right|>\left|\sigma_{a_{i} \theta_{i}}^{\mathrm{N}}\right| \quad \text { and } \quad\left|\sigma_{a_{i} \theta_{j}}^{\mathrm{W}}\right|>\left|\sigma_{a_{i} \theta_{i}}^{\mathrm{N}}\right| .
$$

## 5 Discussion

The key takeaway of Proposition 4.2 is that, when the welfare-maximizing mechanism issues deterministic recommendations, it induces actions that are linear in each player's type, in the state, and in the other players' average type. Therefore, a signal structure that fully reveals the corresponding linear combination of state and competitors' types implements the desired Bayes Correlated Equilibrium action distribution as the unique Bayes Nash Equilibrium (Bergemann and Morris, 2013).

Interestingly, our complete-information benchmark also has a unique BNE which is also linear. In other words, there exists another signal (3) that is deterministic conditional on state and types, which yields the complete-information outcome as the unique BNE. Our comparison results for games with strategic substitutes and complements (Propositions 4.3 and 4.7) show that, relative to the sufficient statistic for the complete information outcome, the welfare-optimal mechanism over-weights the agents' own types and under-weights the common state.

A natural next question is whether this qualitative feature is also present in the seller's revenue-maximizing mechanism? Intuitively, a monopolist information seller wants to limit the players' rents, relative to their welfare-maximizing level. Thus, a monopolist seller will understate the reliance of the mechanism on the private types, but will still qualitatively modify the complete-information signal in the same direction. This is especially surprising in games of strategic complements, where an incomplete intuition would have suggested that over-weighting the common state enables greater coordination than under complete information while also limiting information rents.

Finally, our framework can be tractably used to explore several other extensions. These include asymmetric games, e.g., settings in which a vertically-integrated designer wishes to maximize the payoff of a single player; and regulation interventions, where the players' (firms') actions have implications for downstream players (e.g., consumers) and the designer wishes to limit the firms' market power.

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## A Missing proofs from Section 3

Lemma A. 1 (Lemma 3.3 restated). The covariance matrix K of a symmetric mechanism is positive semidefinite iff

1. $\sigma_{a_{i} \theta_{i}}=\sigma_{a_{i} \theta_{j}}=0$ whenever $\sigma_{\theta_{i}}^{2}=0$, and $\sigma_{a_{i} \omega}=0$ whenever $\sigma_{\omega}^{2}=0$.
2. The following inequality constraints hold with the convention $0 / 0=0$.

$$
\left\{\begin{array}{l}
\frac{1}{\sigma_{\theta_{i}}^{2}}\left(\sigma_{a_{i} \theta_{i}}-\sigma_{a_{i} \theta_{j}}\right)^{2} \leq \sigma_{a_{i}}^{2}-\sigma_{a_{i} a_{j}} \\
\frac{1}{\sigma_{\theta_{i}}^{2}}\left(\sigma_{a_{i} \theta_{i}}+(n-1) \sigma_{a_{i} \theta_{j}}\right)^{2}+\frac{n}{\sigma_{\omega}^{2}} \sigma_{a_{i} \omega}^{2} \leq \sigma_{a_{i}}^{2}+(n-1) \sigma_{a_{i} a_{j}}
\end{array},\right.
$$

Furthermore, there is equality in the first inequality iff $\operatorname{Cov}\left(a_{i}, a_{j} \mid \theta, \omega\right)=\operatorname{Var}\left(a_{i} \mid \theta, \omega\right)$ and in the second inequality iff $\operatorname{Cov}\left(a_{i}, a_{j} \mid \theta, \omega\right)=-\operatorname{Var}\left(a_{i} \mid \theta, \omega\right) /(n-1)$.

Proof. Consider a symmetric mechanism with covariance matrix K. We use the alternative parametrization of the mechanism provided by (5). From $\mathrm{K}_{a \omega}=\sigma_{\omega}^{2} \beta$, we obtain that $\sigma_{\omega}^{2}=0$ implies $\mathrm{K}_{a \omega}=0$. Since $\mathrm{K}_{a \omega} \in \operatorname{span}\left(1_{n}\right)$ by Lemma 3.2, $\sigma_{\omega}^{2} \neq 0$ implies that $\beta \in \operatorname{span}\left(1_{n}\right)$ with

$$
\beta_{i}=\frac{\sigma_{a_{i} \omega}}{\sigma_{\omega}^{2}} .
$$

Similarly, from $\mathrm{K}_{a \theta}=\Gamma \mathrm{K}_{\theta \theta}$ and $\mathrm{K}_{\theta \theta}=\sigma_{\theta_{i}}^{2} I_{n}$ we get that $\sigma_{\theta_{i}}^{2}=0$ implies $\mathrm{K}_{a \theta}=0$. Since furthermore, $\mathrm{K}_{a \theta} \in \operatorname{span}\left(I_{n}, J_{n}\right)$ by Lemma 3.2, $\sigma_{\theta_{i}}^{2} \neq 0$ implies that $\Gamma \in \operatorname{span}\left(I_{n}, J_{n}\right)$ and for all $(i, j) \in[n]^{2}$

$$
\gamma_{i, j}=\frac{\sigma_{a_{i} \theta_{j}}}{\sigma_{\theta_{i}}^{2}}
$$

The only constraint on parametrization (5) is that $\mathrm{K}_{\varepsilon}$ be positive semidefinite. Using that $\mathrm{K}_{\varepsilon}=\mathrm{K}_{a a}-\sigma_{\omega}^{2} \beta \beta^{\top}-\Gamma \Gamma^{\top} \mathrm{K}_{\theta \theta}$, we see that $\mathrm{K}_{\varepsilon}$ is also in $\operatorname{span}\left(I_{n}, J_{n}\right)$ and we compute the onand off-diagonal entries of $\mathrm{K}_{\varepsilon}$.

$$
\begin{gather*}
\left(\mathrm{K}_{\varepsilon}\right)_{i i}=\sigma_{a_{i}}^{2}-\sigma_{\omega}^{2} \beta_{i}^{2}-\sum_{k=1}^{n} \sigma_{\theta_{k}}^{2} \gamma_{i k}^{2}=\sigma_{a_{i}}^{2}-\frac{\sigma_{a_{i} \omega}^{2}}{\sigma_{\omega}^{2}}-\frac{\sigma_{a_{i} \theta_{i}}^{2}}{\sigma_{\theta_{i}}^{2}}-(n-1) \frac{\sigma_{a_{i} \theta_{j}}^{2}}{\sigma_{\theta_{i}}^{2}} \\
\left(\mathrm{~K}_{\varepsilon}\right)_{i j}=\sigma_{a_{i} a_{j}}-\sigma_{\omega}^{2} \beta_{i} \beta_{j}-\sum_{k=1}^{n} \sigma_{\theta_{k}}^{2} \gamma_{i k} \gamma_{j k}=\sigma_{a_{i} a_{j}}-\frac{\sigma_{a_{i} \omega}^{2}}{\sigma_{\omega}^{2}}-2 \frac{\sigma_{a_{i} \theta_{i}} \sigma_{a_{i} \theta_{j}}}{\sigma_{\theta_{i}}^{2}}-(n-2) \frac{\sigma_{a_{i} \theta_{j}}^{2}}{\sigma_{\theta_{i}}^{2}} \tag{24}
\end{gather*}
$$

Note, that the previous expressions remain valid when $\sigma_{\omega}^{2}=0$ (implying $\mathrm{K}_{a \omega}=0$ ), or $\sigma_{\theta_{i}}^{2}=0$ (implying $\mathrm{K}_{a \theta}=0$ ), by adopting the convention $0 / 0=0$. Proposition C. 7 states that $\mathrm{K}_{\varepsilon}$ is positive semidefinite iff $\left(\mathrm{K}_{\varepsilon}\right)_{i i} \geq\left(\mathrm{K}_{\varepsilon}\right)_{i j}$, equivalently with (24),

$$
\sigma_{a_{i}}^{2}-\sigma_{a_{i} a_{j}} \geq \frac{1}{\sigma_{\theta_{i}}^{2}}\left(\sigma_{a_{i} \theta_{i}}-\sigma_{a_{i} \theta_{j}}\right)^{2}
$$

and $\left(\mathrm{K}_{\varepsilon}\right)_{i i} \geq-(n-1)\left(\mathrm{K}_{\varepsilon}\right)_{i j}$, or equivalently,

$$
\begin{aligned}
\sigma_{a_{i}}^{2}+(n-1) \sigma_{a_{i} a_{j}} & \geq n \frac{\sigma_{a_{i} \omega}^{2}}{\sigma_{\omega}^{2}}+\frac{\sigma_{a_{i} \theta_{i}}^{2}}{\sigma_{\theta_{i}}^{2}}+2(n-1) \frac{\sigma_{a_{i} \theta_{i}} \sigma_{a_{i} \theta_{j}}}{\sigma_{\theta_{i}}^{2}}+(n-1)^{2} \frac{\sigma_{a_{i} \theta_{j}}^{2}}{\sigma_{\theta_{i}}^{2}} \\
& =n \frac{\sigma_{a_{i} \omega}^{2}}{\sigma_{\omega}^{2}}+\frac{1}{\sigma_{\theta_{i}}^{2}}\left(\sigma_{a_{i} \theta_{i}}+(n-1) \sigma_{a_{i} \theta_{j}}\right)^{2}
\end{aligned}
$$

Finally, since $\mathrm{K}_{\varepsilon}$ is the covariance matrix of $a \mid \theta, \omega$, the previous two inequalities are equivalent to $\operatorname{Var}\left(a_{i} \mid \theta, \omega\right) \geq \operatorname{Cov}\left(a_{i}, a_{j} \mid \theta, \omega\right)$ and $(n-1) \operatorname{Cov}\left(a_{i}, a_{j} \mid \theta, \omega\right) \geq-\operatorname{Var}\left(a_{i} \mid \theta, \omega\right)$ respectively.

Proposition A. 2 (Prop. 3.10 restated). The mechanism ( $\mu, \mathrm{K}, p$ ) is incentive compatible if and only if it is obedient and for each player $i \in[n]$ :

1. The derivative of the payment function is given by

$$
p_{i}^{\prime}\left(\theta_{i}\right)=\left(\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}-t\right) \mathbb{E}\left[a_{i} \mid \theta_{i}\right]=\left(\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}-t\right)\left(\mu_{a_{i}}+\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}\left(\theta_{i}-\mu_{\theta_{i}}\right)\right)
$$

2. The covariance satisfies $t \sigma_{a_{i} \theta_{i}} \geq t^{2} \sigma_{\theta_{i}}^{2}$, or equivalently by obedience, $r t \sum_{j \neq i} \sigma_{a_{j} \theta_{i}} \geq 0$.

When $\sigma_{\theta_{i}}^{2}=0$, the previous two conditions reduce to $p_{i}^{\prime}\left(\theta_{i}\right)=-t \mu_{a_{i}}$.
Proof. We first compute the action $a_{i}^{\prime}$ that maximizes a player's utility at the second stage conditioned on the received recommendation $a_{i}$ and assuming they reported cost $\theta_{i}^{\prime}$ in the first stage. This yield the following first-order condition, analogous to (9)

$$
a_{i}^{\prime}=s \mathbb{E}\left[\omega \mid a_{i}, \theta_{i}, \theta_{i}^{\prime}\right]+r \sum_{j \neq i} \mathbb{E}\left[a_{j} \mid a_{i}, \theta_{i}, \theta_{i}^{\prime}\right]+t \theta_{i}=s \mathbb{E}\left[\omega \mid a_{i}, \theta_{i}^{\prime}\right]+r \sum_{j \neq i} \mathbb{E}\left[a_{j} \mid a_{i}, \theta_{i}^{\prime}\right]+t \theta_{i}
$$

where the second equality uses that $\omega, a_{j} \perp \theta_{i} \mid a_{i}, \theta_{i}^{\prime}$ for $j \neq i$. But obedience at $\theta_{i}^{\prime}$ implies

$$
s \mathbb{E}\left[\omega \mid a_{i}, \theta_{i}^{\prime}\right]+r \sum_{j \neq i} \mathbb{E}\left[a_{j} \mid a_{i}, \theta_{i}^{\prime}\right]=a_{i}-t \theta_{i}^{\prime} .
$$

The previous two equations combined yield the following expression for the optimal deviation at the second stage

$$
\begin{equation*}
a_{i}^{\prime}=a_{i}+t\left(\theta_{i}-\theta_{i}^{\prime}\right) . \tag{25}
\end{equation*}
$$

Next, we compute $\tilde{u}_{i}\left(\theta_{i}^{\prime} ; \theta_{i}\right)$, the expected utility of player $i$ assuming their true cost is $\theta_{i}$, their reported cost is $\theta_{i}^{\prime}$, and the player deviates optimally (according to (25)) at the second stage. A derivation analogous to the one in the proof of Proposition 3.9 gives

$$
\begin{aligned}
\tilde{u}_{i}\left(\theta_{i}^{\prime} ; \theta_{i}\right) & =\frac{1}{2} \mathbb{E}\left[a_{i}^{\prime 2} \mid \theta_{i}^{\prime}, \theta_{i}\right] \\
& =\frac{1}{2} \mathbb{E}^{2}\left[a_{i} \mid \theta_{i}^{\prime}\right]+t\left(\theta_{i}-\theta_{i}^{\prime}\right) \mathbb{E}\left[a_{i} \mid \theta_{i}^{\prime}\right]+\frac{t^{2}}{2}\left(\theta_{i}-\theta_{i}^{\prime}\right)^{2}+\frac{1}{2} \operatorname{Var}\left(a_{i} \mid \theta_{i}^{\prime}\right) .
\end{aligned}
$$

As for truthfulness, we want $\theta_{i}^{\prime} \mapsto \tilde{u}_{i}\left(\theta_{i}^{\prime} ; \theta_{i}\right)-p_{i}\left(\theta_{i}^{\prime}\right)$ to be maximized at $\theta_{i}$. When $\sigma_{\theta_{i}}^{2}>0$, we recover the same first-order condition

$$
p_{i}^{\prime}\left(\theta_{i}\right)=\left(\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}-t\right) \mathbb{E}\left[a_{i} \mid \theta_{i}\right]=\left(\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}-t\right)\left(\mu_{a_{i}}+\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}\left(\theta_{i}-\mu_{\theta_{i}}\right)\right)
$$

and the second-order condition now becomes

$$
\left(\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}\right)^{2}-2 t \frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}+t^{2}-\left(\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}-t\right) \frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}=t^{2}-t \frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}} \leq 0 .
$$

as desired. Furthermore, since $\sigma_{a_{i} \theta_{i}}-t \sigma_{\theta_{i}}^{2}=r \sum_{j \neq i} \sigma_{a_{j} \theta_{i}}$ by obedience, the previous condition is equivalent to $r t \sum_{j \neq i} \sigma_{a_{j} \theta_{i}} \geq 0$. When $\sigma_{\theta_{i}}^{2}=0, \mathbb{E}\left[a_{i} \mid \theta_{i}^{\prime}\right]=\mu_{a_{i}}$ and the first-order condition reduces to $p_{i}^{\prime}\left(\theta_{i}\right)=-t \mu_{a_{i}}$.

## B Missing proofs from Section 4

Proposition B. 1 (Prop. 4.1 restated). Assume that the prior distribution on players' types $\theta$ is symmetric, then for any Gaussian and obedient mechanism, there exists a symmetric, Gaussian, and obedient mechanism achieving identical welfare.

Proof. Gaussian mechanisms are determined by their mean vector and covariance matrix. Furthermore, the discussion at the beginning of this section shows that the welfare of a Gaussian and obedient mechanism is given by

$$
W(\mu, \mathrm{~K})=\frac{1}{2} \sum_{i=1}^{n}\left(\mu_{a_{i}}^{2}+\sigma_{a_{i}}^{2}\right)
$$

where $\mu$ and K denote respectively the mean and covariance matrix of the mechanism. Denote by $\mathcal{C} \subset \mathbb{R}^{2 n+1} \times \mathcal{M}_{2 n+1}(\mathbb{R})$ the set of pairs $(\mu, \mathrm{K})$ for all Gaussian and obedient mechanisms. This set $\mathcal{C}$ is characterized by:

1. the equality constraint (7) on $\mu$ due to obedience.
2. the $n$ linear equality constraints (8) on K due to obedience.
3. the positive semidefinite constraint that $\mathrm{K} \in \mathcal{S}_{2 n+1}^{+}(\mathbb{R})$.

Since the positive semidefinite cone $\mathcal{S}_{2 n+1}^{+}(\mathbb{R})$ is convex, we easily get from the above characterization that $\mathcal{C}$ is convex. Next, for $(\mu, \mathrm{K}) \in \mathbb{R}^{2 n+1} \times \mathcal{M}_{2 n+1}(\mathbb{R})$, define $(\pi \cdot \mu, \pi \cdot \mathrm{K})$ according to (6). It is easy to see that this defines a linear action of $\mathfrak{S}_{n}$ on $\mathbb{R}^{2 n+1} \times \mathcal{M}_{2 n+1}(\mathbb{R})$. Furthermore, $\mathcal{C}$ is stable under the action of $\mathfrak{S}_{n}$ : indeed, if $(\mu, \mathrm{K})$ is the mean and covariance matrix of a Gaussian and obedient mechanism $(a, \theta, \omega)$, then $(\pi \cdot \mu, \pi \cdot \mathrm{K})$ is the mean and covariance matrix of the permuted mechanism $(\pi \cdot a, \pi \cdot \theta, \omega)$ which is obedient by Lemma 3.4. Finally, for $(\mu, \mathrm{K}) \in \mathcal{C}, \mu$ is constrained to a singleton due to (7), and thus $W(\mu, \mathrm{~K})$ is affine on $\mathcal{C}$.

In summary, the set $\mathcal{C}$ is convex and stable under a linear action of the symmetric group $\mathfrak{S}_{n}$ and $W: \mathcal{C} \rightarrow \mathbb{R}$ is affine. Hence by Lemma C.6, we have $W(\mathcal{C})=W\left(\mathcal{C}_{G}\right)$

Proposition B. 2 (Prop. 4.4 restated). For $r \in\left(0, \frac{1}{n-1}\right)$, there exists a unique symmetric mechanism maximizing social welfare subject to obedience. In this mechanism, the recommended actions are deterministic given $(\theta, \omega)$, hence

$$
a_{i}=\frac{s \mu_{\omega}+t \mu_{\theta_{i}}}{1-(n-1) r}+\frac{\sigma_{a_{i} \omega}}{\sigma_{\omega}^{2}}\left(\omega-\mu_{\omega}\right)+\frac{\sigma_{a_{i} \theta_{i}}}{\sigma_{\theta_{i}}^{2}}\left(\theta_{i}-\mu_{\theta_{i}}\right)+\sum_{j \neq i} \frac{\sigma_{a_{i} \theta_{j}}}{\sigma_{\theta_{j}}^{2}}\left(\theta_{j}-\mu_{\theta_{j}}\right)
$$

Writing $f:=\frac{1}{n-1}$, we have

$$
\begin{gather*}
\sigma_{a_{i} \omega}=\frac{s \sigma_{\omega}^{2}}{2 n} \frac{\lambda n f+1}{\lambda(f-r)-r}, \quad \sigma_{a_{i} \theta_{i}}=t \sigma_{\theta_{i}}^{2} \frac{\lambda n f\left[r^{2}+2(f-r)(1+r)\right]-(2+r) r}{2(1+r)[\lambda n f(f-r)-(1+r) r]},  \tag{26}\\
\sigma_{a_{i} \theta_{j}}=f r t \sigma_{\theta_{i}}^{2} \frac{\lambda n f+(2 r+3)}{2(1+r)[\lambda n f(f-r)-(1+r) r]} . \tag{27}
\end{gather*}
$$

where $\lambda$ is the unique scalar in $\left(\frac{r}{f-r},+\infty\right)$ solution to

$$
\frac{(1+r)^{3} f s^{2} \sigma_{\omega}^{2}}{n^{2}(\lambda(f-r)-r)^{2}}+\frac{r^{2}\left(r^{2}+2 r(1-f)-3 f\right)^{2} t^{2} \sigma_{\theta_{i}}^{2}}{[\lambda n f(f-r)-(1+r) r]^{2}}=f s^{2} \sigma_{\omega}^{2}(1+r)+r^{2} t^{2} \sigma_{\theta_{i}}^{2}
$$

Proof. We follow the same steps as in the proof of Proposition 4.2 and solve for the KKT conditions. The stationarity conditions remain identical and imply the same expressions

$$
\begin{gather*}
\mu=\frac{\lambda(f-r)-r}{1+r}, \quad \sigma_{a_{i} \omega}=\frac{s \sigma_{\omega}^{2}}{2 n} \frac{\lambda n f+1}{\lambda(f-r)-r}, \\
\sigma_{a_{i} \theta_{i}}=t \sigma_{\theta_{i}}^{2} \frac{\lambda n f\left[r^{2}+2(f-r)(1+r)\right]-(2+r) r}{2(1+r)[\lambda n f(f-r)-(1+r) r]}, \tag{28}
\end{gather*}
$$

where $\lambda$ and $\mu$ are the non-negative Lagrange multipliers. Note that the denominator could now vanish in the expression for $\sigma_{a_{i} \omega}$. However, the first stationarity condition implies that $\mu=0$ iff $\lambda=\frac{r}{f-r}$ and the second stationarity condition implies that $s=0$ whenever $\mu=0$. It is easy to see that the choice $\sigma_{a_{i} \omega}=0$ is optimal when $s=0$, hence the expression for $\sigma_{a_{i} \omega}$ in (28) remains valid even for $\lambda=\frac{r}{f-r}$ by adopting the convention $0 / 0=0$.

Since $\mu \geq 0$ we obtain that $\lambda \geq \frac{r}{f-r}>0$ and the first inequality constraint is binding. In contrast to the case $r<0$ in Proposition 4.2, this implies that in order to maximize social welfare, the mechanism aims at maximally positively correlating the players' actions. Using that the first constraint is binding, we can eliminate $\sigma_{a_{i} a_{j}}$ from the second inequality constraint and obtain the same inequality

$$
\begin{equation*}
\frac{(1+r)^{3} f s^{2} \sigma_{\omega}^{2}}{n^{2}[\lambda(f-r)-r]^{2}}+\frac{r^{2}\left(r^{2}+2 r(1-f)-3 f\right)^{2} t^{2} \sigma_{\theta_{i}}^{2}}{[\lambda n f(f-r)-(1+r) r]^{2}} \leq f s^{2} \sigma_{\omega}^{2}(1+r)+r^{2} t^{2} \sigma_{\theta_{i}}^{2} \tag{29}
\end{equation*}
$$

There again, the inequality is valid when $\lambda=\frac{r}{f-r}$ (which implies $s=0$ ) with the convention $0 / 0=0$. Observe that the quantity on the left-hand side in (29) is a continuous and decreasing function for $\lambda>\frac{r}{f-r}$. Denoting by $H$ this function, we see that $\lim _{\lambda \rightarrow+\infty} H(\lambda)=0$ and

- if $s \neq 0$, then $\lim _{\lambda \rightarrow r /(f-r)^{+}} H(\lambda)=+\infty$;
- if $s=0, H(r /(f-r))$ is well-defined but inequality (29) is violated.

In both cases, we have that inequality (29) becomes violated when $\lambda$ is close enough to $\frac{r}{f-r}$. Hence there exists a unique $\lambda>\frac{r}{f-r}$ such that (29) is binding and complementary slackness thus holds at this $\lambda$. The rest of the proof is then identical to the one of Proposition 4.2

Proposition B. 3 (Prop. 4.7 restated). For $r \in\left(0, \frac{1}{n-1}\right)$, the welfare-optimal mechanism described in Proposition 4.4 satisfies

$$
\left|\sigma_{a_{i} \omega}^{\mathrm{W}}\right|<\left|\sigma_{a_{i} \omega}^{\mathrm{N}}\right|, \quad\left|\sigma_{a_{i} \theta_{i}}^{\mathrm{W}}\right|>\left|\sigma_{a_{i} \theta_{i}}^{\mathrm{N}}\right| \quad \text { and } \quad\left|\sigma_{a_{i} \theta_{j}}^{\mathrm{W}}\right|>\left|\sigma_{a_{i} \theta_{i}}^{\mathrm{N}}\right| .
$$

Proof. We follow steps similar to the proof of Proposition 4.3, but as per Proposition 4.4 the minimum possible value for $\lambda$ is now $\lambda_{0}:=r /(f-r)$. We have:

$$
\sigma_{a_{i} \omega}\left(\lambda_{0}\right)=+\infty \quad \text { and } \quad \sigma_{a_{i} \omega}(\infty)=\frac{f}{2(f-r)} s \sigma_{\omega}^{2}
$$

and

$$
\sigma_{a_{i} \theta_{i}}\left(\lambda_{0}\right)=\frac{f(2 f-r)}{2(f-r)^{2}} t \sigma_{\theta_{i}}^{2} \quad \text { and } \quad \sigma_{a_{i} \theta_{i}}(\infty)=\frac{r^{2}+2(f-r)(1+r)}{2(f-r)(1+r)} t \sigma_{\theta_{i}}^{2}
$$

The following inequalities are easy to verify when $r \in(0, f)$ :

$$
\begin{equation*}
\left|\sigma_{a_{i} \theta_{i}}(\infty)\right|<\left|\sigma_{a_{i} \theta_{i}}^{\mathrm{N}}\right|<\left|\sigma_{a_{i} \theta_{i}}\left(\lambda_{0}\right)\right| \quad \text { and } \quad\left|\sigma_{a_{i} \omega}(\infty)\right|<\left|\sigma_{a_{i} \omega}^{\mathrm{N}}\right|<\left|\sigma_{a_{i} \omega}\left(\lambda_{0}\right)\right|, \tag{30}
\end{equation*}
$$

so that in in particular $\left(\sigma_{a_{i} \omega}\left(\lambda_{0}\right), \sigma_{a_{i} \theta_{i}}\left(\lambda_{0}\right)\right)$ lies outside the ellipse and the welfare-maximizing mechanism always lies on the boundary of the ellipse at a point parametrized by some $\lambda^{\mathrm{W}}>$ $\lambda_{0}$. As in Proposition 4.3, let $\lambda^{\mathrm{N}}$ be such that $\sigma_{a_{i} \omega}\left(\lambda^{\mathrm{N}}\right)=\sigma_{a_{i} \omega}^{\mathrm{N}}$ :

$$
\lambda^{\mathrm{N}}=\frac{1+(n+1) r}{n(f-r)} .
$$

Again, we find that for this $\lambda^{\mathrm{N}}$, condition (15) is violated, implying that $\left(\sigma_{a_{i} \omega}\left(\lambda^{\mathrm{N}}\right), \sigma_{a_{i} \theta_{i}}\left(\lambda^{\mathrm{N}}\right)\right.$ ) lies outside the ellipse. Thus, the welfare-optimal $\lambda^{\mathrm{W}}$ satisfies $\lambda^{\mathrm{W}}>\lambda^{\mathrm{N}}$ and as in the proof of Proposition 4.3 this implies that $\left|\sigma_{a_{i} \omega}^{\mathrm{W}}\right|<\left|\sigma_{a_{i} \omega}^{\mathrm{N}}\right|$ and $\left|\sigma_{a_{i} \theta_{i}}^{\mathrm{W}}\right|>\left|\sigma_{a_{i} \theta_{i}}^{\mathrm{N}}\right|$.

## C Linear algebra

The various notions of symmetry considered in this paper (for games, mechanisms, etc.) can be understood in a unified way as a form of invariance under the action of the symmetric group $\mathfrak{S}_{n}$ on the set of players. We first briefly review the definitions and elementary properties of group actions and related notions.

Definition C. 1 (Group action). For a group $G$ and set $S$, a group action of $G$ on $S$ is a map $(g, x) \mapsto g \cdot x$ from $G \times S$ to $S$ such that

1. $e \cdot x$ for all $x \in S$, where $e$ is the identity element of $G$.
2. $g \cdot(h \cdot x)=(g h) \cdot x$ for all $(g, h) \in G^{2}$ and $x \in S$.

It follows from the definition that for all $g \in G$, the map $m_{g}: x \mapsto g \cdot x$ is an element of the symmetric group $\mathfrak{S}_{S}$ with inverse $m_{g^{-1}}$ and that $g \mapsto m_{g}$ is a group homomorphism from $G$ to $\mathfrak{S}_{S}$.

Definition C. 2 (Invariance, stability). Let $G$ be a group acting on a set $S$. We say that $x \in S$ is $G$-invariant if $g \cdot x=x$ for all $g \in G$. Similarly, a function $f$ defined on $S$ is $G$-invariant if $f(g \cdot x)=f(x)$ for all $(x, g) \in S \times G$. Finally, a subset $T \subseteq S$ is $G$-stable if $g \cdot x$ for all $(x, g) \in T \times G$.

Remark C.3. It is easy to check that an action of a group $G$ on a set $S$ induces an action of $G$ on the set of all functions with domain $S$, by defining $g \cdot f: x \mapsto f\left(g^{-1} \cdot x\right)$ for all $g \in G$ and $f$ whose domain is $S$. Consequently, a function $f$ defined on $S$ is $G$-invariant under this induced action iff it is $G$-invariant in the sense of Definition C.2.

When a group $G$ acts on a vector space $V$, it is natural to require that the action be compatible with the linear structure of $V$, as defined next.

Definition C. 4 (Linear action). For a group $G$ acting on a vector space $V$, we say that the action is linear if for all $g \in G$, the map $m_{g}: v \mapsto g \cdot v$ is an endomorphism of $V$. Consequently, $g \mapsto m_{g}$ is a group homomorphism from $V$ to the general linear group GL $(V)$.

Example C.5. For $n \geq 1$, the symmetric group $\mathfrak{S}_{n}$ acts linearly on $\mathbb{R}^{n}$ by defining $\pi \cdot x=$ $\left(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)}\right)$ for $x \in \mathbb{R}^{n}$ and permutation $\pi \in \mathbb{R}^{n}$. For a permutation $\pi \in \mathfrak{S}_{n}$, define the permutation matrix $P_{\pi} \in \mathcal{M}_{n}(\mathbb{R})$ whose $(i, j)$ entry is $\left(P_{\pi}\right)_{i j}=\mathbf{1}\{\pi(j)=i\}$, then the group action $(\pi, x) \mapsto \pi \cdot x$ can equivalently be defined as the (left) multiplication by the matrix $P_{\pi}$.

Similarly the symmetric group $\mathfrak{S}_{n}$ acts linearly on $\mathcal{M}_{n}(\mathbb{R})$ by defining $\pi \cdot A=P_{\pi} A P_{\pi}^{\top}$ for a permutation $\pi \in \mathfrak{S}_{n}$ and matrix $A \in \mathcal{M}_{n}(\mathbb{R})$.

The following lemma captures the core of the symmetrization argument used in the body of a paper, showing that restricting our study to symmetric mechanisms is without loss of generality. Although the lemma is elementary, we were not able to find a suitable formulation in the literature.

Lemma C. 6 (Symmetrization). Let $G$ be a finite group acting linearly on a real vector space $V$, and let $\mathcal{C}$ be a convex and $G$-stable subset of $V$. Consider $x \in \mathcal{C}$ and define

$$
x_{G}:=\frac{1}{|G|} \sum_{g \in G} g \cdot x .
$$

Then $x_{G}$ is a $G$-invariant element of $\mathcal{C}$, and $f\left(x_{G}\right) \leq f(x)$ for every convex and $G$-invariant function $f: \mathcal{C} \rightarrow \mathbb{R}$. Consequently, for such a function $f, \inf f(\mathcal{C})=\inf f\left(\mathcal{C}_{G}\right)$, where $\mathcal{C}_{G}$ denotes the $G$-invariant elements of $\mathcal{C}$. If $f$ is furthermore affine, we have $f(\mathcal{C})=f\left(\mathcal{C}_{G}\right)$.

Proof. Because $\mathcal{C}$ is $G$-stable, $g \cdot x \in \mathcal{C}$ for all $g \in G$, hence $x_{G} \in \mathcal{C}$ by convexity of $\mathcal{C}$. Furthermore, $x_{G}$ is $G$-invariant: indeed, for each $h \in G$

$$
h \cdot x_{G}=\frac{1}{|G|} \sum_{g \in G}(h g) \cdot x=\frac{1}{|G|} \sum_{g \in G} g \cdot x=x_{G}
$$

where the first equality uses that $G$ acts linearly on $V$ and the second equality uses that $g \mapsto h g$ is a permutation of $G$. Finally,

$$
\begin{equation*}
f\left(x_{G}\right)=f\left(\frac{1}{|G|} \sum_{g \in G} g \cdot x\right) \leq \frac{1}{|G|} \sum_{g \in G} f(g \cdot x)=f(x) \tag{31}
\end{equation*}
$$

where the inequality is by convexity of $f$ and the second equality uses that $f$ is $G$-invariant.
The claim $\inf f(\mathcal{C})=\inf f\left(\mathcal{C}_{G}\right)$ then follows immediately when $f$ is convex and $G$-invariant, and when $f$ is affine, (31) becomes an equality, implying that $f(\mathcal{C})=f\left(\mathcal{C}_{G}\right)$.

Proposition C.7. For $n \geq 2$ and $(a, b) \in \mathbb{R}^{2}$, define $J_{n}(a, b):=(a-b) I_{n}+b J_{n}$, the matrix in $\mathcal{M}_{n}(\mathbb{R})$ whose diagonal entries are all equal to a and off-diagonal entries equal to $b$.

1. The determinant of $J_{n}(a, b)$ is $\operatorname{det} J_{n}(a, b)=(a-b)^{n-1}(a+(n-1) b)$ and its inverse is

$$
J_{n}^{-1}(a, b)=\frac{J_{n}(a+(n-2) b,-b)}{(a-b)(a+(n-1) b)}
$$

whenever $a \neq b$ and $a \neq-(n-1) b$.
2. For a matrix $A \in \mathcal{M}_{n}(\mathbb{R}), P_{\pi} A=A P_{\pi}$ for each $\pi \in \mathfrak{S}_{n}$ iff $A=J_{n}(a, b)$ for some $(a, b) \in \mathbb{R}^{2}$. In other words, $\operatorname{span}\left(I_{n}, J_{n}\right)$ is the commutant of $\left\{P_{\pi} \mid \pi \in \mathfrak{S}_{n}\right\}$.
3. $\operatorname{span}\left(I_{n}, J_{n}\right)$ is a commutative algebra.
4. The matrix $J_{n}(a, b)$ is positive semidefinite iff $-a /(n-1) \leq b \leq a$.

Proof. The matrix $J_{n}$ is symmetric, hence we can diagonalize it as $J_{n}=U D U^{\top}$ where $U \in$ $\mathcal{M}_{n}(\mathbb{R})$ is orthogonal and $D \in \mathcal{M}_{n}(\mathbb{R})$ is diagonal. Furthermore, $J_{n}$ has rank 1 , hence its kernel has dimension $n-1$ and $D$ has $n-1$ zeros on its diagonal. Finally, $J_{n} 1_{n}=n 1_{n}$ shows that the remaining eigenvalue of $J_{n}$ is $n$, with an associated eigenspace of dimension 1 . We can therefore write $D=\operatorname{diag}(0, \ldots, 0, n)$. Hence, for each $(a, b) \in \mathbb{R}^{2}$
$J_{n}(a, b)=(a-b) I_{n}+b J_{n}=U\left((a-b) I_{n}+b D\right) U^{\top}=U\left[\begin{array}{cccc}a-b & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & a-b & 0 \\ 0 & \cdots & 0 & a+(n-1) b\end{array}\right] U^{\top}$

1. We directly compute from (32), det $J_{n}(a, b)=(a-b)^{n-1}(a+(n-1) b)$, showing that $J_{n}(a, b)$ is nonsingular iff $a \neq b$ and $a \neq-(n-1) b$. For such a pair $(a, b)$, we look for an inverse of $J_{n}(a, b)$ of the form $c I_{n}+d J_{n}$ for some $(c, d) \in \mathbb{R}^{2}$. Using the identity $J_{n}^{2}=n J_{n}$, we get

$$
\left((a-b) I_{n}+b J_{n}\right)\left(c I_{n}+d J_{n}\right)=(a-b) c I_{n}+(b c+d(a+(n-1) b)) J_{n} .
$$

Hence, $c I_{n}+d J_{n}$ is an inverse of $J_{n}(a, b)$ iff

$$
c=\frac{1}{a-b} \quad \text { and } \quad d=-\frac{b}{(a-b)(a+(n-1) b)}
$$

yielding the stated expression for $J_{n}^{-1}(a, b)$.
2. Let $A$ be a matrix commuting with all permutation matrices. For $i \in[n]$ and $j \in[n] \backslash\{i\}$, let $\tau$ be the transposition that swaps $i$ and $j$, and let $e_{i}$ be the $i$ th standard basis vector. The condition $P_{\tau} A e_{i}=A P_{\tau} e_{i}$ implies $a_{\tau(k) i}=a_{k j}$ for all $k \in[n]$. Writing this for $k \in\{i, j\}$ and $k \in[n] \backslash\{i, j\}$ (when $n \geq 3$ ) yields

$$
\begin{equation*}
a_{i i}=a_{j j}, \quad a_{i j}=a_{j i}, \quad a_{k i}=a_{k j} . \tag{33}
\end{equation*}
$$

The first equality shows that all diagonal entries are equal and the last equality shows that for each row $k$, all off-diagonal entries in row $k$ are equal. To compare off-diagonal entries in different rows, consider $i^{\prime} \neq i$ and $j^{\prime} \neq i^{\prime}$, then

$$
a_{i j}=a_{i i^{\prime}}=a_{i^{\prime} i}=a_{i^{\prime} j^{\prime}}
$$

where the first and last equalities used the third equality in (33) and the middle equality used the second equality in (33).
3. By definition, $\operatorname{span}\left(I_{n}, J_{n}\right)$ is a subspace of $\mathcal{M}_{n}(\mathbb{R})$. The fact that is a commutative algebra follows immediately from the fact that $I_{n}$ and $J_{n}$ commute and the identity $J_{n}^{2}=n J_{n}$.
4. The eigenvalues of $J_{n}(a, b)$ can be read directly from (32). The matrix $J_{n}(a, b)$ is positive semidefinite iff all its eigenvalues are nonnegative, that is, $a \geq b$ and $a \geq-(n-1) b$.


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[^1]:    ${ }^{1}$ Transfers inside the organization can be interpreted as formal or relational continuation-value transfers within a relationship. Literal monetary transfers are more intuitive in the case of a data broker selling information to an industry.

[^2]:    ${ }^{2}$ In what follows, we use the variance $\sigma_{a_{i}}^{2}$, and not the standard variation $\sigma_{a_{i}}$ as the optimization variable. Note that we do not need to add the constraint $\sigma_{a_{i}}^{2} \geq 0$ to the optimization problem, since it is implied by the covariance matrix being positive semidefinite. This can be seen, for example, by multiplying the first inequality constraint by $n-1$ and adding it to the second inequality constraint.

