Lecture 22:
Linearity Testing
Sparse Fourier Transform

Administrivia, Plan

- Thu: no class. Happy Thanksgiving!
- Tue, Dec 1^{st}:
  - Sergei Vassilvitskii (Google Research) on MapReduce model and algorithms
- I'm away until next Thu, Dec 3rd
  - Office hours: Tue 2:30-4:30, Wed 4-6pm
- Plan:
  - Linearity Testing (finish)
  - Sparse Fourier Transform

Last lecture

- Linearity Testing:
  - $f: \{-1, +1\}^n$ is linear iff for any $x, y \in \{-1, +1\}^n$, we have:
    - $f(x) \cdot f(y) = f(x \oplus y)$
- Test: repeat $O(1/\epsilon)$ times
  - Pick random x, y
  - Verify that $f(x) \cdot f(y) = f(x \oplus y)$
- Main Theorem:
  - If f is ϵ-far from linearity, then $\Pr[test\ fails] \geq \epsilon$

Linearity Testing

- Remaining Lemma:
  - Let $T_{xy} = 1$ iff $f(x) \cdot f(y) = f(x \oplus y)$
  - $\Pr[T_{xy} = 1] = \frac{1}{2} + \frac{1}{2} \sum_{S \subseteq [n]} \hat{f}_S^3$
  - Where $\hat{f}_S = \langle f, \chi_S \rangle$ for $\chi_S(x) = \prod_{i \in S} x_i$

Discrete Fourier Transform

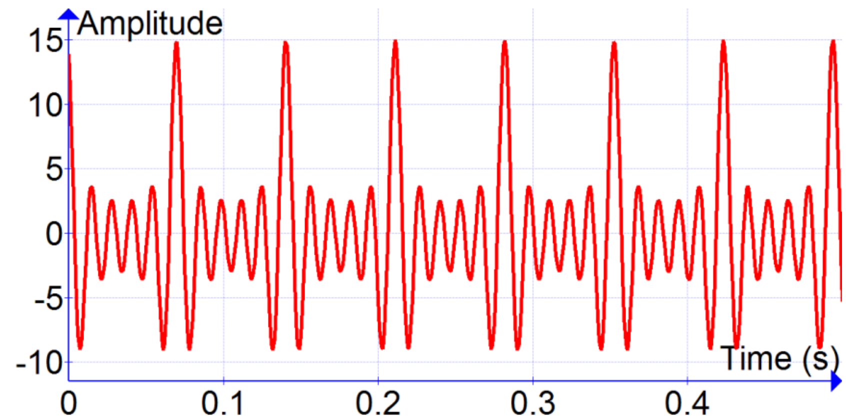
$\hat{x}_i = \frac{1}{n}\sum_{j\in[n]} x_j \omega^{-ij}$,
$\omega = e^{-2\pi\mathbf{i}/n}$ (n^{th} root of unity)
$x_i = \sum_{j\in[n]} \hat{x}_j\, \omega^{ij}$

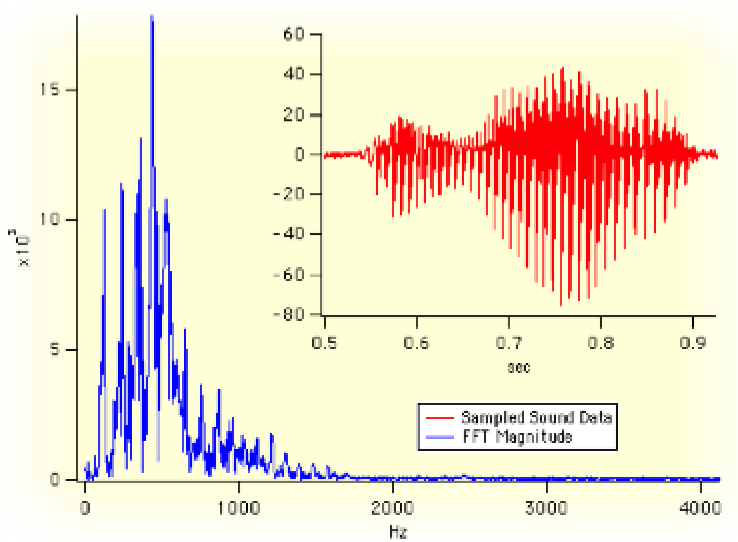
- Consider
  - $f: [n] \to \Re$
    - also special case of more general setting:
    - $f: [n]^d \to \Re$
- Will call such function: $x = (x_1, \ldots x_n)$
- Fourier transform:
  - $\hat{x}_i = \frac{1}{n}\sum_{j\in[n]} x_j \omega^{-ij}$,
  - where $\omega = e^{-2\pi\mathbf{i}/n}$ is the n^{th} root of unity
  - $x_i = \sum_{j\in[n]} \hat{x}_j\, \omega^{ij}$
  - Assume: n is power of 2

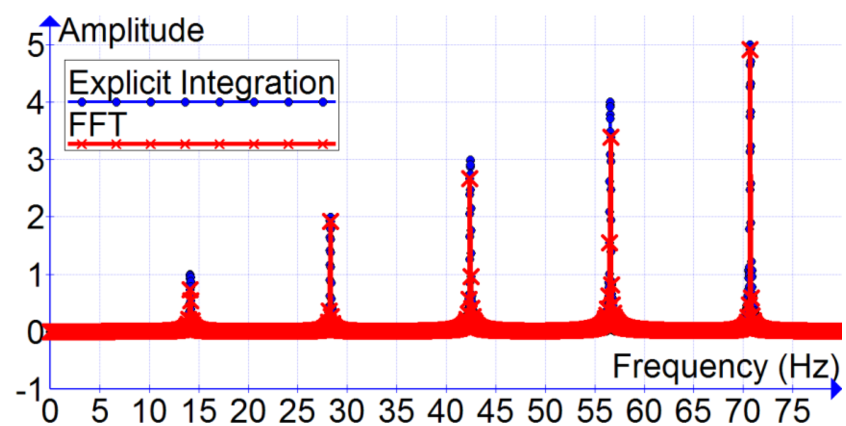
Why important?

$$\hat{x}_i = \frac{1}{n} \sum_{j \in [n]} x_j \omega^{-ij},$$
$$\omega = e^{-2\pi \mathbf{i}/n} \text{ (n^{th} root of unity)}$$
$$x_i = \sum_{j \in [n]} \hat{x}_j \, \omega^{ij}$$

- Imaging
  - MRI, NMR
- Compression:
  - JPEG: retain only high Fourier coefficients
- Signal processing
- Data analysis
- …







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Computing

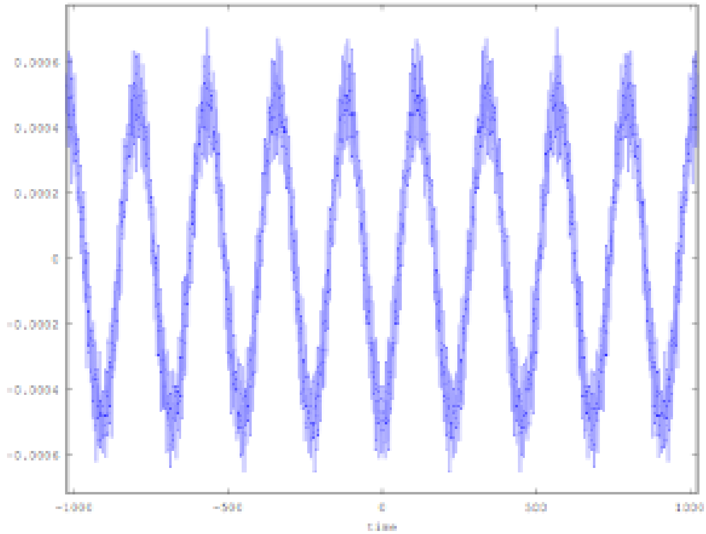
$\hat{x}_i = \frac{1}{n}\sum_{j\in[n]} x_j \omega^{-ij}$,
$\omega = e^{-2\pi\mathbf{i}/n}$ (n^{th} root of unity)
$x_i = \sum_{j\in[n]} \hat{x}_j \, \omega^{ij}$

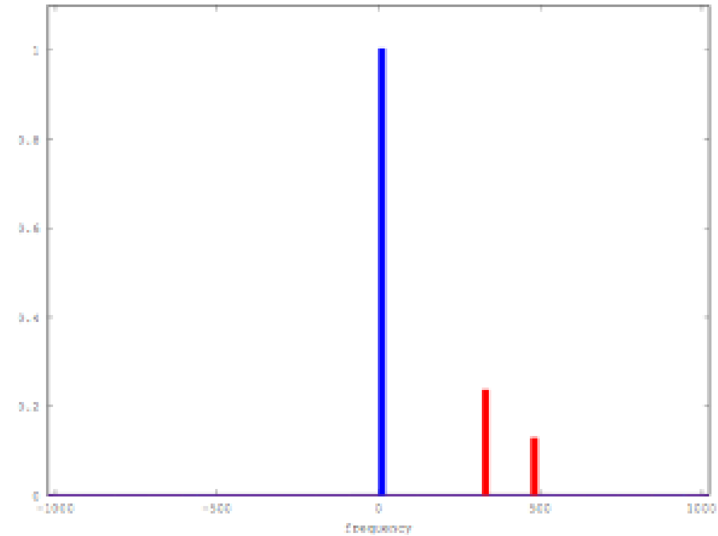
- Naively:
  - $O(n^2)$
- Fast Fourier Transform:
  - $O(n \log n)$ time
  - [Cooley-Tukey 1964]
  - [Gauss 1805]
- One of the biggest open questions in CS:
  - Can we do in $O(n)$ time?

Sparse Fourier Transform

$$\hat{x}_i = \frac{1}{n}\sum_{j\in[n]} x_j \omega^{-ij},$$
$$\omega = e^{-2\pi\mathbf{i}/n} \text{ } (n^{th} \text{ root of unity})$$
$$x_i = \sum_{j\in[n]} \hat{x}_j\, \omega^{ij}$$

- Many signals represented well by sparse Fourier transform
- If $\hat{x}$ is sparse,
  - Can we do better?
- YES!
  - $O(k \cdot \log^2 n)$ time possible, assuming k non-zero Fourier coefficients!
  - Sublinear time: just sample a few positions in x
- Even when $\hat{x}$ is approximately sparse





Similar to Compressed Sensing

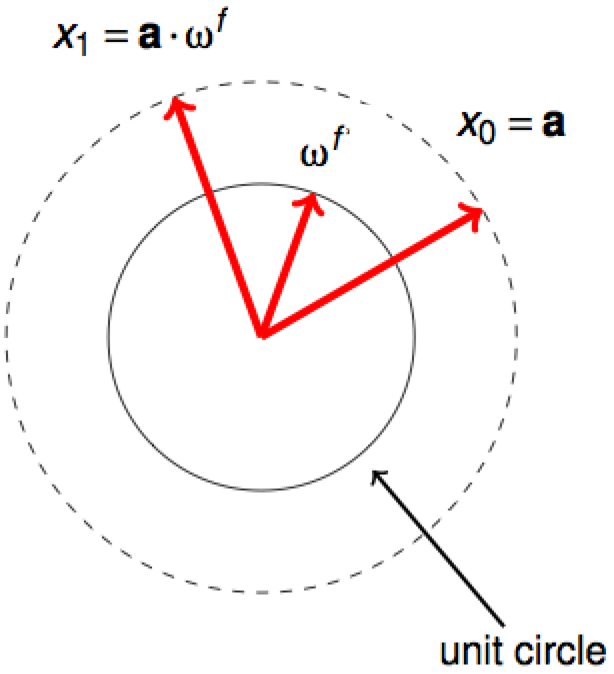
$\hat{x}_i = \frac{1}{n}\sum_{j\in[n]} x_j \omega^{-ij}$,
$\omega = e^{-2\pi\mathbf{i}/n}$ (n^{th} root of unity)
$x_i = \sum_{j\in[n]} \hat{x}_j\, \omega^{ij}$

- Sparse Fourier Transform
  - Sparse: $\hat{x} = Fx$
  - Access: $x = F^{-1}\hat{x}$ of dimension n
  - $F, \hat{F}$ = concrete matrix
- Compressed Sensing
  - Sparse: $x \in \Re^n$
  - Access: $y = Ax \in \Re^m$, where $m = O(k \log n)$
  - A is usually designed (though sometimes: random rows of the Fourier matrix)

Warm-up: $k = 1$

$$\hat{x}_i = \frac{1}{n}\sum_{j\in[n]} x_j \omega^{-ij},$$
$$\omega = e^{-2\pi \mathbf{i}/n} \ (n^{th} \text{ root of unity})$$
$$x_i = \sum_{j\in[n]} \hat{x}_j \, \omega^{ij}$$

- Assume $\hat{x}$ is exactly 1-sp.
  - $\hat{x}_f \neq 0$
- Problem:
  - How many queries into x ?
- Algorithm:
  - Sample x_0, x_1
- $x_0 = a\omega^{0f} = a$
- $x_1 = a\omega^f$
- $\omega^f = x_1/x_0$
  - Can read off the frequency f



What about noise?

$\hat{x}_i = \frac{1}{n} \sum_{j \in [n]} x_j \omega^{-ij}$,
$\omega = e^{-2\pi \mathbf{i}/n}$ (n^{th} root of unity)
$x_i = \sum_{j \in [n]} \hat{x}_j \, \omega^{ij}$

- $x_i = a\omega^{if} + noise$
- Problem:
  - Find y s.t. 1-sparse (in Fourier domain)
  - Best approximation to x
  - $||\hat{x} - \hat{y}||_2 \leq C \cdot \min_{\hat{y}:\, 1-sparse} ||\hat{x} - \hat{y}||$
  - $||x - y||_2 \leq C \cdot \min_{y:\, 1-sparse} ||x - y||$

Parseval's:

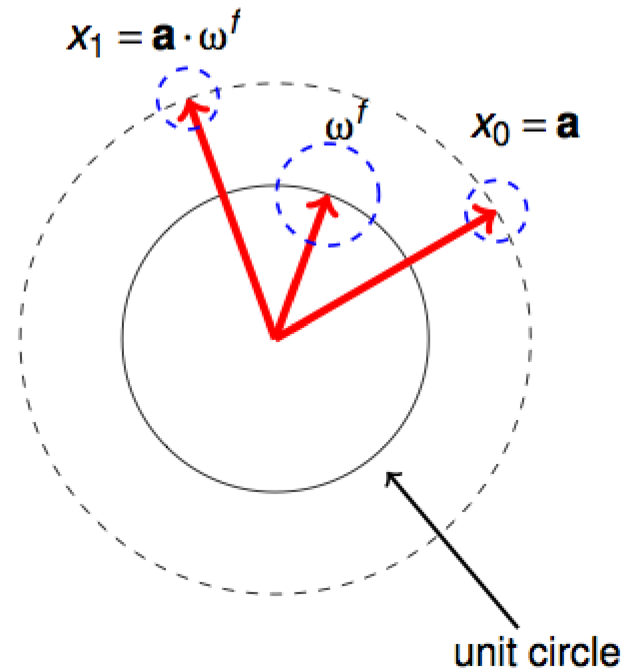
$$||\hat{z}||^2 = \frac{1}{n} ||z||^2 = E_j\left[\left|z_j\right|^2\right]$$

Use for $z = x - y$!

- Will assume "error" is ϵ fraction:
  - $\min_{\hat{y}:\, 1-sparse} ||\hat{x} - \hat{y}||^2 = \sum_{j \neq f} |\hat{x}_j|^2 \leq \epsilon^2 \cdot \widehat{x_f}^2 = \epsilon^2 a^2$
  - $E_j\left[\left|x_j - y_j\right|^2\right] \leq \epsilon^2 a^2$ (Parserval's)
  - Interesting when $\epsilon \ll 1$

Re-use $k = 1$ algorithm?

- Suppose: $a = 1$
- $x_0 = 1 + \epsilon$
- $x_1 = \omega^f + \epsilon\omega^q$
- So: $\frac{x_1}{x_0} = \frac{1}{1+\epsilon}(\omega^f + \epsilon\omega^q)$
- Error in frequency!
  - Will recover $y = \omega^g$ for $g \neq f$
  - Thus $||\hat{x} - \hat{y}|| \geq ||\hat{x}_f|| = 1$ instead of $O(\epsilon)$...
- Good news: error bounded, up to ϵn



Algorithm for $k = 1$ + noise

- $x_i = \omega^{if} + noise$
- Will find f by binary search!
- Bit 0:
  - $f = 2f_1 + b$ for $b \in \{0,1\}$
- Claim: for pure signal $y_i = \omega^{if}$:
  - $y_{n/2} = y_0 \cdot (-1)^b$
  - $y_{n/2+r} = y_r(-1)^b$
- Proof:
  - $y_{n/2} = \omega^{f \cdot n/2} = (-1)^f = (-1)^{2f_1} \cdot (-1)^b = (-1)^b y_0$
  - $y_{n/2+r} = \omega^{f \cdot n/2 + fr} = (-1)^f \omega^{fr} = (-1)^b y_r$
- What about noise?

Bit 0 with noise

- We have:
  - $x_i = \omega^{i\cdot(2f_1+b)} + noise$
  - $y_i = \omega^{i\cdot(2f_1+b)}$
  - Claim: $y_{n/2+r} = y_r(-1)^b$
  - $E_j\left[\left|x_j - y_j\right|^2 \le \epsilon^2\right]$ (Parseval's)

If $b = 0$:

$$\left|y_{n/2+r} + y_r\right| = 2$$
$$\left|y_{n/2+r} - y_r\right| = 0$$

If $b = 1$:

$$\left|y_{n/2+r} + y_r\right| = 0$$
$$\left|y_{n/2+r} - y_r\right| = 2$$

- Algorithm:
  - For t times:
    - Pick random $r \in [n]$
    - Check $\left|x_{n/2+r} + x_r\right| > |x_{n/2+r} - x_r|$: then $b = 0$
    - Otherwise $b = 1$
  - Take majority vote
- Claim: output the right b with $1 - 2^{-\Omega(t)}$ probability
- Proof:
  - Each test:
    - $x_{n/2+r}, x_r$ are within $5\epsilon^2$ of $y_{n/2+r}, y_r$ with probability $1 - 2 \cdot 1/5$ (Markov)
    - Hence test works with at least 0.6 probability
  - Majority of t tests work with $1 - 2^{-\Omega(t)}$ probability (Chernoff bound concentration)

Bit 1

- Reduce to bit 0 case!
- We have
  - $x_i = \omega^{i(2f_1+b)} + noise$
  - $y_i = \omega^{i(2f_1+b)}$
- Suppose $b = 0$:
  - $y_i = \omega^{i \cdot 2f_1} = (\omega^2)^{if_1} = \left(\omega_{n/2}\right)^{if_1}$
    - where $\omega_{n/2}$ is the $(n/2)^{th}$ root of unity
  - Same problem as for Fourier transform over $[n/2]$!
- Suppose $b = 1$?
  - Define $y_i' = y_i \omega^{-i}$
    - Then $y_i' = \omega^{if-i} = \omega^{i(2f_1+1-1)} = \omega^{i\cdot(2f_1)}$
    - Just shifts all frequencies down by one!
  - Continue as above for $x_i' = x_i \omega^{-i}$
    - Note: we compute x_i' on the fly when whenever we query some x_i

Overall algorithm to recover f

- $x_i = \omega^{if} + noise$
  - Where $f = b_0 + b_1 \cdot 2^1 + b_2 \cdot 2^2 + \cdots + b_{\lg\frac{n}{2}} \cdot \frac{n}{2}$
- Algorithm:
  - Learn b_0: take majority of t trials of
    - Pick random r
    - Check: $\left|x_{n/2+r} + x_r\right| > \left|x_{n/2+r} - x_r\right|$
      - Then set $b_0 = 0$
  - Learn b_1: take majority of t trials of
    - Pick random r
    - Check: $\left|\omega^{n/4 \cdot b_0} x_{n/4+r} + x_r\right| > \left|\omega^{n/4 \cdot b_0} x_{n/4+r} - x_r\right|$
      - Then set $b_1 = 0$
  - Learn b_2: take majority of t trials of
    - Pick random r
    - Check: $\left|\omega^{n/8 \cdot (b_0 + 2b_1)} x_{n/8+r} + x_r\right| > \left|\omega^{n/8 \cdot (b_0 + 2b_1)} x_{n/8+r} - x_r\right|$
      - Then set $b_2 = 0$
  - …

Wrap-up of the algorithm $k = 1$

- Correctness:
  - We learn $O(\log n)$ bits
  - Each needs to succeed with probability $1 - O(1/\log n)$
  - Hence set $t = O(\log\log n)$
- Overall performance:
  - Number of samples: $O(\log n \cdot \log\log n)$
  - Same run-time

$k > 1$

- $x_i = a_1\omega^{if_1} + a_2\omega^{if_2} + \cdots a_k\omega^{if_k} + noise$
- Main ideas:
  - Isolate each frequency
    - Like in CountSketch or compressed sensing!
    - "Throw" frequencies in "buckets"
    - Hope have 1 frequency per "bucket"
  - Throw in buckets:
    - permute the frequencies (pseudo-)randomly
      - Can have frequencies go as $i \to ai + b$ for random a, b
    - partition in blocks: $\left[1, \frac{n}{k}\right], \left[\frac{n}{k} + 1, \frac{2n}{k}\right], \ldots$
    - Apply a filter that keeps only the correct block