COMS E6998-9: Algorithmic Techniques for Massive Data	Oct 15, 2015

Lecture 12 – More LSH, Data-Dependent Hashing

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1 Time-space Trade-offs

Below we present a table of LSH algorithms using different space and time.

Type	Space	Time	Comment	Ref
Low space and High query time	$\approx n$	n^σ	$\sigma = 2.09/c$	[Ind'01,Pan'06]
	$\approx n$	n^σ	$\sigma = O(1/c^2)$	[AI'06]
Medium space and Medium query time	$n^{1+\rho}$	n^ρ	$\rho = 1/c$	[IM'98,DIIM'04]
	$n^{1+\rho}$	n^ρ	$\rho = 1/c^2$	[AI'06]
	$n^{1+\rho}$	n^ρ	$\rho \geq 1/c^2$	[MNP'06,OWZ'11]
	$n^{1+o(1+1/c^2)}$	$\omega(1)$ memory lookup		[PTW'08,PTW'10]
High space and Low query time	n^{4/ϵ^2}	$O(dlogn)$(1 mem lookup)	$c = 1 + \epsilon$	[KOR'98,IM'08,Pan'06]
	$n^{o(1/\epsilon^2)}$	$\omega(1)$ memory lookup		[AIP'06]

2 Near-linear Space for $\{0,1\}^d$

[Indyk'01,PanIgrahy'06]

- General idea: Sample a few bucket in the same hash table.
- Setting:
  - Close: $r = \frac{d}{2c}$ [Note that from last lecture $P_1 = 1 - \frac{r}{d} = 1 - \frac{1}{2c}$]
  - Far: $cr = \frac{d}{2}$ [Note that from last lecture $P_2 = 1 - \frac{cr}{d} = \frac{1}{2}$]
- Algorithm:
  - Use on hash table with $k = \frac{\log n}{\log 1/P_2} = \alpha \ln n$

    [Note that since $P_2 = 1/2$ here α is a constant]
  - On query q:
    * Compute $w = g(p) \in \{0,1\}^k$
    * Define w' such that starting from w, flip each w_j with probability $1 - P_1$
    * Lookup bucket $g(w')$ and compute distance to all points there
    * Repeat $R = n^\sigma$ times, stop if found an approximate near neighbor

Theorem. *For* $\sigma = \Theta(\frac{\log c}{c})$, *we have*

- *Pr[find an approximate near neighbor]* ≥ 0.1
- *Expected runtime:* $O(n^{\sigma})$

Proof. Let p^* be the near neighbor, then we know that $\|q - p^*\| \leq r$. Define $w = g(q), t = \|w - g(p^*)\|_1$.

Claim 1. $Pr[t \leq \frac{k}{c}] \geq \frac{1}{2}$

Proof. Note that $E[t] = \frac{r}{d}k = \frac{k}{2c}$.

Hence by Markov Inequality, $Pr[t \leq \frac{k}{c}] \geq 1 - \frac{k}{2c}/\frac{k}{c} = \frac{1}{2}$ □

Claim 2. $Pr[w' = g(p) | \|q - p\|_1 \geq \frac{d}{2}] \leq \frac{1}{n}$

Proof.

$$\begin{aligned} Pr[\text{Collision}] &\leq (P_1P_2 + (1 - P_1)(1 - P_2))^k \\ &= (P_2(P_1 + 1 - 1) + (1 - P_1)(1 - P_2))^k \\ &= (P_2 + (1 - P_1)(1 - 2P_2))^k \\ &\leq P_2^k = 1/n \end{aligned}$$

□

Claim 3. $Pr[w' = g(p^*) |$ *Claim 1*$] \geq n^{-\sigma}$

Proof.

$$\begin{aligned} Pr[w' = g(p^*)|\text{Claim 1}] &= (1 - P_1)^t P_1^{k-t} \\ &\geq (1 - (1 - \frac{1}{2c}))^{k/c}(1 - \frac{1}{2c})^{k(1-1/c)} \\ &\geq (\frac{1}{2c})^{\frac{k}{c}} e^{-\frac{1}{2c}k} \\ &\geq n^{-\frac{\Theta(1)\lg c}{c}} n^{-\frac{\alpha}{2c}} \\ &\geq n^{-\sigma} \end{aligned}$$

□

Since if $w' = g(p^*)$ for at least one w', we are guaranteed to output either p^* or an approximate near neighbor, we are done by Claim 3. □

3 Beyond LSH

Below we give a contrast of LSH algorithms and other algorithm.

In Hamming Space

Type	Space	Time	Comment	$c = 2$	Reference
LSH	$n^{1+\rho}$	n^{ρ}	$\rho = 1/c$	$\rho = 1/2$	[IM'98]
			$\rho \geq 1/c$		[MNP'06,OWZ'11]
Non-LSH	$n^{1+\rho}$	n^{ρ}	$\rho \approx \frac{1}{2c-1}$	$\rho = 1/3$	[AINR'14,AR'15]

In Euclidean Space

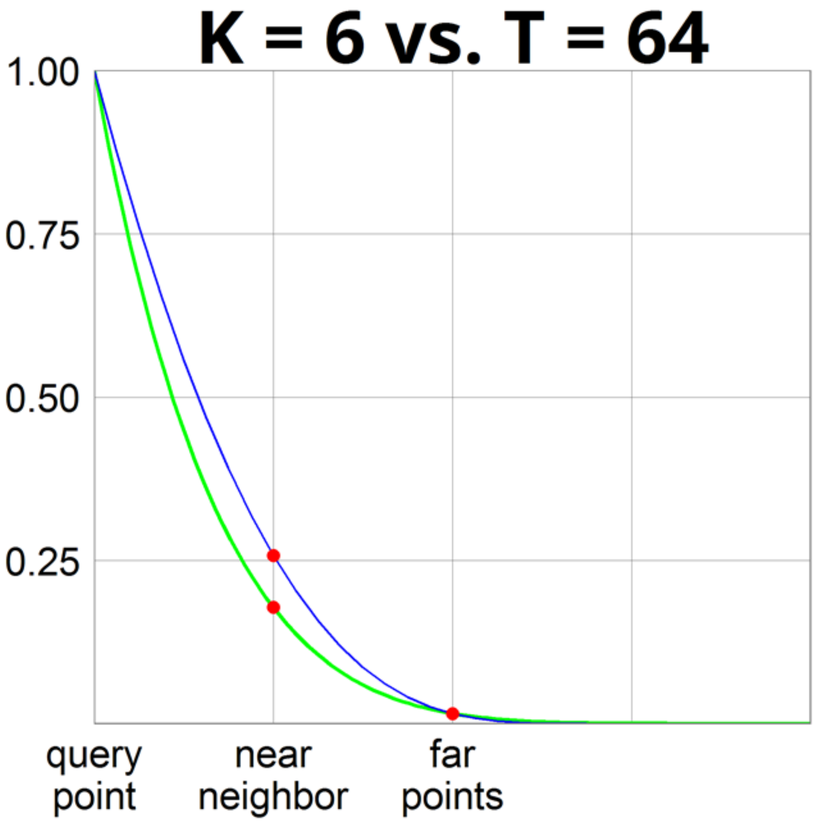
Type	Space	Time	Comment	$c = 2$	Reference
LSH	$n^{1+\rho}$	n^{ρ}	$\rho \approx 1/c^2$	$\rho = 1/4$	[AI'06]
			$\rho \geq 1/c^2$		[MNP'06,OWZ'11]
Non-LSH	$n^{1+\rho}$	n^{ρ}	$\rho \approx \frac{1}{2c^2-1}$	$\rho = 1/7$	[AINR'14,AR'15]

4 Data-dependent hashing

[A.-Indyk-Nguyen-Razenshteyn'14,A.-Razenshteyn'15]

- General idea: Using a random has function, which is chosen after seeing the given dataset
- Feature: Efficiently computable
- Components:
  - Nice geometric structure (has better data partition)
  - Reduction to such structure (depends on the data)
- Nice geometric structure:
  - Like a random dataset on a sphere s.t. random points at distance $\approx cr$
  - Query: At angle 45' from near-neighbor
- Alg 1: Hyperplanes[Charikar'02]
  - We sample unit r uniformly, hash p into $sgn < r, p >$,
    $Pr[h(p) = h(q)] = 1 - \alpha/\pi$, where α is the angle between p and q
  - $P_1 = 3/4, P_2 = 1/2$
  - $\rho \approx 0.42$
- Alg 2: Voronoi[A.-Indyk-Nguyen-Razenshteyn'14] based on [Karger-Motwani-Sudan'94]
  - Sample T i.i.d. standard d-dimensional Gaussians $g_2, g_2, .., g_T$.
  - Hash p into $h(p) = argmax_{1 \leq i \leq T} < p, g_i >$
  - Note that it is simply Hyperplane LSH when $T = 2$

- Hyperplane VS Voronoi
  - Hyperplane with $k = 6$ hyperplanes , which means we partition space into $2^6 = 64$ pieces
  - Voronoi with $T = 2^k = 64$ vectors. $\rho = 0.18$



  - In Hyperplane algorithm we partition into grids while in Voronoi we partition into sphere

5 Nearest Neighbor Search: Conclusion

- Approach 1: Via sketches
- Approach 2: Locality Sensitive Hashing
  - Use Random Space Partitions
  - Algorithm with Better Space Bound
  - Use Data-dependent hashing