Lecture 3: Frequency Moments: F_2, Heavy Hitters

COLUMBIA | ENGINEERING
The Fu Foundation School of Engineering and Applied Science

Administrivia, Plan

- Piazza: sign-up!
- PS1 releazed
- Scriber?

- Plan:
  - Frequency Moments
  - Heavy Hitters

Part 1: Frequency Moments

- Let f_i be frequency of i
  - Lecture 1: count one f_i
  - Lecture 2: count # of non-zeros
- Moment 1:
  - $\sum_i f_i$
  - Estimator with low space?
    - Just count
- Moment 2:
  - $\sum_i f_i^2$

IP	Frequency
1	3
2	2
3	0
4	9
5	0
…	0
n	1

$$\sum_i f_i = 15$$

$$\sum_i f_i^2 = 95$$

2nd Moment: F_2

[Alon-Matias-Szegedy 1996]

- **Idea**: *Rademacher* random variables

  hash function $r: [n] \to \{-1, +1\}$

- Algorithm (Tug-of-War):

  store $z = \sum_i r(i) \cdot f_i$

- Estimator:

  z^2

Algorithm TOW (F_2):

- Init: $z = 0$
- when see element i:

  $z = z + r(i)$

Estimator:

z^2

Rademacher r.v.

Algorithm TOW (F_2):
- Init: $z = 0$
- when see element i:
  $z = z + r(i)$

Estimator:
z^2

- What if we have m ones ?
  sum of m random ± 1's

- How much is $z = \sum r(i)$ roughly ?
  - Say, $|z|$?
  - $E[z] = 0$
  - $Var[z] = m$
  - Apply Chebyshev:
    - $|z| \le O(\sqrt{m})$ with constant probability
  - In fact tight

Analysis

- $E[z^2] = \cdots$
  $= \sum_i f_i^2$
- $Var[z^2] \le E[z^4] = \cdots$
  $\le O\left(\sum f_i^2\right)^2$
- Randomness?
  - $O(\log n)$ for h that is 4-wise independent
- Can apply the average trick:
  - Take $k = O\left(\frac{1}{\epsilon^2}\right)$ counters
  - Obtain: $1+\epsilon$ approximation in $O\left(\frac{1}{\epsilon^2}\log n\right)$ space

Algorithm TOW (F_2):
- Init: $z = 0$
- when see element i:
  $z = z + r(i)$

Estimator:
z^2

Linearity

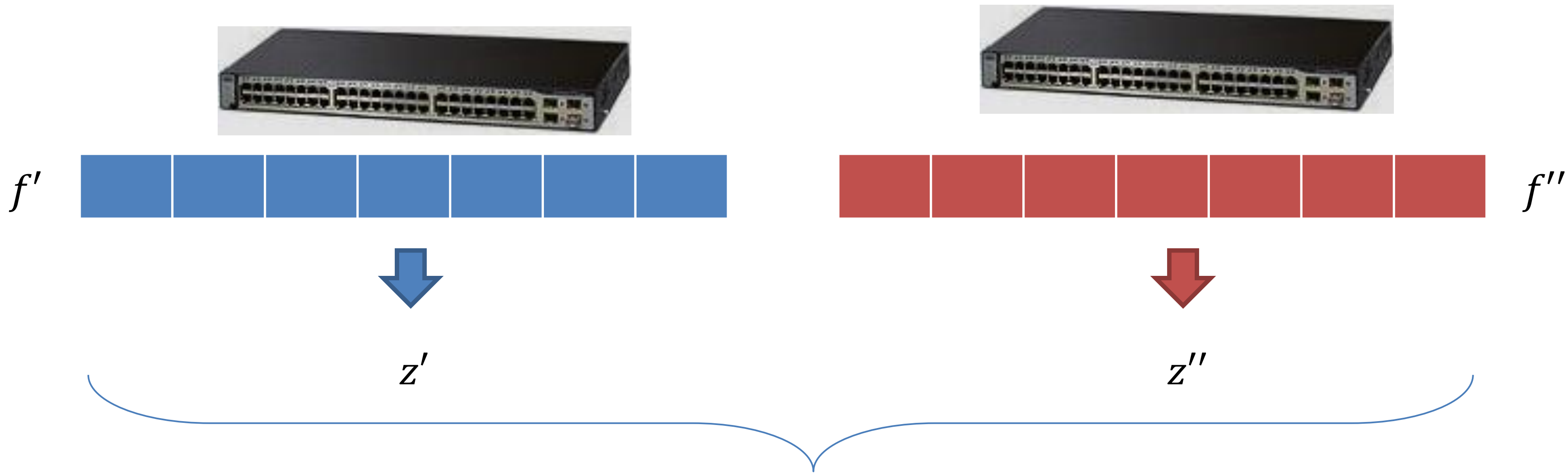
- Important property

Algorithm TOW (F_2):
- Init: $z = 0$
- when see element i:

  $z = z + r(i)$

Estimator:

z^2



$z = z' + z''$ (for $f = f' + f''$)

Similarly for difference!

- Estimate for $\sum(f_i' - f_i'')^2$

  $(z' - z'')^2$

- How about $\sum|f_i' - f_i''|$?
  - will see later in the class

IP	Frequency
131.107.65.14	1
18.9.22.69	1
35.8.10.140	1

z'

IP	Frequency
131.107.65.14	1
18.9.22.69	2

z''

General streaming model

- At each moment, an update is:
  (i, δ_i) : increase i^{th} entry by δ_i (may be negative!)

- Linear algorithm S handles easily:
  – $S(f + e_i\delta_i) = S(f) + S(e_i\delta_i)$
  – We'll call S a *sketch*

- [Nguyen-Li-Woodruff'14]: in fact any algorithm for general streamin might as well be linear!

Part 2: Heavy Hitters

- How about max frequency?

- Impossible to approximate in sublinear space!

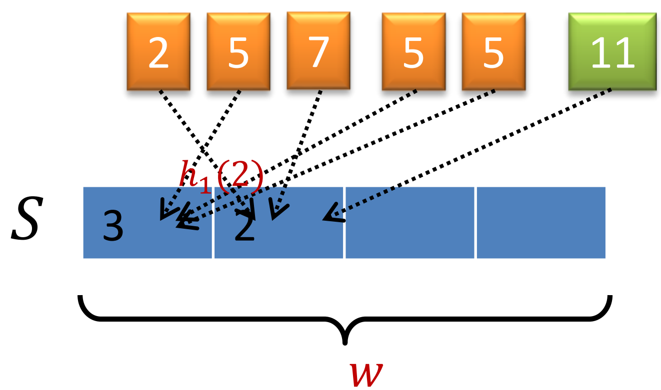
- Will settle for an even more modest goal:
  - can detect the max-frequency element if it is *very heavy*

IP	Frequency
1	3
2	2
3	0
4	9
5	0
…	0
n	1

Heavy Hitters: Iteration 1

[Charikar-Chen-FarachColton'04, Cormode-Muthukrishnan'05]

- **Definition**: i is ϕ-heavy if $f_i \geq \phi \sum_j f_j$
- Will find them in space $O(1/\phi)$
- **Idea:** hash functions!
  - $h: [n] \to [w]$ random $\quad w = O(1/\phi)$
  - Element i goes to bucket $h(i)$
  - In a bucket?
    - Sum frequencies there



Estimator for f_i ?

$$\hat{f}_i = S(h(i))$$

$\widehat{f_2} = 2$
$\widehat{f_5} = 3$
$\widehat{f_7} = 2$
$\widehat{f_{11}} = 2$

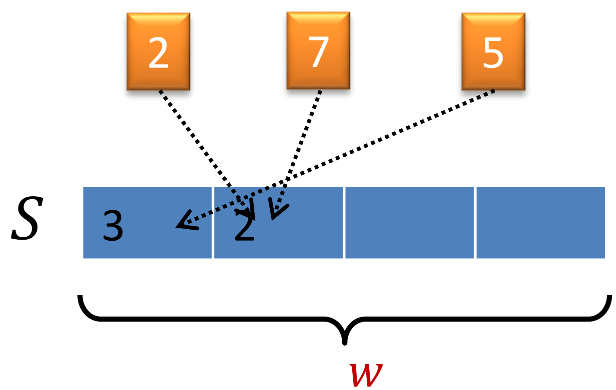
Iteration 1: analysis

- Let’s analyze:
  - Estimator of frequency for element i

$$\widehat{f_i} = S(h(i))$$
$$= f_i + \sum_{\{j:h(j)=h(i)\}} f_j$$

Extra “chaff”

- How much extra “chaff” is there?



Iteration 1: extra chaff

- $S(h(i)) = f_i + \underbrace{\sum_{\{j:h(j)=h(i)\}} f_j}_{C}$
- Extra "chaff":
  - $E[C] = \sum_j \Pr[h(j) = h(i)] \cdot f_j = \frac{\sum_{j \neq i} f_j}{w}$
- Is $S(h(i))$ an unbiased estimator?
  - No!
  - Bias is at most $\frac{\sum_j f_j}{w}$: small for $f_i \gg \frac{\sum_j f_j}{w}$
- Done?
  - Yes: by Markov $C \leq \frac{10 \sum_j f_j}{w}$ with 90% prob.

Iteration 1: really done?

- Estimator:

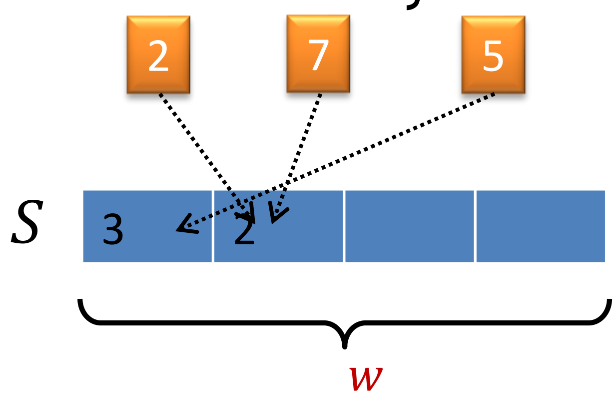
$$\widehat{f_i} = S(h(i)) = f_i + \sum_{\{j: h(j)=h(i)\}} f_j$$
$$= f_i + C$$

where $C \leq O(\sum_j f_j / w)$ with 90% prob

  - for $w = O\left(\frac{1}{\epsilon\phi}\right)$, and $f_i \geq \phi \sum_j f_j$

    $C \leq \epsilon f_i \Rightarrow \widehat{f_i}$ is a $1+\epsilon$ approximation!

- Issues?
  - Only constant probability
  - For many indices, it is an overestimate!

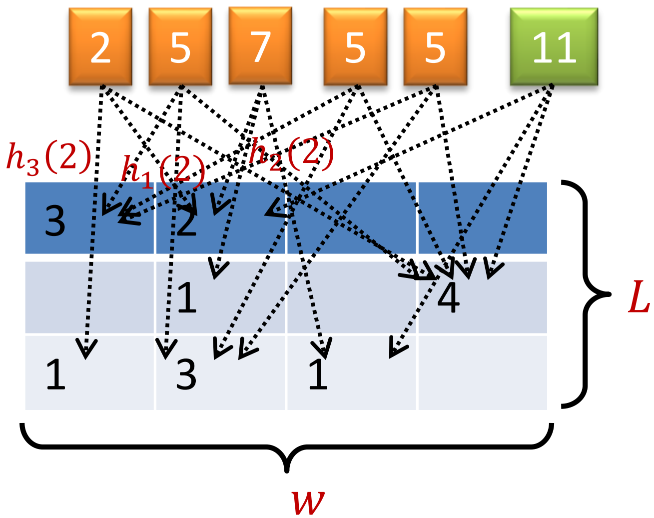


Fundamental issue: if i and j collide, can't know if it's i or j with high frequency;

but must have many collisions to reduce space

Iteration 2: CountMin

- Median trick!
  - Use $L = O(\log n)$ hash tables with hash functions h_j



$\widehat{f_2} = 2$
$\widehat{f_5} = 3$
$\widehat{f_7} = 1$
$\widehat{f_{11}} = 2$

```
Algorithm CountMin:

Initialize(r, L):
  array S[L][w]
  L hash functions h_1 … h_L, into {1,…w}

Process(int i):
  for(j=0; j<L; j++)
    S[j][ h_j(i) ] += 1;

Estimator:
  foreach i in PossibleIP {
    f̂_i = median_j(S[j][h_j(i)]);
  }
```

CountMin: analysis

- Consider an index i
- Each table gives
  - $\widehat{f_i} = f_i \pm \epsilon\phi$ with 90% probability
- Median is a $\pm\epsilon\phi$ with $1 - 1/n^2$ probability
  - Apply union bound over all $i \in [n]$
  - All are $\pm\epsilon\phi$, with $1 - 1/n$ probability
- Alternative estimator?
  - Take MIN instead of median

```
Algorithm CountMin:

Initialize(r, L):
  array S[L][w]
  L hash functions h_1 … h_L, into {1,…w}

Process(int i):
  for(j=0; j<L; j++)
    S[j][ h_j(i) ] += 1;

Estimator:
  foreach i in PossibleIP {
    f̂_i = ~~median~~_j(S[j][h_j(i)]);
  }      min
```

CountMin: overall

- Iterate over all i's
- Heavy hitters: $\frac{\widehat{f_i}}{\sum f_j} \geq \phi$
  - If $\frac{f_i}{\sum f_j} \leq \phi(1-\epsilon)$, not in the output
  - If $\frac{f_i}{\sum f_j} \geq \phi(1+\epsilon)$, reported as heavy hitter
- Space: $O\left(\frac{\log^2 n}{\epsilon\phi}\right)$ bits
- Issues?
  - Time: to iterate $\Omega(n)$

```
Algorithm CountMin:

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  array S[L][w]
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Process(int i):
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Estimator:
  foreach i in PossibleIP {
    f̂_i = ~~median~~_j(S[j][h_j(i)]);
  }      min
```
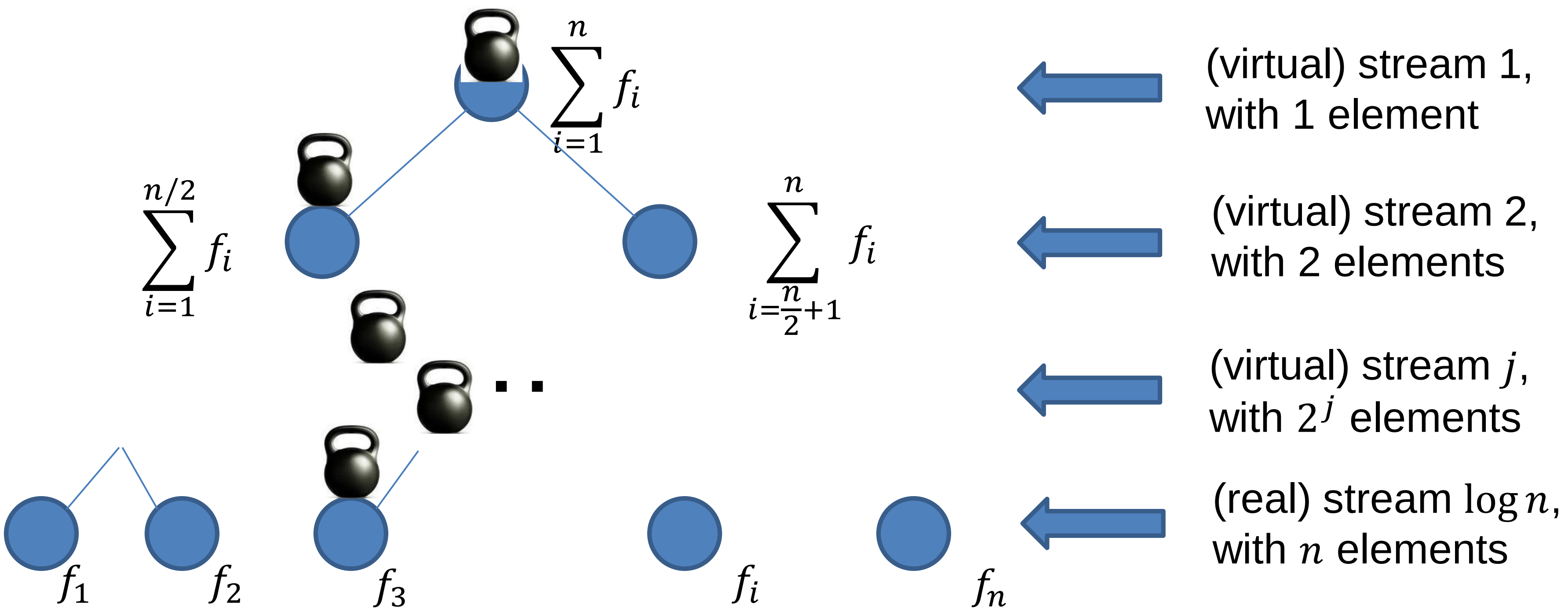

CountMin: time

- Can improve time; space degrades to $O\left(\frac{\log^3 n}{\epsilon\phi}\right)$ bits
- Idea: dyadic intervals
  - Each level with its own sketch
  - Find heavy hitters by following down the tree all the heavy hitters (in intermediary)



A variant: CountSketch

- Is CountMin linear?
  - CountMin($f' + f''$) from CountMin(f') and CountMin(f'') ?
  - Just sum the two!
    - sum the 2 arrays, assuming we use the same hash function h_j
- What about $f = f' - f''$?
  - "Heavy hitter": if $|f_i| \geq \phi \sum_j |f_j| = \phi \cdot ||f||_1$
  - "min" is an issue
  - But median is still ok
  - Ideas to improve it further?
    - Use Tug of War r in each bucket => CountSketch
    - Better in certain cases

Recap

- 2nd moment:
  - Tug-Of-War (sum of random ± 1's)
- Linearity:
  - Can add/subtract sketches easily
- Max-frequency:
  - Can only do heavy hitters
  - Hash functions to distribute elements
  - CountMin
    - https://sites.google.com/site/countminsketch/
  - CountSketch: CountMedian+TugOfWar